J. Appl. Prob. **42,** 879–882 (2005)
Printed in Israel

JACKSON NETWORKS WITH UNLIMITED SUPPLY OF WORK

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Abstract

We consider a Jackson network in which some of the nodes have an infinite supply of work: when all the customers queued at such a node have departed, the node will process a customer from this supply. Such nodes will be processing jobs all the time, so they will be fully utilized and experience a traffic intensity of 1. We calculate flow rates for such networks, obtain conditions for stability, and investigate the stationary distributions. Standard nodes in this network continue to have product-form distributions, while nodes with an infinite supply of work have geometric marginal distributions and Poisson inflows and outflows, but their joint distribution is not of product form.

Keywords: Queue; Jackson network; infinite virtual buffer

2000 Mathematics Subject Classification: Primary 60K25; 90B22; 68M20; 90B15

Consider a Jackson network [5] with nodes i, service rates μ_i, exogenous input rates α_i, and routing probabilities $P_{ij},\ i, j = 1, \dots, I,\ j \neq i$. Denote by $Q_i(t),\ t > 0$, the number of items in node i at time t. In addition, assume that a subset of the nodes $E \subseteq \{1, \dots, I\}$ have an infinite supply of work, by which we mean the following. For $i \in E$, when $Q_i(t) = 0$, a new item is picked from an infinite supply of items and is processed by the node at the rate μ_i. Upon completion of processing, it is routed according to P_{ij}. At each of the nodes $i \in E$, items that arrive at the node have preemptive priority over items from the infinite supply. After its initial processing, an item from the infinite supply is treated like any other item.

We believe that such Jackson networks with an infinite supply of work are a useful and realistic model for some situations; for example, consider a communications network, where each node is transmitting messages originating at this node, with an unlimited supply of material to transmit. In addition, each node also serves to transmit messages that are in transit between other nodes. Assume that each node gives preemptive priority to messages in transit over its own messages. When only messages in transit are counted as congestion, this is exactly our system. A particular computer communication system that works in this way is a MAN (metropolitan area network) Ethernet RPR (resilient packet ring), in which ring traffic has priority over the traffic generated at nodes.

The idea of an infinite supply (or backlog) of lower-priority work in a system has been used frequently, e.g. in [3] and [6]. Jackson networks with an infinite supply of work are a special case of multiclass queueing networks with virtual infinite buffers, introduced in [2], [8], [9], and [10].

Recall that all service and interarrival times in a Jackson network are independent and exponentially distributed. Also, assume that the routing matrix $\boldsymbol{P}$ has spectral radius less than 1,

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Received 16 November 2004; revision received 21 February 2005.
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Supported in part by Israel Science Foundation grant number 249/02.

so that, with probability 1, every item leaves the system after a finite number of processing steps. Let $\bar{E} = \{i : i \notin E\}$. We will use $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, and $\boldsymbol{\alpha}$ to denote vectors, and E, $\bar{E}$, and $E\bar{E}$, as subscripts, to denote subvectors or submatrices.

Let λ_i denote the rate at which items arrive into node i, counting exogenous input or routeing from other nodes. In equilibrium, the rate at which items depart from node i is λ_i, $i \in \bar{E}$, and μ_i, $i \in E$. Hence, the traffic equations for this system are

$$\lambda_i = \alpha_i + \sum_{j \in \bar{E} \neq i} \lambda_j P_{ji} + \sum_{j \in E \neq i} \mu_j P_{ji}.$$

These equations are solved by

$$\begin{aligned} \boldsymbol{\lambda}_{\bar{E}} &= (\boldsymbol{I} - \boldsymbol{P}_{\bar{E}\bar{E}}^{\top})^{-1}(\boldsymbol{\alpha}_{\bar{E}} + \boldsymbol{P}_{E\bar{E}}^{\top}\boldsymbol{\mu}_E), \\ \boldsymbol{\lambda}_E &= \boldsymbol{\alpha}_E + \boldsymbol{P}_{EE}^{\top}\boldsymbol{\mu}_E + \boldsymbol{P}_{\bar{E}E}^{\top}\boldsymbol{\lambda}_{\bar{E}} \\ &= \boldsymbol{\alpha}_E + \boldsymbol{P}_{\bar{E}E}^{\top}(\boldsymbol{I} - \boldsymbol{P}_{\bar{E}\bar{E}}^{\top})^{-1}\boldsymbol{\alpha}_{\bar{E}} + (\boldsymbol{P}_{EE}^{\top} + \boldsymbol{P}_{\bar{E}E}^{\top}(\boldsymbol{I} - \boldsymbol{P}_{\bar{E}\bar{E}}^{\top})^{-1}\boldsymbol{P}_{E\bar{E}}^{\top})\boldsymbol{\mu}_E, \end{aligned} \tag{1}$$

where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{P}^{\top}$ denotes the transpose of $\boldsymbol{P}$. Formulae similar to (1) were derived by Goodman and Massey [4], in a paper on transient Jackson networks, in which nodes $i \in E$ were unstable.

A necessary condition for stability is $\boldsymbol{\mu} \geq \boldsymbol{\lambda}$. The nodes with the infinite supply of work introduce new items into the system at the following rates:

$$\boldsymbol{\eta}_E = \boldsymbol{\mu}_E - \boldsymbol{\lambda}_E.$$

It is easy to establish that $\boldsymbol{\mu} > \boldsymbol{\lambda}$ is sufficient for stability, and indeed to partially derive steady-state distributions.

Proposition 1. *Assume that* $\rho_i = \lambda_i/\mu_i < 1$, $i = 1, \ldots, I$.

(i) *For nodes* $i \in \bar{E}$, *the joint steady-state distribution is of product form:*

$$\lim_{t\to\infty} \mathrm{P}\{Q_i(t) = n_i,\ I \in \bar{E}\} = \prod_{i \in \bar{E}} (1 - \rho_i)(\rho_i)^{n_i}.$$

(ii) *For nodes* $i \in E$, *the marginal steady-state distribution is*

$$\lim_{t\to\infty} \mathrm{P}\{Q_i(t) = m\} = (1 - \rho_i)(\rho_i)^m, \qquad i \in E.$$

(iii) *The departure streams from node* $i \in E$ *to all other nodes* $j \neq i$ *are independent Poisson streams of rates* $\mu_i P_{ij}$.

(iv) *The arrival streams into node* $i \in E$ *from all other nodes* $j \neq i$ *are independent Poisson streams of rates* $\lambda_j P_{ji}$, $j \in \bar{E}$ *and* $\mu_j P_{ji}$, $j \in E$.

The key observation for the proof of this proposition is that each of the nodes $i \in E$ works non-stop, processing items for independent and identically exponentially distributed times at rate μ_i. Hence, departures from the nodes with the infinite supply of work consist of independent Poisson streams. Thus, the subnetwork of nodes $i \in \bar{E}$ behaves like a Jackson network with Poisson inputs.

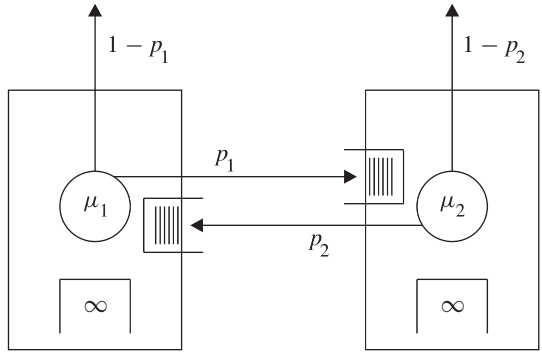

FIGURE 1: A two-node Jackson network with an infinite supply of work.

The difference between the nodes with an infinite supply of work and the standard nodes is intriguing. For the nodes with an infinite supply of work, all the input and output streams are Poisson. It is well known [7] that the streams of items moving between nodes in the Jackson part of the network, $i \in \bar{E}$, are not necessarily Poisson. (All the streams of items entering and leaving the subnetwork are Poisson, and streams between, say, node i and node j may be Poisson in some special cases, for example in overtake-free networks. However, in general, the stream of customers moving from node i to node j in a Jackson network need not be Poisson.) On the other hand, the product form, i.e. the independence of the queue lengths at different nodes when observed all at the same time, t, is lost for nodes $i \in E$. It is no longer true that $Q_i(t)$ and $Q_j(t)$, $i, j \in E$, in steady state are independent. This loss of independence can be observed in the following example.

In a recent paper, Adan and Weiss [1] analyzed a two-node Jackson network with an infinite supply of work, and no exogenous input. This system is depicted in Figure 1. In Table 1, we list the values of the stationary probabilities for such a symmetric two-node Jackson network with an infinite supply of work, with $\mu_1 = \mu_2$, $p_1 = p_2 = 0.5$, and

TABLE 1: Stationary distribution of the infinite-supply Jackson network.

n_2	$n_1 = 0$	$n_1 = 1$	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$
0	0.151 946	0.151 946	0.094 020	0.050 354	0.025 752
1	0.151 946	0.057 926	0.021 833	0.009 457	0.004 471
2	0.094 020	0.021 833	0.005 688	0.001 918	0.000 802
3	0.050 354	0.009 457	0.001 918	0.000 481	0.000 161
4	0.025 752	0.004 471	0.000 802	0.000 161	0.000 040

TABLE 2: Product-form joint distribution.

n_2	$n_1 = 0$	$n_1 = 1$	$n_1 = 2$	$n_1 = 3$	$n_1 = 4$
0	0.25	0.125	0.062 5	0.031 25	0.015 625
1	0.125	0.062 5	0.031 25	0.015 625	0.007 813
2	0.062 5	0.031 25	0.015 625	0.007 813	0.003 906
3	0.031 25	0.015 625	0.007 813	0.003 906	0.001 953
4	0.015 625	0.007 813	0.003 906	0.001 953	0.000 977

$\rho_1 = \rho_2 = 0.5$. The occupancies of the two queues are clearly not independent under this distribution. For comparison, in Table 2 we give the corresponding probabilities for a standard symmetric two-node product-form Jackson network, with $\rho = 0.5$. The calculations use the results of [1].

The following intuitive thought may provide an explanation of the dependence between the two nodes in the system of Figure 1. A large number of items in node 1 indicates that there were more arrivals than departures in the recent past. Arrivals come from node 2, and departures go to node 2, so this may indicate more departures and fewer arrivals at node 2 in the recent past. Thus, observing many customers in node 1 may indicate a small number of customers in node 2. This form of negative correlation is indeed observed in Table 1.

It is a challenging question to derive similar results in the general case.

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https://doi.org/10.1239/jap/1127322036 Published online by Cambridge University Press