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CONJUGACY CLASSES OF MAXIMAL TORI IN SIMPLE REAL ALGEBRAIC GROUPS AND APPLICATIONS

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ABSTRACT. Let G be an almost simple complex algebraic group defined over $\mathbf{R}$, and let $G(\mathbf{R})$ be the group of real points of G. We enumerate the $G(\mathbf{R})$-conjugacy classes of maximal $\mathbf{R}$-tori of G. Each of these conjugacy classes is also a single $G(\mathbf{R})^\circ$-conjugacy class, where $G(\mathbf{R})^\circ$ is the identity component of $G(\mathbf{R})$, viewed as a real Lie group. As a consequence we also obtain a new and short proof of the Kostant–Sugiura's theorem on conjugacy classes of Cartan subalgebras in simple real Lie algebras.

A connected real Lie group P is said to be *weakly exponential* (w.e.) if the image of its exponential map is dense in P. This concept was introduced in [HM] where also the question of identifying all w.e. almost simple real Lie groups was raised. By using a theorem of A. Borel and our classification of maximal $\mathbf{R}$-tori we answer the above question when P is of the form $G(\mathbf{R})^\circ$.

0. **Introduction.** Let G be an almost simple complex algebraic $\mathbf{R}$-group and $G(\mathbf{R})$ the group of real points of G. There are only finitely many $G(\mathbf{R})$-conjugacy classes of maximal $\mathbf{R}$-tori of G, see [Ser, p. III-34]. We describe a general method for finding representatives of these conjugacy classes. Our method is based on Borel-Tits theory of reductive algebraic groups over arbitrary fields and is presented in Section 1. If $G(\mathbf{R})$ is compact then there is only one conjugacy class (see [H, p. 248]). Hence we will exclude this case from our consideration.

If G is of classical type we use a different method which enables us to give an explicit description of all maximal $\mathbf{R}$-tori of G. This method uses the standard representation $G \longrightarrow \mathrm{GL}(V)$, where V is a real, complex or quaternionic vector space and is presented in Section 2.

The exceptional groups G are treated in Section 3. We have enumerated the $G(\mathbf{R})$-conjugacy classes of maximal $\mathbf{R}$-tori of G and provided some information about their maximal split and maximal anisotropic subtori.

The above mentioned results can be used to obtain the classification of Cartan subalgebras of absolutely simple finite dimensional real Lie algebras $\mathfrak{g}$, with respect to the action of the adjoint group of $\mathfrak{g}$. This classification is due to Kostant [K] and Sugiura [Su], where the Lie algebra techniques are used. Recently Helminck [He] has generalized these results to reductive algebraic groups, equipped with an involutorial automorphism, over an arbitrary field of characteristic $\neq 2$, and has simplified some of the proofs.

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We may assume that $\mathfrak{g}$ is the Lie algebra of $G(\mathbf{R})$ for a suitable almost simple $\mathbf{R}$-group G. If T is a maximal $\mathbf{R}$-torus of G, then the Lie algebra of $T(\mathbf{R})$ is a Cartan subalgebra of $\mathfrak{g}$ and all Cartan subalgebras of $\mathfrak{g}$ arise in this manner. This establishes a one to one correspondence between $G(\mathbf{R})^\circ$-conjugacy classes of maximal $\mathbf{R}$-tori in G and the conjugacy classes of Cartan subalgebras of $\mathfrak{g}$ under the adjoint action. L. P. Rothschild [R, Corollary 2.4] has shown that if two maximal $\mathbf{R}$-tori of G are $G(\mathbf{R})$-conjugate then they are also $G(\mathbf{R})^\circ$-conjugate. Consequently if H is another such group which is $\mathbf{R}$-isogeneous to G then we obtain a one to one correspondence between the $G(\mathbf{R})$-conjugacy classes of maximal $\mathbf{R}$-tori of G and those of H.

A connected Lie group P is called *exponential* (resp. *weakly exponential*) if the image of the exponential map is equal to P (resp. is dense in P). We abbreviate "weakly exponential" by w.e. The systematic study of w.e. groups was begun in [HM]. Recall that a *Cartan subgroup* of P is the centralizer in P of a Cartan subalgebra of the Lie algebra of P. A. Borel (see [HM]) has shown that a connected semisimple Lie group is w.e. iff all of its Cartan subgroups are connected. Hofmann and Mukherjea [HM] have raised the question of finding the complete list of w.e. almost simple real Lie groups. In the case when $P = G(\mathbf{R})^\circ$ the Cartan subgroups of P have the form $P \cap T(\mathbf{R})$ where T is a maximal $\mathbf{R}$-torus of G. By using our classification of maximal $\mathbf{R}$-tori, we answer, in Section 4, the above question for the groups of type $G(\mathbf{R})^\circ$.

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1. **A general method for classifying maximal R-tori.** Our main reference for notions pertaining to algebraic groups will be [Bo] and [BT1] where the reader may find all necessary definitions. Let k be a field of characteristic 0, $\bar{k}$ its algebraic closure and G an algebraic group defined over k (a *k-group*). In this section we describe a method for classifying maximal k-tori in connected reductive k-groups, which will be used in the sequel. Fix a maximal k-torus T of G containing a maximal k-split torus S of G. Denote by Φ (resp. by ${}_k\Phi$) the root (resp. k-root) system of G with respect to T (resp. S) and by Δ (resp. ${}_k\Delta$) a basis of Φ (resp. ${}_k\Phi$). We assume that the orderings of Φ (resp. ${}_k\Phi$) determined by Δ (resp. ${}_k\Delta$) are compatible. This means that if $\alpha \in \Phi$ is a positive root then $\alpha|_S$ is either 0 or a positive root in ${}_k\Phi$. Let θ be a subset of ${}_k\Delta$ and S_θ the connected component of identity of $\bigcap_{\alpha\in\theta} \operatorname{Ker}(\alpha)$. Then S_θ is a k-split subtorus of S and is called the *standard k-split torus of type* θ. Let W be the Weyl group of G with respect to T, *i.e.*, $W = N_G(T)/Z_G(T)$, where we use $N_G(\cdot)$ (resp. $Z_G(\cdot)$) to denote the normalizers (resp. centralizers) in G. We denote by ${}_kW$ the k-Weyl group of ${}_k\Phi$, *i.e.*, ${}_kW = N_G(S)/Z_G(S)$.

We recall briefly the definition of the k-index of a reductive k-group G. For more details see [Bo, p. 270], [T1], [Se]. The Galois group Γ of $\bar{k}/k$ acts on Φ. If $\sigma \in \Gamma$ then $\sigma(\Delta)$ is a base of Φ, and so there exists a unique $w_\sigma \in W$ such that $w_\sigma\sigma(\Delta) = \Delta$. This defines an action of Γ on Δ, called the **-action*. An orbit of Γ in Δ is called *distinguished* if it is contained in $\Delta \setminus \Delta_0$, where Δ_0 is the set of roots $\alpha \in \Delta$ which are trivial on S. The *k-index* of G is the pair (Δ, Δ_0) together with the *-action of Γ.

The following proposition establishes some properties of $Z_G(S_\theta)$ which are well known in the case $S_\theta = S$, see [T1], [Se].

PROPOSITION 1.1. *Let S_θ be a standard k-split torus of G and $Z_G(S_\theta) = S_\theta S_0 H$ (almost direct product), where S_0 is a k-torus and H a semisimple k-subgroup of G. Then the Tits index of H is obtained from that of G by removing all vertices not belonging to the preimage $\tilde{\theta}$ of $\theta \cup \{0\}$ under the restriction map $\rho\colon \Delta \longrightarrow_k \Delta \cup \{0\}$. Furthermore S_0 is k-anisotropic.*

PROOF. By [Bo, Section 21.11] we know that $Z_G(S_\theta)$ is generated by T and the root groups U_α, for roots α which are integral linear combinations of elements of $\tilde{\theta}$. This proves the first assertion.

The k-rank, r, of G is equal to the number of distinguished orbits in the k-index of G. By the first assertion, the k-rank, s, of H is the number of distinguished orbits in $\tilde{\theta}$. Hence $r - s$ is the number of distinguished orbits in $\Delta \setminus \tilde{\theta}$. By [BT1, Section 6], two roots $\alpha, \beta \in \Delta \setminus \tilde{\theta}$ have the same image under ρ if and only if they belong to the same distinguished orbit. Thus $r - s$ is the cardinality of ${}_k\Delta \setminus \theta$, *i.e.*, $r - s = \dim S_\theta$. Since S_θ is split, it follows that S_0 is anisotropic. ∎

PROPOSITION 1.2. *Any maximal k-torus T_1 of G is G(k)-conjugate to a maximal k-torus of G having a standard k-split torus as its k-split part.*

PROOF. The assertion is trivial if T_1 is k-anisotropic, so we assume that the k-split part S_1 of T_1 has positive dimension. Let P_1 be a parabolic k-subgroup of G, with $Z_G(S_1)$ as a Levi subgroup (see [Bo, Proposition 20.6]). It is well-known (see [Bo, Proposition 21.12]), that P_1 is k-conjugate to a standard k-parabolic subgroup P_θ of G, corresponding to a subset θ of ${}_k\Delta$, and we may assume that $Z_G(S_1) = Z_G(S_\theta)$. Since $T_1 \supset S_1$, T_1 is a maximal k-torus of $Z_G(S_\theta)$, and $S_1 \supset S_\theta$ since S_1 is the k-split part of T_1. From Proposition 1.1 it follows that S_θ is the k-split part of the connected center of $Z_G(S_\theta)$, hence $S_1 = S_\theta$. ∎

We call the torus T_1 in the above proposition a *standard maximal k-torus of type θ* if $S_1 = S_\theta$. It is clear that if S_θ is the k-split part of a maximal k-torus then the derived group of $Z_G(S_\theta)$ has a k-anisotropic maximal torus. We recall (see [Bo, Section 24.6]) that the group of real points of an anisotropic reductive $\mathbf{R}$-group is compact.

PROPOSITION 1.3. *Let $k = \mathbf{R}$ and let T_1 (resp. T_1') be a standard maximal $\mathbf{R}$-torus of G of type θ (resp. θ'). Then T_1 is $G(\mathbf{R})$-conjugate to T_1' if and only if θ is ${}_{\mathbf{R}}W$-conjugate to θ'.*

PROOF. Let $T_1 = S_\theta S_1$ and $T_1' = S_{\theta'} S_1'$ where S_1 and S_1' are anisotropic $\mathbf{R}$-tori. Assume that $w(\theta') = \theta$ for some $w \in_{\mathbf{R}} W$. By [Bo, Theorem 21.2] we can choose a representative $n \in N_G(S)(\mathbf{R})$ of w. Then $S_\theta = nS_{\theta'}n^{-1}$, and so we may assume that $S_\theta = S_{\theta'}$. By Proposition 1.1, $Z_G(S_\theta) = S_\theta T_0 H$ where H is a semisimple $\mathbf{R}$-subgroup of G and T_0 is an anisotropic $\mathbf{R}$-torus. By Cartan's theorem (see [H, Theorem 2.2, p. 256]) all maximal compact subgroups of $H(\mathbf{R})$ are conjugate. Thus we may assume that $T_2(\mathbf{R})$ and $T_2'(\mathbf{R})$ are maximal tori of a fixed maximal compact subgroup of $H(\mathbf{R})$. Consequently they are conjugate in $H(\mathbf{R})$, and so T_1 and T_1' are $G(\mathbf{R})$-conjugate.

Conversely, if $a \in G(\mathbf{R})$, and $T_1 = aT_1' a^{-1}$, then $aS_{\theta'}a^{-1} = S_\theta$. Since $S_\theta, S_{\theta'}$ are subtori of S, by [BT1, Corollary 4.22] there is $b \in G(\mathbf{R})$ such that $bSb^{-1} = S$ and $bsb^{-1} = asa^{-1}$ for all $s \in S_{\theta'}$. If w is the image of b in ${}_{\mathbf{R}}W$, then $w(\theta') = \theta$ as required. ∎

Type	$\mathfrak{g}_c$	$\mathfrak{k}_c$	Restrictions	Inner/ outer	$G(\mathbf{R})$
A I	A_l	$B_{l/2}$, l even; $D_{(l+1)/2}$, l odd.	$l \geq 2$	$-$	$\mathrm{SL}_{l+1}(\mathbf{R})$
A II	A_{2l-1}	C_l	$l \geq 1$	$-$	$\mathrm{SL}_l(\mathbf{H})$
A III	A_l	$A_{p-1} + A_{q-1} + \mathbf{C}$	$l \geq 1$ $p+q=l+1$ $p \geq q \geq 1$	$+$	$\mathrm{SU}(p,q)$
B I	B_l	$D_{p/2} + B_{(q-1)/2}$, p even ; $B_{(p-1)/2} + D_{q/2}$, p odd .	$l \geq 2$ $p+q=2l+1$ $p > q \geq 1$	$+$	$\mathrm{SO}(p,q)$
C I	C_l	$A_{l-1} + \mathbf{C}$	$l \geq 3$	$+$	$\mathrm{Sp}_{2l}(\mathbf{R})$
C II	C_l	$C_p + C_q$	$l \geq 3$ $p+q=l$ $p \geq q \geq 1$	$+$	$\mathrm{Sp}(p,q)$
D I	D_l	$D_{p/2} + D_{q/2}$, p even; $B_{(p-1)/2} + B_{(q-1)/2}$, p odd.	$l \geq 4$ $p+q=2l$ $p \geq q \geq 1$	$(-)^p$	$\mathrm{SO}(p,q)$
D III	D_l	$A_{l-1} + \mathbf{C}$	$l \geq 4$	$+$	$\mathrm{SO}^*(2l)$

TABLE I Non-compact real forms of classical simple complex Lie algebras

From this proposition we derive immediately the following.

COROLLARY 1.4. *If $k = \mathbf{R}$, then all maximal $\mathbf{R}$-tori of G containing S are $G(\mathbf{R})$-conjugate.* ∎

Tables I and II provide the list of all non-compact real forms of simple complex Lie algebras $\mathfrak{g}$. In the first column of these tables we list the types of the real forms by using Cartan's notation. We denote by $\mathfrak{k}$ a maximal compact subalgebra of $\mathfrak{g}$, and by $\mathfrak{g}_c$ (resp. $\mathfrak{k}_c$) the complexification of $\mathfrak{g}$ (resp. $\mathfrak{k}$). Let G be an almost simple $\mathbf{R}$-group such that the Lie algebra of $G(\mathbf{R})$ is isomorphic to $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, and $\mathfrak{g}_u = \mathfrak{k} + i\mathfrak{p}$ a compact real form of $\mathfrak{g}_c$. The linear map α which is 1 on $\mathfrak{k}$ and -1 on $i\mathfrak{p}$ is an involutorial automorphism of $\mathfrak{g}_u$. If α is an inner (resp. outer) automorphism of $\mathfrak{g}_u$, then $\mathfrak{g}$, and also G, is said to be of *inner* (resp. *outer*) type. It follows from [GG, Proposition 8.5.2] that G is of inner type if and only if it has an anisotropic maximal

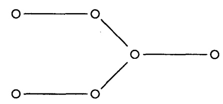
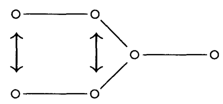
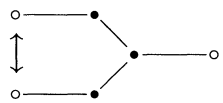
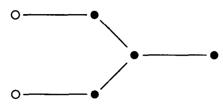
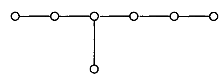
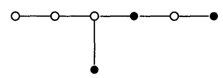
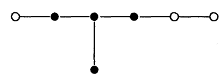
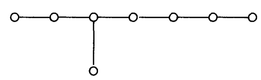
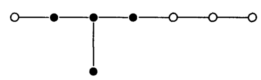
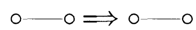
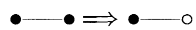
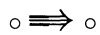
Type	$\mathfrak{g}_c$	$\mathfrak{k}_c$	Inner/Outer	Satake diagram
E I	E_6	C_4	$-$	
E II	E_6	$A_5 + A_1$	$+$	
E III	E_6	$D_5 + \mathbf{C}$	$+$	
E IV	E_6	F_4	$-$	
E V	E_7	A_7	$+$	
E VI	E_7	$D_6 + A_1$	$+$	
E VII	E_7	$E_6 + \mathbf{C}$	$+$	
E VIII	E_8	D_8	$+$	
E IX	E_8	$E_7 + A_1$	$+$	
F I	F_4	$C_3 + A_1$	$+$	
F II	F_4	B_4	$+$	
G I	G_2	$2A_1$	$+$	

TABLE II Non-compact real forms of exceptional complex Lie algebras

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R-torus. In Tables I and II we indicate by "+" sign the groups of inner type, and by "−" sign those of outer type. In the last column of Table II we give the Satake diagrams for the indicated algebras. For more details concerning these diagrams we refer the reader to [H], [Sa], or [W]. We also use the conventions $A_0 = B_0 = 0$, $B_1 = C_1 = A_1$, $B_2 = C_2$, $D_1 = \mathbf{C}$, $D_2 = 2A_1$, and $D_3 = A_3$.

2. **Maximal tori in classical real Lie groups.** We denote by V a finite dimensional vector space over $F = \mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$ (real quaternions). Sometimes V will be equipped with a non-degenerate form $f\colon V \times V \longrightarrow F$. We denote by $G(V,f)$ the group of F-automorphisms of V which preserve the form f. If f is absent, then $G(V,.)$ is the general linear group of V over F. We refer to the groups $G(V,f)$ as the *classical real Lie groups*. Each $G(V,f)$ is the group of real points of some **R**-group. Accordingly the term "torus" will refer to the group of real points of some **R**-torus. The classical groups for which the corresponding **R**-group is almost simple fall into seven classes summarized in the following table. Thus the complex general linear, orthogonal, and symplectic groups are excluded. The numbering in this table will be used also in Tables IV and V.

No.	F	f	Signature	$\dim_F V$	$G(V,f)$
1.	**R**			n	$GL_n(\mathbf{R})$
2.	**H**			n	$GL_n(\mathbf{H})$
3.	**C**	hermitian	(p,q)	$p+q$	$U(p,q)$
4.	**R**	symmetric	(p,q)	$p+q$	$O(p,q)$
5.	**H**	skew-hermitian		n	$SO^*(2n)$
6.	**R**	skew-symmetric		$2n$	$Sp_{2n}(\mathbf{R})$
7.	**H**	hermitian	(p,q)	$p+q$	$Sp(p,q)$

TABLE III Classical real Lie groups

If f is hermitian or symmetric, we define its *signature* to be (p,q), where p (resp. q) is the maximum dimension of a positive (resp. negative) definite subspace of V. We shall consider pairs (G,T) where G is a classical real Lie group and T is a torus of G. We shall say that such a pair is *maximal* if T is a maximal torus of G, and that it is *indecomposable* if $G = G(V,f)$, $V \neq 0$, and V has no non-zero proper T-invariant f-non-degenerate (if f is present) F-subspaces

PROPOSITION 2.1. *The isomorphism classes of maximal indecomposable pairs (G,T) are listed in Table IV below for each of the seven types of classical real Lie groups. In this table d_s (resp. d_a) denotes the dimension of the maximal split (resp. anisotropic) subtorus of T. With (p,q) we denote the signature of f when appropriate.*

PROOF. The proofs are similar in all 7 cases and we shall give details only for real orthogonal groups. In the cases 4(a) and 4(b) we have $G = \{\pm 1\}$, $T = \{1\}$, and since $\dim V = 1$, (G,T) is maximal and indecomposable. In the case 4(c), we have $G = O(1,1)$ and $T = SO(1,1)$. The two isotropic lines in V are non-isomorphic irreducible T-modules and so (G,T) is maximal and indecomposable. In the cases 4(d) and 4(e), we

No.	$\dim_F V$	d_s	d_a	p	q	$T \hookrightarrow G$
1 (a)	1	1	0			$T = \mathbf{R}^* = G$
(b)	2	1	1			$T = \mathbf{C}^* \hookrightarrow \mathrm{GL}_2(\mathbf{R}) = G$
2 (a)	1	1	1			$T = \mathbf{C}^* \hookrightarrow \mathbf{H}^* = G$
3 (a)	1	0	1	1	0	$T = \mathrm{U}(1) = G$
(b)	1	0	1	0	1	$T = \mathrm{U}(1) = G$
(c)	2	1	1	1	1	$T = \mathbf{C}^* \hookrightarrow \mathrm{U}(1,1) = G$
4 (a)	1	0	0	1	0	$T = 1$
(b)	1	0	0	0	1	$T = 1$
(c)	2	1	0	1	1	$T = \mathbf{R}^* \hookrightarrow \mathrm{O}(1,1) = G$
(d)	2	0	1	2	0	$T = \mathrm{SO}(2) \hookrightarrow \mathrm{O}(2) = G$
(e)	2	0	1	0	2	$T = \mathrm{SO}(2) \hookrightarrow \mathrm{O}(2) = G$
(f)	4	1	1	2	2	$T = \mathbf{C}^* \hookrightarrow \mathrm{GL}_2(\mathbf{R}) \hookrightarrow \mathrm{O}(2,2) = G$
5 (a)	1	0	1			$T = \mathrm{U}(1) = G$
(b)	2	1	1			$T = \mathbf{C}^* \hookrightarrow \mathrm{U}(1,1) \hookrightarrow \mathrm{SO}^*(2) = G$
6 (a)	2	1	0			$T = \mathbf{R}^* \hookrightarrow \mathrm{SL}_2(\mathbf{R}) = G$
(b)	2	0	1			$T = \mathrm{SO}(2) \hookrightarrow \mathrm{SL}_2(\mathbf{R}) = G$
(c)	4	1	1			$T = \mathbf{C}^* \hookrightarrow \mathrm{GL}_2(\mathbf{R}) \hookrightarrow \mathrm{Sp}_4(\mathbf{R}) = G$
7 (a)	1	0	1	1	0	$T = \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1) = G$
(b)	1	0	1	0	1	$T = \mathrm{U}(1) \hookrightarrow \mathrm{Sp}(1) = G$
(c)	2	1	1	1	1	$T = \mathbf{C}^* \hookrightarrow \mathrm{U}(1,1) \hookrightarrow \mathrm{Sp}(1,1) = G$

TABLE IV Maximal indecomposable pairs (G,T)

have $G = \mathrm{O}(2)$ and $T = \mathrm{SO}(2)$. Hence V is an irreducible T-module and so (G,T) is maximal and indecomposable. In the case 4(f) we have $G = \mathrm{O}(2,2)$ and we have obvious embeddings: $\mathbf{C}^* \hookrightarrow \mathrm{GL}_2(\mathbf{R}) \hookrightarrow \mathrm{O}(2,2)$. The image of $\mathbf{C}^*$ is a maximal torus T of G. As a T-module, V is a direct sum of two non-isomorphic irreducible submodules, each of which is a maximal totally isotropic subspace of V. Thus (G,T) is maximal and indecomposable.

Conversely, let (G,T) be a maximal indecomposable pair where $G = G(V,f)$. If there exists a 1-dimensional T-invariant subspace $V_1 \subset V$ such that V_1 is non-degenerate, then $V = V_1$ and (G,T) is of type 4(a) or 4(b). Thus we may assume that every 1-dimensional T-invariant subspace of V is isotropic. Assume that such a subspace, say V_1 exists. Clearly V_1 must be orthogonal to every 2-dimensional irreducible T-submodule of V. As V is a direct sum of 1- or 2-dimensional irreducible T-submodules, it follows that there is a 1-dimensional T-invariant subspace $V_2 \subset V$, not orthogonal to V_1. Hence $V_1 \oplus V_2$ is non-degenerate, and so $V = V_1 \oplus V_2$, and (G,T) is of type 4(c). From now on we may assume that there are no 1-dimensional T-invariant subspaces of V. If V

contains a 2-dimensional T-invariant subspace V_1 such that V_1 is non-degenerate, then V_1 is either positive or negative definite, and so $V = V_1$, and (G, T) is either of type 4(d) or 4(e). It remains to consider the case where $V = V_1 \oplus \cdots \oplus V_m$ and each V_i is a 2-dimensional irreducible and totally isotropic T-invariant subspace. We may assume that V_1 is not orthogonal to V_2. Then $V_1 \oplus V_2$ is non-degenerate and so $m = 2$. It follows that $G = \mathrm{O}(2, 2)$ and that $V_2 \simeq V_1^*$ as T-modules. Hence (G, T) is of type 4(f). ∎

Let (G_1, T_1) and (G_2, T_2) be two pairs belonging to the same series of classical real Lie groups, say $G_i = G(V_i, f_i)$, $i = 1, 2$. We set $V = V_1 \oplus V_2, f = f_1 \perp f_2$, $G = G(V, f)$ and $T = T_1 \times T_2$. Then we say that (G, T) is the sum of these two pairs and we write $(G, T) = (G_1, T_1) \sqcup (G_2, T_2)$.

PROPOSITION 2.2. *Let $G = G(V, f)$ be a classical real Lie group and $T \subset G$ a maximal torus. Then the pair (G, T) is a sum of maximal indecomposable pairs (G_i, T_i), $i = 1, \ldots, m$, which are unique up to ordering.*

PROOF. We prove first the existence of a sum decomposition by induction on $n = \dim_F V$. We may assume that (G, T) is decomposable. Hence there exists a non-zero proper F-subspace $V_1 \subset V$, which is T-invariant and f-non-degenerate (if f is present). There is a T-invariant subspace V_2 , complementary to V_1. In the case when f is present, we choose $V_2 = V_1^{\perp}$, and set $f_i = f|_{V_i \times V_i}$ for $i = 1, 2$. We have $T \hookrightarrow G_1 \times G_2 \hookrightarrow G$, where $G_i = G(V_i, f_i)$, and consequently $T = T_1 \times T_2$, where T_i is a maximal torus of G_i. We have therefore $(G, T) = (G_1, T_1) \sqcup (G_2, T_2)$ and we can use induction hypothesis to conclude the proof.

Next we prove the uniqueness assertion. We can write

$$(G, T) = \bigsqcup_{i=1}^{m} (G_i, T_i),$$

where each (G_i, T_i) is a maximal indecomposable pair, say $G = G(V, f)$, $G_i = G(V_i, f_i)$, $f = f_1 \perp \cdots \perp f_m$ and

$$V = V_1 \bigoplus \cdots \bigoplus V_m. \tag{2.1}$$

Each V_i is either irreducible as T-module (this is certainly the case if f is absent), or it is a direct sum of two totally isotropic irreducible T-submodules of the same dimension, say $V_i = V_i' \oplus V_i''$. We may assume that the former alternative holds for $i \leq r$ and the latter for $i > r$. The irreducible T-modules $V_1, \ldots, V_r, V_{r+1}', V_{r+1}'', \ldots, V_m', V_m''$ are pairwise non-isomorphic. Hence the decomposition

$$V = V_1 \oplus \cdots \oplus V_r \oplus V_{r+1}' \oplus V_{r+1}'' \oplus \cdots \oplus V_m' \oplus V_m''$$

is unique up to ordering. Since $V_i' \perp V_j$, $V_i' \perp V_j'$, and, if $i \neq j$, $V_i' \perp V_j''$, the decomposition (2.1) is also unique up to ordering. ∎

We shall write the decomposition (2.1) as

$$V = n_a V_a \bigoplus n_b V_b \bigoplus \cdots,$$

where the indices $a, b, \ldots$ refer to the isomorphism types $(G_a, T_a), (G_b, T_b), \ldots$ of maximal indecomposable pairs as given in Table IV, and $n_a, n_b, \ldots$ are the corresponding multiplicities. Therefore every maximal torus T of $G = G(V,f)$ has the form

$$T = T_a^{n_a} \times T_b^{n_b} \times \cdots .$$

We shall refer to $n_a, n_b, \ldots$ as the *parameters* of T.

PROPOSITION 2.3. *Two maximal tori of $G = G(V,f)$ are G-conjugate if and only if they have the same parameters. The number N of G-conjugacy classes of maximal tori in G is given in the last column of Table V.*

No.	$\dim_F V$	Equations for T	N
1	n	$n_a + 2n_b = n$	$[n/2] + 1$
2	n	$n_a = n$	1
3	$p + q = n$, $p \geq q \geq 0$	$n_a + n_c = p$ $n_b + n_c = q$	$q + 1$
4	$p + q = 2n + 1$, $p \geq q \geq 0$	$n_a + n_b = 1$ $n_e + n_f = (q - n_b - n_c)/2$ $n_d = n_e + (p - q - n_a + n_b)/2$	$(q+2)^2/4$, q even; $(q+1)(q+3)/4$, q odd.
4	$p + q = 2n$, $p \geq q \geq 0$	$n_a = n_b = 0$ $n_d = n_e + (p - q)/2$ $n_e + n_f = (q - n_c)/2$	$(q+2)(q+4)/8$, q even; $(q+1)(q+3)/8$, q odd.
5	n	$n_a + 2n_b = n$	$[n/2] + 1$
6	$2n$	$n_a + n_b + 2n_c = n$	$(n+2)^2/4$, n even; $(n+1)(n+3)/4$, n odd.
7	$p + q = n$, $p \geq q \geq 0$	$n_a + n_c = p$ $n_b + n_c = q$	$q + 1$

TABLE V Conjugacy classes of maximal tori in classical real Lie groups

PROOF. Each maximal torus T of G defines uniquely (up to ordering) a direct decomposition of type (2.1). It is clear that G-conjugate maximal tori of G give rise to G-equivalent decompositions.

In order to prove the converse, we may assume that two maximal tori, say T and T', determine the same decomposition of V. We have

$$T = \prod_{i=1}^{m} T_i, \quad T' = \prod_{i=1}^{m} T_i'$$

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with (G_i, T_i) and (G_i, T_i') maximal and indecomposable. It is easy to verify that T_i and T_i' are conjugate in G_i.

The number N of G-conjugacy classes of maximal tori is obtained by counting the number of non-negative integral solutions $n_a, n_b, \ldots$ of equations in column 4. For instance, let $G = \mathrm{O}(p, q)$, with $p + q = 2n + 1, p \geq q \geq 0$. We have either

(i) $n_a = 1, n_b = 0$, or

(ii) $n_a = 0, n_b = 1$.

In case (i), the number of solutions is

$$([q/2] + 1)([q/2] + 2)/2,$$

and in case (ii) ,

$$[(q + 1)/2]\big([(q + 1)/2] + 1\big)/2.$$

By adding these two numbers we obtain N given in the last column. ∎

If (G, T) is a pair of type 1(a) or 1(b) then $N_G(T)$ is not contained in G°. This implies that $\mathrm{SL}_n(\mathbf{R})$-conjugacy classes of maximal tori of $\mathrm{GL}_n(\mathbf{R})$ are the same as $\mathrm{GL}_n(\mathbf{R})$-conjugacy classes. The description of the conjugacy classes of maximal tori in the groups $\mathrm{SL}_n(\mathbf{H})$ (resp. $\mathrm{SU}(p, q)$) follows immediately from that of $\mathrm{GL}_n(\mathbf{H})$ (resp. $\mathrm{U}(p, q)$). The case of $\mathrm{SO}(p, q)$ and $\mathrm{SO}(p, q)^\circ$ is more delicate and we consider it in the next proposition.

PROPOSITION 2.4. *Let $G = G(V,f) = \mathrm{O}(p, q)$, $p \geq q \geq 1$, and $H = \mathrm{SO}(p, q)$. Then every G-conjugacy class of maximal tori of G is also an H-conjugacy class, with only one exception: $p = q$ is even, $n_a = n_b = n_c = n_d = n_e = 0$, and $n_f = p/2$, in which case this G-conjugacy class is a union of two H-conjugacy classes. Furthermore the H-conjugacy classes and the H°-conjugacy classes coincide.*

PROOF. Let T be a maximal torus of G and N its normalizer in G. Let

$$(G, T) = \bigsqcup_{i=1}^{m} (G_i, T_i),$$

where (G_i, T_i) are indecomposable and $G_i = G(V_i, f_i)$. Set $H_i = \{x \in G_i : \det(x) = 1\}$ and let N_i be the normalizer of T_i in G_i. It is easy to verify that $N_i \subset H_i$ iff (G_i, T_i) is of type 4(f).

In order to prove the first assertion, observe that

$$\{xTx^{-1} : x \in G\} = \{xTx^{-1} : x \in H\} \iff N \not\subset H.$$

If at least one pair (G_i, T_i) is not of type 4(f) then $N_i \not\subset H_i$, and consequently $N \not\subset H$. We now consider the exceptional case where all pairs (G_i, T_i) are of type 4(f) and so $p = q = 2m$. Each V_i is a direct sum of irreducible totally isotropic T-submodules V_i' and V_i''. We have to prove that $N \subset H$. An arbitrary $a \in N$ permutes the pairs $\{V_i', V_i''\}$. Since $N \cap H$ induces all possible permutations of such pairs and N_i contains an element that interchanges V_i' and V_i'', we may assume that all subspaces V_i' and V_i'' are a-invariant. It follows that $\det(a) = 1$ and so $N \subset H$. The second assertion is a special case of the result of Rothschild mentioned earlier. ∎

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Type	d_s	c	H	Type	d_s	c	H
E I	6	1	0	E II	4	1	0
	5	1	A_1		3	1	A_1
	4	1	$2A_1$		2	1	A_3
	3	1	$3A_1$		1	1	A_5
$N = 5$	2	1	D_4	$N = 5$	0	1	E_6
E III	2	1	A_3	E IV	2	1	D_4
	1	1	A_5				
$N = 3$	0	1	E_6	$N = 1$			
E V	7	1	0	E VI	4	1	$3A_1$
	6	1	A_1		3	1	$4A_1$
	5	1	$2A_1$		2	1	$D_4 + A_1$
	4	2	$3A_1, (3A_1)'$		1	1	D_6
	3	2	$4A_1, D_4$	$N = 5$	0	1	E_7
	2	1	$D_4 + A_1$	E VII	3	1	D_4
	1	1	D_6		2	1	$D_4 + A_1$
	0	1	E_7		1	1	D_6
$N = 10$				$N = 4$	0	1	E_7
E VIII	8	1	0	E IX	4	1	D_4
	7	1	A_1		3	1	$D_4 + A_1$
	6	1	$2A_1$		2	1	D_6
	5	1	$3A_1$		1	1	E_7
	4	2	$4A_1, D_4$	$N = 5$	0	1	E_8
	3	1	$D_4 + A_1$	F I	4	1	0
	2	1	D_6		3	2	$A_1, \tilde{A}_1$
	1	1	E_7		2	2	$B_2, 2A_1$
	0	1	E_8		1	2	B_3, C_3
$N = 10$				$N = 8$	0	1	F_4
F II	1	1	B_3	G I	2	1	0
	0	1	F_4		1	2	$A_1, \tilde{A}_1$
$N = 2$				$N = 4$	0	1	G_2

TABLE VI Conjugacy classes of maximal **R**-tori in exceptional groups

3. **Maximal tori in exceptional real groups.** In this section G will denote an exceptional almost simple **R**-group. We study the $G(\mathbf{R})$-conjugacy classes of maximal **R**-tori in G, by using the results and notations of Section 1. As mentioned in the Introduction, we may assume that $G(\mathbf{R})$ is not compact. Let T' be a maximal **R**-torus of G. By d_s we denote the dimension of the **R**-split part S' of T', by c the number of $G(\mathbf{R})$-conjugacy

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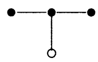
classes of maximal $\mathbf{R}$-tori T' with d_s fixed, and by H the semisimple part of $Z_G(S')$. The Dynkin diagram of H can be identified with a subdiagram of Δ. If Δ has roots of different lengths then $\tilde{A}_1$ denotes an A_1 made up of short roots. We shall show later by an example how one can determine the Satake diagram of H. Note that if G is simply connected then $G(\mathbf{R}) = G(\mathbf{R})^\circ$ (see [BT2, Corollary 4.7.]).

THEOREM 3.1. *With one exception in the case of E V, each $G(\mathbf{R})$-conjugacy class of maximal $\mathbf{R}$-tori T' is uniquely determined by d_s and the type of H. The $G(\mathbf{R})$-conjugacy classes of maximal $\mathbf{R}$-tori T' of G are listed in Table VI.*

PROOF. Since the proofs in all cases are of similar nature, we give only the proof s for some typical cases, including the case of split type E_7 which requires additional arguments. In all cases we use (explicitly or not) Proposition 1.1 in order to find the type of H.

a) Type E I. If $d_s = 6$, then $c = 1$ by Corollary 1.4. Let $d_s = 5$. Since all roots have the same length, they are permuted transitively by the Weyl group. Hence in this case $c = 1$. If $d_s = 4$, then H cannot be a split A_2, since the latter is of outer type. Thus H is of type $2A_1$, which corresponds to a pair of non-adjacent simple roots. As above, any two such pairs are W-conjugate and $c = 1$. If $d_s = 3$, the same argument as above shows that H is of type $3A_1$, which corresponds to a 3-element set of mutually orthogonal simple roots, and similarly $c = 1$. If $d_s = 2$, D_4 is the only possibility for H (we exclude some groups of outer type, like the case A_2 above). We have again $c = 1$, and we are done.

b) Type E III. If $d_s = 2$, then $c = 1$ (Corollary 1.4). If $d_s = 1$, then A_5 is the only possible type for H, since the group with the Satake diagram

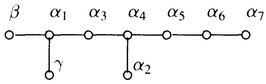


is of outer type. Hence $c = 1$.

c) Type E V. The argument used above shows that $c = 1$ if $d_s = 7, 6$ or 5. If $d_s = 4$, H must be of type $3A_1$. It is known (*cf.* [Dy]), that there are two W-conjugacy classes of root subsystems of type $3A_1$ in a root system of type E_7. Lemma 3.2 below implies that $c = 2$. If $d_s = 3$ then $4A_1$ and D_4 are the only possibilities for H. From [Dy] we know that there are two conjugacy classes of regular subgroups of type $4A_1$. As we are looking only for those corresponding to subsets of the given base of the root system, it is easy to see that there is only one conjugacy class of type $4A_1$ and one of type D_4. Thus $c = 2$. If $d_s = 2$ then H must be of type $D_4 + A_1$ and $c = 1$. Indeed D_5 is of outer type. If $d_s = 1$, then the only possible type for H is D_6. Since E V is of inner type, we also have $c = 1$ when $d_s = 0$. ∎

LEMMA 3.2. *The Dynkin diagram of the root system of type E_7 contains two subdiagrams of type $3A_1$, which are not W-conjugate.*

PROOF. We use the notation for roots as in [Bou]. Extend the Dynkin diagram of E_7 as follows



where

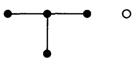
$$-\beta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7,$$
$$\gamma = \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7.$$

The roots orthogonal to α_7 form a root system of type D_6 with a base $\{\beta, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ and the roots in this D_6 which are orthogonal to α_5 form a root system of type $D_4 + A_1$, with the base $\{\beta, \gamma, \alpha_1, \alpha_3\}$ and $\{\alpha_2\}$. We claim that $\theta := \{\alpha_2, \alpha_5, \alpha_7\}$ and $\theta' := \{\alpha_3, \alpha_5, \alpha_7\}$ are not W-equivalent. Assume that there is a $w \in W$ such that $w(\theta) = \theta'$. In a root system of type A_3, say the one with the base $\{\alpha_5, \alpha_6, \alpha_7\}$, there is an element of its Weyl group which interchanges α_5 and α_7. It follows that in the root system of type A_5 with base $\{\alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7\}$, the roots $\alpha_3, \alpha_5, \alpha_7$ can be permuted arbitrarily by the Weyl group of that A_5. Hence we may assume that $w(\alpha_5) = \alpha_5$, $w(\alpha_7) = \alpha_7$ and so w belongs to the Weyl group of the root system $D_4 + A_1$, and we have a contradiction with $w(\alpha_2) = \alpha_3$. ∎

From the above it is clear that in order to find representatives of conjugacy classes of maximal **R**-tori, it is sufficient to find:

(a) representatives of ${}_{\mathbf{R}}W$-orbits of subsets θ of ${}_{\mathbf{R}}\Delta$,
(b) the corresponding standard split tori S_θ,
(c) a maximal **R**-torus T containing S_θ as its **R**-split part.

EXAMPLE. Let G be of type E VII. If $d_s = 2$ then $c = 1$ and H is of type $D_4 + A_1$. From Table II we see that the Satake diagram of H is



4. **Weakly exponential groups of type $G(\mathbf{R})^\circ$.** Hofmann and Mukherjea [HM] have raised the problem of deciding which almost simple Lie groups are w.e. In particular they have shown that all connected complex Lie groups are w.e. In this section we solve this problem for the groups $G(\mathbf{R})^\circ$, where G is an almost simple **R**-group. In the case of classical real Lie groups, as defined in Section 2, the answer to the above question is contained in the more general results obtained in [D2].

We denote by σ the complex conjugation of G. The universal covering group of G will be denoted by $\tilde{G}$ and the corresponding adjoint group by $\bar{G}$. As in [T2] we set $G^* = \tilde{G}(\mathbf{R})$. Recall that this group is always connected as a real Lie group.

We fix a maximal **R**-torus T of G containing a maximal **R**-split torus S of G. The symbols Φ and $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ will have the same meaning as in Section 1. For each simple root α_i we denote by $h_i\colon \mathbf{C}^* \longrightarrow T$ the corresponding multiplicative 1-parameter subgroup as defined in [St, Chapter 3]. We set $\epsilon_i = h_i(-1)$. If $\alpha = k_1\alpha_1 + \cdots + k_l\alpha_l$, where $k_i \in \mathbf{Z}$, we define $S_\alpha(t) = h_1(t)^{k_1} \cdots h_l(t)^{k_l}$. If $\beta_1, \ldots, \beta_r$ are integral linear combinations of simple roots then we denote by $S(\beta_1, \ldots, \beta_r)$ the subtorus of T generated by the images of $S_{\beta_1}, \ldots, S_{\beta_r}$. If $G = \tilde{G}$, then T is the direct product of 1-dimensional tori $S(\alpha_i)$, $i = 1, \ldots, l$ (see [St]).

The conjugation σ also acts on Φ and on the root lattice. Matsumoto [M, p. 421] explains how to compute this action of σ from the knowledge of the Satake diagram. One can show that if $\sigma(\alpha) = \varepsilon\alpha$, $\varepsilon = \pm 1$, then $\sigma S_\alpha(t)\sigma^{-1} = S_\alpha(\bar{t}^\varepsilon)$. Hence if $\sigma(\beta_i) = \beta_i$ (resp. $\sigma(\beta_i) = -\beta_i$) for all i then the torus $S(\beta_1, \ldots, \beta_r)$ is a split (resp. anisotropic) $\mathbf{R}$-torus.

We need the following theorem of A. Borel (see [HM]):

THEOREM 4.1. *A connected semisimple real Lie group is* w.e. *if and only if all of its Cartan subgroups are connected.* ∎

In the case of groups $G(\mathbf{R})^\circ$ the Cartan subgroups are $Q(\mathbf{R}) \cap G(\mathbf{R})^\circ$, where Q runs through all maximal $\mathbf{R}$-tori of G.

Let Q be an $\mathbf{R}$-torus of G and let $Q = Q_0Q_1$, where Q_0 (resp. Q_1) is the maximal $\mathbf{R}$-split (resp. $\mathbf{R}$-anisotropic) subtorus of Q. This is an almost direct product and so $F = Q_0 \cap Q_1$ is a finite group. Denote by Q_0^f the maximal compact subgroup of $Q_0(\mathbf{R})$, which is in fact a finite 2-group. We shall need the following lemma.

LEMMA 4.2. *With the above notation we have:*

(i) $Q(\mathbf{R}) = Q_0(\mathbf{R})Q_1(\mathbf{R}) = Q_0^f \cdot Q(\mathbf{R})^\circ$;

(ii) $Q(\mathbf{R})/Q(\mathbf{R})^\circ \simeq Q_0^f/\big(Q_0^f \cap Q_1(\mathbf{R})\big)$.

PROOF. (i) Since $Q_0(\mathbf{R}) = Q_0^f \cdot Q_0(\mathbf{R})^\circ$ and $Q_1(\mathbf{R})$ is connected, the second equality follows from the first. We now prove the first equality. Let $x \in Q(\mathbf{R})$ and write $x = st$, with $s \in Q_0$, $t \in Q_1$. Then $\sigma(x) = x$, *i.e.*, $\sigma(s)\sigma(t) = st$. It follows that $y := s\sigma(s)^{-1} = t^{-1}\sigma(t) \in F$. Hence $\sigma(t) = ty$. We claim that this implies $y = 1$. For this it suffices to consider the case when $\dim Q_1 = 1$. Then $Q_1 = \mathbf{C}^*$ with $\sigma(t) = \bar{t}^{-1}$ for $t \in \mathbf{C}^*$. Hence $y = t^{-1}\sigma(t) = (t\bar{t})^{-1} > 0$, and $y = 1$ since $y \in F$ and so y has finite order.

(ii) We have $Q_0(\mathbf{R}) = Q_0^f \cdot Q_0(\mathbf{R})^\circ$, and $Q(\mathbf{R})^\circ = Q_0(\mathbf{R})^\circ \cdot Q_1(\mathbf{R})$. Note that both of these products are direct products. By (i)

$$Q(\mathbf{R})/Q(\mathbf{R})^\circ \simeq Q_0^f/\big(Q_0^f \cap Q(\mathbf{R})^\circ\big).$$

Since $Q_0(\mathbf{R})^\circ$ is torsion-free, $Q_0^f \cap Q(\mathbf{R})^\circ = Q_0^f \cap Q_1(\mathbf{R})$. ∎

Next we show that, when applying Borel's theorem to $G(\mathbf{R})^\circ$, we need only to check whether $T(\mathbf{R}) \cap G(\mathbf{R})^\circ$ is connected.

LEMMA 4.3. *Let G, S and T be as above. If $T(\mathbf{R}) \cap G(\mathbf{R})^\circ$ is connected then $G(\mathbf{R})^\circ$ is w.e.*

PROOF. By Borel's theorem, it suffices to show that $Q(\mathbf{R}) \cap G(\mathbf{R})^\circ$ is connected for all maximal $\mathbf{R}$-tori Q of G. Let $Q = Q_0Q_1$ and $T = ST_1$ where Q_0 is the split part of Q, and Q_1 (resp. T_1) is the anisotropic part of Q (resp. T). We may assume that $Q_0 \subset S$ and $Q_1 \supset T_1$. By Lemma 4.2 we have

$$T(\mathbf{R}) \cap G(\mathbf{R})^\circ = S^f T(\mathbf{R})^\circ \cap G(\mathbf{R})^\circ = \left(S^f \cap G(\mathbf{R})^\circ\right) T(\mathbf{R})^\circ.$$

Since $T(\mathbf{R}) \cap G(\mathbf{R})^\circ$ is connected and $T_1(\mathbf{R})$ is the maximal compact subgroup of $T(\mathbf{R})^\circ$, we have

$$Q_0^f \cap G(\mathbf{R})^\circ \subset S^f \cap G(\mathbf{R})^\circ \subset T_1(\mathbf{R}) \subset Q_1(\mathbf{R}).$$

Hence $Q(\mathbf{R}) \cap G(\mathbf{R})^\circ = \left(Q_0^f \cap G(\mathbf{R})^\circ\right) Q(\mathbf{R})^\circ \subset Q_1(\mathbf{R}) Q(\mathbf{R})^\circ = Q(\mathbf{R})^\circ$. ∎

LEMMA 4.4. *Let Z be a central subgroup of G and $\phi\colon G \longrightarrow H = G/Z$ the canonical projection and assume that $Z \subset G(\mathbf{R})$. Then, if T is a maximal $\mathbf{R}$-torus of G, $\phi\big(T(\mathbf{R})\big) \cap H(\mathbf{R})^\circ$ is a Cartan subgroup of $H(\mathbf{R})^\circ$.*

PROOF. We have to show that $\phi\big(T(\mathbf{R})\big) \cap H(\mathbf{R})^\circ = \phi(T) \cap H(\mathbf{R})^\circ$. In order to prove the non-trivial inclusion, let $x \in \phi(T) \cap H(\mathbf{R})^\circ$. Since $H(\mathbf{R})^\circ \subset \phi\big(G(\mathbf{R})\big)$, we have $x = \phi(y) = \phi(z)$ for some $y \in T$ and $z \in G(\mathbf{R})$. It follows that $yz^{-1} \in Z \subset G(\mathbf{R})$, and consequently $y \in G(\mathbf{R})$. Thus $y \in T(\mathbf{R})$, and so $x \in \phi\big(T(\mathbf{R})\big) \cap H(\mathbf{R})^\circ$. ∎

THEOREM 4.5. *Let G be an almost simple complex algebraic group defined over $\mathbf{R}$. Then $G(\mathbf{R})^\circ$ is w.e. if and only if it is either compact or isomorphic to a quotient of a group listed in Table VII. (The elements z and z' of the center of* $\mathrm{Spin}_{2n}(\mathbf{C})$ *are defined below in the proof of this theorem.)*

A I	$\mathrm{PSL}_2(\mathbf{R})$
A II	$G^* = \mathrm{SL}_n(\mathbf{H}),\ n \geq 2$
A III	$G^* = \mathrm{SU}(p,q),\ p > q \geq 1$ $\mathrm{SU}(p,p)/Z_{2^m},\ 2p = 2^m r,\ r$ odd
B I	$\mathrm{SO}(2n,1)^\circ,\ n \geq 1$
C II	$G^* = \mathrm{Sp}(p,q),\ p \geq q \geq 1$
D I	$G^* = \mathrm{Spin}(2n-1,1),\ n \geq 3$ $\mathrm{PSO}(2n-2,2)^\circ,\ n \geq 3,\ n$ odd $\mathrm{Spin}(2n-2,2)/\langle z\rangle,\ n \geq 4,\ n$ even
D III	$G^* = \mathrm{Spin}^*(2n),\ n \geq 3,\ n$ odd $\mathrm{SO}^*(2n),\ n \geq 4,\ n$ even $\mathrm{Spin}^*(2n)/\langle z'\rangle,\ n \geq 4,\ n$ even
E III, IV	G^*
E VII	G^*/Z_2
F II	G^*

TABLE VII Maximal non-compact almost simple w.e. groups $G(\mathbf{R})^\circ$

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PROOF. If G is split over $\mathbf{R}$, then $G(\mathbf{R})^\circ$ is not w.e. except in the case $G = \mathrm{PSL}_2(\mathbf{C})$, see [HM, Proposition 2.12]. Hence we may assume that G is not split over $\mathbf{R}$. In particular we can exclude the types A I, C I, E I, E V, E VIII, F I, and G I from further considerations. Since connected compact Lie groups are exponential, we may assume that $G(\mathbf{R})^\circ$ is not compact. It is well known that the groups $\mathrm{SL}_n(\mathbf{H})$, $\mathrm{Sp}(p,q)$ and $\mathrm{SO}^*(2n)$ are exponential (see *e.g.* [D1]). The first two groups are also simply connected, and so we are done with types A II and C II.

In the remaining cases, in view of Lemma 4.3, we examine the connectedness of $T(\mathbf{R}) \cap G(\mathbf{R})^\circ$. Let S and S^f be defined as above. The centralizer $Z_G(S)$ is an almost direct product ST_1H, where H is the derived group of $Z_G(S)$ and T_1 is an anisotropic $\mathbf{R}$-torus (which may be trivial). By Lemma 4.2,

$$[T(\mathbf{R}) : T(\mathbf{R})^\circ] = [S^f : S^f \cap T_1 H].$$

In particular $T(\mathbf{R})$ is connected if and only if $S^f \subset T_1H$. We shall now consider each of the remaining Cartan types separately. By $\tilde{\alpha}$ we denote the highest root of Φ.

CASE A III. Let $G(\mathbf{R}) = \mathrm{SU}(p,q)$, $p \geq q \geq 1$, $p+q = n \geq 3$. We may assume that $T(\mathbf{R})$ consists of all diagonal matrices

$$\mathrm{diag}(\lambda_1, \ldots, \lambda_{p-q}, \mu_1, \bar{\mu}_1^{-1}, \ldots, \mu_q, \bar{\mu}_q^{-1})$$

where $\lambda_i, \mu_j \in \mathbf{C}^*$, $|\lambda_i| = 1$, and

$$\lambda_1 \cdots \lambda_{p-q} \mu_1 \bar{\mu}_1^{-1} \cdots \mu_q \bar{\mu}_q^{-1} = 1.$$

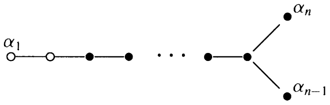
If $p > q$, $T(\mathbf{R})$ is connected and $\mathrm{SU}(p,q)$ is w.e. If $p = q$ the above equation reduces to $\mu_1 \cdots \mu_q \in \mathbf{R}^*$. Then $T(\mathbf{R})$ has two connected components and so $\mathrm{SU}(p,p)$ is not w.e. We now investigate the factor groups $\mathrm{SU}(p,p)/\langle \zeta^s \rangle$, where ζ is a primitive $2p$-th root of 1. We have $\zeta^s \in T(\mathbf{R})^\circ$ if and only if $\zeta^{ps} = 1$, *i.e.* s is even. Hence this factor group is w.e. if and only if s is odd.

CASE B I. Let $G(\mathbf{R}) = \mathrm{SO}(p,q)$, $p+q = 2n+1 \geq 3$, $p > q \geq 1$. The maximal torus $T(\mathbf{R})$ of $\mathrm{SO}(p,q)$ has parameters $n_a = 1$, $n_b = n_e = n_f = 0$, $n_c = q$, and $n_d = (p-q-1)/2$, see Section 2. It follows that $T(\mathbf{R})$ has 2^q connected components. Since $q > 0$, $T(\mathbf{R})$ is not contained in $\mathrm{SO}(p,q)^\circ$ and so $T(\mathbf{R}) \cap \mathrm{SO}(p,q)^\circ$ has 2^{q-1} connected components. Hence $\mathrm{SO}(p,q)^\circ$ is w.e. if and only if $q = 1$. It remains to consider the group G^* when $q = 1$. Let T be a maximal $\mathbf{R}$-torus of $\tilde{G} = \mathrm{Spin}_{2n+1}(\mathbf{C})$ containing a maximal split $\mathbf{R}$-torus S. We have $S = S(\alpha_1 + \cdots + \alpha_n)$ and $Z_G(S) = SH$, where $\Delta_H = \Delta \setminus \{\alpha_1\}$. As $S^f = \langle \epsilon_1 \epsilon_2 \cdots \epsilon_n \rangle \not\subset H$, $T(\mathbf{R})$ is not connected and G^* is not w.e.

CASE D I. Let $G(\mathbf{R}) = \mathrm{SO}(p,q)$, $p \geq q \geq 1$, $p+q = 2n \geq 6$. Let $T(\mathbf{R})$ be a maximally split maximal torus of $\mathrm{SO}(p,q)$. As in the previous case, $T(\mathbf{R})$ has 2^q connected components.

If $q \geq 3$ we claim that the adjoint group $\mathrm{PSO}(p,q)^\circ$ is not w.e. Indeed, the image of $T(\mathbf{R})$ under the canonical projection $\phi\colon \mathrm{SO}(p,q) \longrightarrow \mathrm{PSO}(p,q)$ has 2^{q-1} components. As $\mathrm{PSO}(p,q)$ has at most two components, the claim follows from Lemmas 4.3 and 4.4.

Let $q = 2$. Then $\phi\big(T(\mathbf{R})\big)$ has two components. Since $\phi\big(T(\mathbf{R})\big) \not\subset \mathrm{PSO}(p,2)^\circ$, $\phi\big(T(\mathbf{R})\big) \cap \mathrm{PSO}(p,2)^\circ$ is connected, and $\mathrm{PSO}(p,2)^\circ$ is w.e. Since $T(\mathbf{R}) \cap \mathrm{SO}(p,2)^\circ$ has two components, $\mathrm{SO}(p,2)^\circ$ is not w.e. It remains to consider the factor group $G^*/\langle z\rangle$ where n is even and z is a central element mapped to -1 by the projection map $G^* \longrightarrow \mathrm{SO}(p,2)^\circ$. The Satake diagram of G^* is



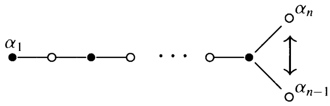
We may assume that $S = S(\tilde\alpha, \alpha_1)$ and $\Delta_H = \Delta \setminus \{\alpha_1, \alpha_2\}$. Since $\tilde\alpha = \alpha_1 + 2(\alpha_2 + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n$, we have $S^f = \langle \epsilon_1, \epsilon_{n-1}\epsilon_n\rangle$ and $[S^f : S^f \cap H] = 2$. Therefore $T(\mathbf{R})$ has two components. By using [St, Chapter 3] we find that the center of G^* is $\langle z, z'\rangle$, where

$$z = \epsilon_1\epsilon_3 \cdots \epsilon_{n-3}\epsilon_{n-1}, \quad z' = \epsilon_1\epsilon_3 \cdots \epsilon_{n-3}\epsilon_n.$$

As $zz' \in H$ and $z \notin H$, the image of $T(\mathbf{R})$ in $G^*/\langle z\rangle$ is connected. Hence $G^*/\langle z\rangle$ is w.e.

Finally let $q = 1$. Then the Satake diagram has only one white vertex, namely α_1. We have $S = S(\alpha_1 + \tilde\alpha)$ and $\Delta_H = \Delta \setminus \{\alpha_1\}$. Since $S^f = \langle \epsilon_{n-1}\epsilon_n\rangle \subset H$, $T(\mathbf{R})$ is connected, and so G^* is w.e.

CASE D III, A. Let $G = \mathrm{Spin}_{2n}(\mathbf{C})$, n odd. The Satake diagram of G, as an $\mathbf{R}$-group of this type, is



We have

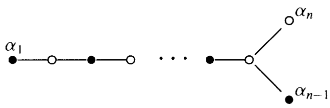
$$S = S\big(\tilde\alpha, \alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1};\ 1 \leq i \leq (n-3)/2\big),$$

and $\Delta_H = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-2}\}$. It follows that $Z_G(S) = ST_1H$, where $T_1 = S(\alpha_{n-1} - \alpha_n)$. Indeed, by using Matsumoto's recipe [M] we can easily check that σ fixes $\tilde\alpha$ and each of $\alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1}$, $1 \leq i \leq (n-3)/2$. Since $\sigma(\alpha_{n-1} - \alpha_n) = \alpha_n - \alpha_{n-1}$, the torus $S(\alpha_{n-1} - \alpha_n)$ is anisotropic. Since

$$S^f = \langle \epsilon_1\epsilon_{n-1}\epsilon_n, \epsilon_{2i-1}\epsilon_{2i+1};\ 1 \leq i \leq (n-3)/2\rangle$$

is contained in T_1H, $T(\mathbf{R})$ is connected. Hence $G(\mathbf{R})$ is w.e.

CASE D III, B. Let $G = \mathrm{Spin}_{2n}(\mathbf{C})$, n even. The Satake diagram in this case is



We have

$$S = S\big(\tilde{\alpha}, \alpha_{2i-1} + 2\alpha_{2i} + \alpha_{2i+1};\, 1 \le i \le (n-2)/2\big),$$

and $\Delta_H = \{\alpha_1, \alpha_3, \ldots, \alpha_{n-1}\}$. Now

$$S^f = \langle \epsilon_1\epsilon_{n-1}\epsilon_n, \epsilon_{2i-1}\epsilon_{2i+1};\, 1 \le i \le (n-2)/2 \rangle$$

and $S^f \cap H = \langle \epsilon_{2i-1}\epsilon_{2i+1};\, 1 \le i \le (n-2)/2\rangle$. Hence $T(\mathbf{R})$ has two connected components, and $G(\mathbf{R})$ is not w.e. In this case $z \in H$ while $z' \notin H$, where z, z' are as in case D I. Since $\epsilon_1\epsilon_{n-1}\epsilon_n z' \in H$, z' is not in $T(\mathbf{R})^\circ$. Hence $G(\mathbf{R})/\langle z\rangle$ is not w.e. while $G(\mathbf{R})/\langle z'\rangle$ is.

In the remaining cases G will denote a simply connected $\mathbf{R}$-group of the corresponding type.

CASE E II. The maximal standard split torus is $S = S(\alpha_1 + \alpha_6, \alpha_2, \alpha_3 + \alpha_5, \alpha_4)$ and $S^f = \langle \epsilon_2, \epsilon_4, \epsilon_1\epsilon_6, \epsilon_3\epsilon_5\rangle$. We have $T = ST_1$, with $T_1 = S(\alpha_1 - \alpha_6, \alpha_3 - \alpha_5)$ anisotropic. As $S^f \cap T_1 = \langle \epsilon_1\epsilon_6, \epsilon_3\epsilon_5\rangle$, $T(\mathbf{R})$ has four connected components. By using [St, Chapter 3] we find that the center of G is

$$Z = \langle h_1(\zeta^{-1})h_6(\zeta)h_3(\zeta)h_5(\zeta^{-1})\rangle,$$

where ζ is a primitive 3-rd root of 1. As $Z \subset T_1(\mathbf{R}) \subset T(\mathbf{R})^\circ$, the image of $T(\mathbf{R})$ in $G(\mathbf{R})/Z$ is not connected. Hence $G(\mathbf{R})/Z$ is not w.e.

CASE E III. In this case we need to compute the action of σ on Φ. Since $\alpha_3, \alpha_4, \alpha_5$ are the black vertices in the Satake diagram (see Table II), we have $\sigma(\alpha_i) = -\alpha_i$, $i = 3, 4, 5$. Let w_0 be the unique element of the Weyl group of the Dynkin diagram of $\{\alpha_3, \alpha_4, \alpha_5\}$ such that $w_0(\alpha_3) = -\alpha_5$, $w_0(\alpha_4) = -\alpha_4$, and $w_0(\alpha_5) = -\alpha_3$. By using the recipe of Matsumoto [M, p. 421] we find that

$$\sigma(\alpha_1) = w_0(\alpha_6) = \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6,$$
$$\sigma(\alpha_2) = w_0(\alpha_2) = \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5,$$
$$\sigma(\alpha_6) = w_0(\alpha_1) = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5.$$

Hence the maximal standard split torus S is $S(\tilde{\alpha}, \beta)$, where $\tilde{\alpha} = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ and $\beta = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$. It follows that $Z_G(S) = ST_1H$, where $T_1 = S(2\alpha_1 + \alpha_3 - \alpha_5 - 2\alpha_6)$ is an anisotropic torus and $\Delta_H = \{\alpha_3, \alpha_4, \alpha_5\}$. We have $S^f = \langle \epsilon_1\epsilon_4\epsilon_6, \epsilon_3\epsilon_5\rangle \subset T_1H$ because $\epsilon_1\epsilon_6 h_3(i)h_5(-i) \in T_1$ and $h_3(i)h_5(-i) \in H$, where $i = \sqrt{-1}$. Consequently $T(\mathbf{R})$ is connected and $G(\mathbf{R})$ is w.e.

CASE E IV. The elements $\beta = 4\alpha_1 + 3\alpha_2 + 5\alpha_3 + 6\alpha_4 + 4\alpha_5 + 2\alpha_6$ and $\gamma = 2\alpha_1 + \alpha_3 - \alpha_5 - 2\alpha_6$ generate the orthogonal complement of $\Delta_H = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$ in the root lattice. Hence $S = S(\beta, \gamma)$. As $S^f = \langle \epsilon_2\epsilon_3, \epsilon_3\epsilon_5\rangle \subset H$, $T(\mathbf{R})$ is connected and $G(\mathbf{R})$ is w.e.

CASE E VI. In this case $S = S(\alpha_1, \alpha_3, \alpha_2 + 2\alpha_4 + \alpha_5, \alpha_5 + 2\alpha_6 + \alpha_7)$, $Z_G(S) = SH$ and $\Delta_H = \{\alpha_2, \alpha_5, \alpha_7\}$. Since $S^f = \langle \epsilon_1, \epsilon_3, \epsilon_2\epsilon_5, \epsilon_5\epsilon_7\rangle$ and $S^f \cap H = \langle \epsilon_2\epsilon_5, \epsilon_5\epsilon_7\rangle$, $T(\mathbf{R})$ has four connected components. As the center of G has order 2, the image of $T(\mathbf{R})$ under the map $G(\mathbf{R}) \longrightarrow \bar{G}(\mathbf{R})^\circ$ is not connected. Hence $\bar{G}(\mathbf{R})^\circ$ is not w.e. (In fact $\bar{G}(\mathbf{R})$ is connected.)

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CASE E VII. In this case $S = S(\tilde{\alpha}, \beta, \alpha_7)$, where $\tilde{\alpha} = 2\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+3\alpha_5+2\alpha_6+\alpha_7$, $\beta = \alpha_2+\alpha_3+2\alpha_4+2\alpha_5+2\alpha_6$, and $\Delta_H = \Delta \setminus \{\alpha_1, \alpha_6, \alpha_7\}$. Since $S^f = \langle \epsilon_7, \epsilon_2\epsilon_3, \epsilon_3\epsilon_5 \rangle$ and $S^f \cap H = \langle \epsilon_2\epsilon_3, \epsilon_3\epsilon_5 \rangle$, $T(\mathbf{R})$ has two components and $G(\mathbf{R})$ is not w.e.

A simple computation based on [St, Chapter 3] shows that the center of G is generated by the element $\epsilon_2\epsilon_5\epsilon_7$. Since this element is not in $T(\mathbf{R})^\circ$, the image of $T(\mathbf{R})$ in $\bar{G}(\mathbf{R})$ is connected and so $\bar{G}(\mathbf{R})$ is w.e.

CASE E IX. In this case $S = S(\alpha_7, \alpha_8, \tilde{\alpha}, \beta)$ where $\beta = \alpha_2+\alpha_3+2(\alpha_4+\alpha_5+\alpha_6)$. We have $Z_G(S) = SH$ with $\Delta_H = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5\}$. Since $\tilde{\alpha} = 2\alpha_1+3\alpha_2+4\alpha_3+6\alpha_4+5\alpha_5+4\alpha_6+3\alpha_7+2\alpha_8$, we have $S^f = \langle \epsilon_7, \epsilon_8, \epsilon_2\epsilon_3, \epsilon_2\epsilon_5 \rangle \not\subset H$. Hence $T(\mathbf{R})$ is not connected and $G(\mathbf{R})$ is not w.e.

CASE F II. Now $S = S(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4)$, $\Delta_H = \Delta \setminus \{\alpha_4\}$, and $S^f = \langle \epsilon_1\epsilon_3 \rangle \subset H$. Thus $T(\mathbf{R})$ is connected and $G(\mathbf{R})$ is w.e. ∎

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