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ON SOME NORMING PROPERTIES OF SUBFACTORS

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Abstract We investigate the role of projections in norming a C^*-algebra by a type II_1 subfactor. Applications are given for factors with property Γ and for free-product factors.

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1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-subalgebras of $\mathcal{B}(H)$. We denote by $\mathrm{Row}_n(\mathcal{A})$ and $\mathrm{Col}_n(\mathcal{A})$, respectively, the spaces $M_{1,n}(\mathcal{A})$ and $M_{n,1}(\mathcal{A})$ of row and column matrices over $\mathcal{A}$, normed as subspaces of $M_n(\mathcal{A})$. We say that $\mathcal{A}$ norms (or is norming for) $\mathcal{B}$ if, for each $n \geqslant 1$ and for every $X \in M_n(\mathcal{B})$,

$$\|X\| = \sup\{\|RXC\| : R \in \mathrm{Row}_n(\mathcal{A}),\ C \in \mathrm{Col}_n(\mathcal{A}),\ \|R\|, \|C\| \leqslant 1\}.$$

More generally, $\mathcal{A}$ is λ-norming for $\mathcal{B}$, for some $0 < \lambda \leqslant 1$, if

$$\lambda\|X\| \leqslant \sup\{\|RXC\| : R \in \mathrm{Row}_n(\mathcal{A}),\ C \in \mathrm{Col}_n(\mathcal{A}),\ \|R\|, \|C\| \leqslant 1\}.$$

Norming is, therefore, the same as 1-norming.

The concept of a norming C^*-subalgebra of a C^*-algebra was introduced in [**11**], where the basic theory was developed and applications were given, particularly for cohomology of von Neumann algebras and the bounded projection problem.

One of the main roles of norming concerns automatic complete boundedness of modular linear and bilinear maps on von Neumann algebras [**13**, **15**]. To illustrate, let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{B}(H)$ be an inclusion of von Neumann algebras. If $\mathcal{N}$ norms $\mathcal{M}$, then every bounded $\mathcal{N}$-modular map $\varphi : \mathcal{M} \to \mathcal{M}$ is completely bounded. If, in addition, $\mathcal{M}$ is finite and $\mathcal{N}$ has trivial relative commutant, then every bounded, $\mathcal{N}$-modular bilinear map $\varphi : \mathcal{M} \times \mathcal{M} \to \mathcal{M}$ is completely bounded. Modularity, we recall, means that $\varphi(nm) = n\varphi(m)$ and $\varphi(mn) = \varphi(m)n$ for linear maps and

$$n\varphi(m_1, m_2) = \varphi(nm_1, m_2), \quad \varphi(m_1 n, m_2) = \varphi(m_1, nm_2), \quad \varphi(m_1, m_2 n) = \varphi(m_1, m_2)n$$

for bilinear maps, $m, m_1, m_2 \in \mathcal{M}$, $n \in \mathcal{N}$.

In the case of a type II_1 factor, the existence of a hyperfinite, norming subfactor with trivial relative commutant would provide affirmative answers to the elusive problem of the vanishing of the second cohomology group $H^2(\mathcal{M},\mathcal{M})$, as well as to the bounded projection conjecture, namely that the existence of a bounded projection $P:\mathcal{B}(H)\to\mathcal{M}$ implies injectivity for $\mathcal{M}$.

The interest in norming subalgebras originates in [**14**], where it was proved that Cartan subalgebras of type II_1 factors are norming. In [**11**] additional classes of subalgebras were proved to be norming. For example,

(i) type II_1 factors are normed by any subfactor of finite index, and

(ii) a type II_1 factor $\mathcal{M}$ norms $\mathcal{M}\mathbin{\bar{\otimes}}\mathcal{B}(H)$.

This paper continues the investigation along the lines initiated in [**11**]. We exploit the role played by projections in norming subfactors and obtain several new results. We show that type II_1 factors with property $\boldsymbol{\Gamma}$ and separable predual contain hyperfinite, norming subfactors with trivial relative commutant (Corollary 3.2) and present a characterization of property $\boldsymbol{\Gamma}$ from the norming viewpoint (Proposition 3.3). By contrast, the class of free-product factors provides examples of subfactors with trivial relative commutants which fail to be norming (Proposition 4.1). The norming properties of projections are also used in relation to free complements (Proposition 4.2). Property $\boldsymbol{\Gamma}$ factors are known to have vanishing cohomology groups of any order [**2**] and the bounded projection conjecture was also verified for these factors in [**1**]. Our Corollary 3.2, combined with Theorem 6.1 in [**11**], provides an alternate proof of the vanishing of the second cohomology group (see [**1**] and [**13**] for different proofs).

We believe that the line of investigation adopted here sheds new light on the concept of norming and establishes connections with other topics in operator algebras. On the other hand, we hope that the geometry of projections may provide a useful approach, via norming subfactors, to the second cohomology problem.

2. Norming and projections

In this section we present some properties equivalent to norming, but relying on projections. The next result is probably known, but we could not find a reference for it.

Lemma 2.1. *In a type* II_1 *factor* $\mathcal{M}$*, the set of invertible elements is norm-dense.*

Proof. Let $x\in\mathcal{M}$ be non-invertible with polar decomposition $x=v|x|$. Then v is a partial isometry between two equivalent projections p and q in $\mathcal{M}$, $v^*v=p$, $vv^*=q$. Since $I-p$ and $I-q$ are also equivalent, let w be a partial isometry such that $w^*w=I-p$, $ww^*=I-q$. If $\varepsilon>0$ is arbitrary, then $v+\varepsilon w$ is invertible. It follows that $(v+\varepsilon w)(|x|+\varepsilon I)$ is invertible and close to x for ε close enough to 0. □

Lemma 2.2. *Let* $\mathcal{M}$ *be a type* II_1 *factor and let* $C=(x_1,\dots,x_n)^{\mathrm{T}}\in\mathrm{Col}_n(\mathcal{M})$, $\|C\|=1$. *For any* $\varepsilon>0$*, there exists* $C'=(y_1,\dots,y_n)^{\mathrm{T}}\in\mathrm{Col}_n(\mathcal{M})$ *such that* $\|C'\|=1$, $\|C-C'\|<\varepsilon$ *and* $\sum_{i=1}^n y_i^*y_i$ *is invertible.*

Proof. By Lemma 2.1, let $y_0 \in \mathcal{M}$ be invertible and $\|x_1 - y_0\| < \delta$. Then $y_0^* y_0 + \sum_{i=2}^n x_i^* x_i$ is invertible. Define

$$C' = \left\| y_0^* y_0 + \sum_{i=2}^n x_i^* x_i \right\|^{-1/2} (y_0, x_2, \dots, x_n)^{\mathrm{T}} = (y_1, \dots, y_n)^{\mathrm{T}}.$$

We have $\|C'\| = 1$, $\sum_{i=1}^n y_i^* y_i$ is invertible, and, for small enough δ, $\|C - C'\| < \varepsilon$. □

Proposition 2.3. *Let $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{B}(H)$, where $\mathcal{N}$ is a type II_1 factor, $\mathcal{M}$ is a C^*-algebra, and fix $n \geqslant 1$, $\varepsilon > 0$ and $X \in \mathcal{M} \otimes M_n$. Then there exists a column $C \in \mathrm{Col}_n(\mathcal{N})$ such that $\|C\| = 1$ and $\|XC\| > \|X\| - \varepsilon$ if and only if there exists a projection $P \in \mathcal{N} \otimes M_n$ of trace $1/n$ such that $\|XP\| > \|X\| - \varepsilon$.*

Proof. There is no loss of generality in assuming that $\|X\| = 1$. From Lemma 2.2, we may also assume that the entries of $C = (y_1, \dots, y_n)^{\mathrm{T}}$ are such that $\sum_{i=1}^n y_i^* y_i$ is invertible. Choose a vector $\xi \in H$, $\|\xi\| = 1$, such that $\|XC\xi\| > 1 - \varepsilon$ and define a new column $C_1 = C(\sum_{i=1}^n y_i^* y_i)^{-1/2}$. Notice that the norm of $\eta = (\sum_{i=1}^n y_i^* y_i)^{1/2} \xi$ is at most one and that $C_1^* C_1 = I$. Let D_1 be the element of $\mathcal{N} \otimes M_n$ having the first column equal to C_1 and the remaining $n - 1$ columns having all entries 0. Then $D_1^* D_1$ is a projection of trace $1/n$ in $\mathcal{N} \otimes M_n$, so $P = D_1 D_1^*$ is also a projection of trace $1/n$. We have

$$\begin{aligned} (1 - \varepsilon)^2 < \|XC\xi\|^2 = \|XC_1\eta\|^2 \leqslant \|XC_1\|^2 = \|XD_1\|^2 \\ = \|XD_1 D_1^* X^*\| = \|XPX^*\| = \|XP\|^2, \end{aligned}$$

therefore $\|XP\| > 1 - \varepsilon$.

Conversely, suppose that $\|XP\| > 1 - \varepsilon$ for some projection $P \in \mathcal{N} \otimes M_n$ of trace $1/n$. Since $\mathcal{N} \otimes M_n$ is a factor, P is equivalent to $I \otimes e_{11}$, so choose a partial isometry $V \in \mathcal{N} \otimes M_n$ satisfying $V^* V = I \otimes e_{11}$ and $VV^* = P$. We have

$$(1 - \varepsilon)^2 < \|XP\|^2 = \|XPX^*\| = \|XVV^*X^*\| = \|XV\|^2$$

hence $\|XV\| > 1 - \varepsilon$. But $V^* V = I \otimes e_{11}$, so the last $n - 1$ columns of V must have all entries 0. If C denotes the first column of V, then $\|XC\| = \|XV\| > 1 - \varepsilon$, which completes the proof. □

We are ready to state the main result of this section.

Theorem 2.4. *Let $\mathcal{N} \subseteq \mathcal{M} \subseteq B(H)$, where $\mathcal{N}$ is a type II_1 factor and $\mathcal{M}$ is a C^*-algebra. The following are equivalent.*

(a) *$\mathcal{N}$ norms $\mathcal{M}$.*

(b) *For every $n \geqslant 1$, $\varepsilon > 0$, and every $X \in \mathcal{M} \otimes M_n$, there exists a projection $Q \in \mathcal{N} \otimes M_n$ of trace $1/n$ such that $\|XQ\| > \|X\| - \varepsilon$.*

If, in addition, $\mathcal{M}$ is a von Neumann algebra, then (a) *and* (b) *are also equivalent to the following.*

(c) *For every $n \geqslant 1$, $\varepsilon > 0$, and every non-zero projection $P \in \mathcal{M} \otimes M_n$, there exists a projection $Q \in \mathcal{N} \otimes M_n$ of trace $1/n$ such that $\|PQ\| > 1 - \varepsilon$.*

Proof. The equivalence of (a) and (b) follows from Proposition 2.3 combined with [**11**, Theorem 2.4], and (c) is a particular case of (b). To prove (c) $\Rightarrow$ (b), fix $X \in \mathcal{M} \otimes M_n$, $\|X\| = 1$ and $\varepsilon > 0$. Since X^*X is positive and of norm one, the spectral projection $\chi_{[1-\varepsilon,1]}(X^*X) = P$ is non-zero and, by functional calculus, $X^*X \geqslant (1-\varepsilon)P$. Choose $Q \in \mathcal{N} \otimes M_n$ of trace $1/n$ such that $\|PQ\| > \sqrt{1-\varepsilon}$. Then

$$\|XQ\|^2 = \|QX^*XQ\| \geqslant (1-\varepsilon)\|QPQ\| = (1-\varepsilon)\|PQ\|^2 > (1-\varepsilon)^2,$$

hence $\|XQ\| > 1 - \varepsilon$. □

Remark 2.5. If norming is replaced by λ-norming, for some $0 < \lambda < 1$, then the inequalities in (b) and (c) become $\|XQ\| > \lambda\|X\| - \varepsilon$ and $\|PQ\| > \lambda - \varepsilon$, respectively.

We conclude this section with a result which will be used in § 4.

Corollary 2.6. *Let*

$$\mathcal{N} = \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N}_n \subseteq \cdots \subseteq \mathcal{B}(H)$$

be an increasing sequence of type II_1 factors such that $[\mathcal{N}_{i+1} : \mathcal{N}_i] < \infty$ for all $i \geqslant 0$ and let $\mathcal{M} = (\bigcup_{n \geqslant 1} \mathcal{N}_n)''$. Then for every $x \in \mathcal{M}$ and every $\varepsilon, \delta > 0$, there exists a projection $q \in \mathcal{N}$ of trace $\tau(q) < \delta$ such that $\|xq\| > \|x\| - \varepsilon$.

Proof. Assume first that the sequence is stationary, that is, $[\mathcal{M} : \mathcal{N}] < \infty$. If $k > 1/\delta$ is a positive integer and $e \in \mathcal{N}$ is a projection of trace $\tau(e) = 1/k$, then $\mathcal{N}$ and $\mathcal{M}$ are isomorphic to $e\mathcal{N}e \otimes M_k$ and $e\mathcal{M}e \otimes M_k$, respectively. Since $[e\mathcal{M}e : e\mathcal{N}e] < \infty$, $e\mathcal{N}e$ norms $e\mathcal{M}e$ [**11**, 3.3]. The conclusion follows from Theorem 2.4 (b).

To finish the proof, let E_n denote the unique faithful, normal, trace-preserving conditional expectation of $\mathcal{M}$ onto $\mathcal{N}_n$. Since $x = \lim_{n\to\infty} E_n(x)$ ultraweakly, we have $\|E_n(x)\| > \|x\| - \varepsilon/2$ for some $n \geqslant 1$. By applying the first part of the proof, there is a projection $q \in \mathcal{N}$ of trace $\tau(q) < \delta$ such that $\|E_n(x)q\| > \|E_n(x)\| - \varepsilon/2 > \|x\| - \varepsilon$. Then $\|xq\| \geqslant \|E_n(xq)\| = \|E_n(x)q\| > \|x\| - \varepsilon$. □

3. Factors with property *Γ*

A type II_1 factor $\mathcal{M}$ has property ***Γ*** [**10**] if, for every $x_1, \ldots, x_n \in \mathcal{M}$ and $\varepsilon > 0$, there exists a unitary $u \in \mathcal{M}$ of trace $\tau(u) = 0$, such that

$$\|ux_i - x_iu\|_2 < \varepsilon, \quad 1 \leqslant i \leqslant n.$$

Property ***Γ*** is equivalent to an asymptotic commutativity property involving projections [**5**], a stronger version of which was proved in [**2**].

If $\mathcal{M}$ has property ***Γ*** and separable predual, then there exists a hyperfinite subfactor $\mathcal{R} \subseteq \mathcal{M}$ with trivial relative commutant such that, given $x_1, \ldots, x_k \in \mathcal{M}$, $n \in \mathbb{N}$, and

$\varepsilon > 0$, there exist mutually orthogonal projections $p_1, \dots, p_n \in \mathcal{R}$, each of trace $1/n$, such that

$$\|p_i x_j - x_j p_i\|_2 < \varepsilon, \quad 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant k.$$

Proposition 3.1. *Let $\mathcal{M}$ be a type II_1 factor with property $\boldsymbol{\Gamma}$ and separable predual. With the notation above, $\mathcal{R}$ is norming for $C^*(\mathcal{M}, \mathcal{M}')$.*

Proof. Fix $X = (X_{i,j}) \in C^*(\mathcal{M}, \mathcal{M}') \otimes M_n$, $1 \leqslant i, j \leqslant n$. We may clearly assume that each $X_{i,j}$ is a finite sum of terms of the form mm' for some $m \in \mathcal{M}$ and $m' \in \mathcal{M}'$, and denote by $\mathcal{S}$ the set of all the operators m and m' appearing in the expressions of $X_{i,j}$, $1 \leqslant i, j \leqslant n$. For every $k \geqslant 1$, there exist mutually orthogonal projections $p_1^{(k)}, \dots, p_n^{(k)} \in \mathcal{R}$, $\sum_{i=1}^n p_i^{(k)} = I$, $\tau(p_i^{(k)}) = 1/n$, such that

$$\|p_i^{(k)} y - y p_i^{(k)}\|_2 < 1/k$$

for all $y \in \mathcal{S} \cap \mathcal{M}$ and $1 \leqslant i \leqslant n$. This shows that, for every $y \in \mathcal{S} \cap \mathcal{M}$ and $i \neq j$,

$$\lim_{k \to \infty} \|p_i^{(k)} y p_j^{(k)}\|_2 = 0$$

and, consequently,

$$\lim_{k \to \infty} \biggl\| \sum_{i=1}^n p_i^{(k)} y p_i^{(k)} - y \biggr\|_2 = 0.$$

It follows that, if $e_i^{(k)} = p_i^{(k)} \otimes I_n \in \mathcal{R} \otimes M_n$ and ξ and η are fixed, but arbitrary, vectors in $H \oplus \cdots \oplus H$, then

$$\lim_{k \to \infty} \biggl\langle \biggl(\sum_{i=1}^n e_i^{(k)} X e_i^{(k)} - X \biggr) \xi, \eta \biggr\rangle = 0.$$

Ultraweak lower semicontinuity of the norm implies that, for fixed $\varepsilon > 0$, there exists $k \geqslant 1$ such that

$$\biggl\| \sum_{i=1}^n e_i^{(k)} X e_i^{(k)} \biggr\| > \|X\| - \varepsilon,$$

therefore

$$\max_{1 \leqslant i \leqslant n} \|e_i^{(k)} X e_i^{(k)}\| > \|X\| - \varepsilon,$$

hence

$$\|X e_{i_0}^{(k)}\| \geqslant \|e_{i_0}^{(k)} X e_{i_0}^{(k)}\| > \|X\| - \varepsilon$$

for some i_0. Since $e_{i_0}^{(k)}$ has trace $1/n$, $\mathcal{R}$ is norming for $C^*(\mathcal{M}, \mathcal{M}')$ by Theorem 2.4 (b). $\square$

Corollary 3.2. *If $\mathcal{M}$ has property $\boldsymbol{\Gamma}$ and separable predual, then it contains a hyperfinite norming subfactor with trivial relative commutant.*

It was proved in [**11**] that a separably acting C^*-algebra $\mathcal{A} \subseteq \mathcal{B}(H)$ has a cyclic vector if and only if it norms $\mathcal{B}(H)$. (We must point out that [**11**, 2.7] actually requires $\mathcal{A}$ to be only *locally cyclic*, but in separable Hilbert spaces the two notions are equivalent [**9**, **11**].)

A closer inspection of the proof of Proposition 2.7 in [**11**] shows that the key argument is the norming of a certain rank-one projection, therefore the same proof makes the following hold true:

$$\mathcal{A} \text{ has a cyclic vector} \quad \Longleftrightarrow \quad \mathcal{A} \text{ norms } \mathcal{B}(H) \quad \Longleftrightarrow \quad \mathcal{A} \text{ norms } \mathcal{K}(H).$$

Let H be a separable Hilbert space. The above remarks show that, at least from the norming viewpoint, the existence of a cyclic vector for $\mathcal{A}$ (an otherwise desirable property) has the potential to obscure other intrinsic properties of $\mathcal{A}$. For this reason, in the next result we must assume the absence of a cyclic vector. We are now in the position to prove a characterization of property $\boldsymbol{\Gamma}$.

Proposition 3.3. *Let $\mathcal{M} \subseteq \mathcal{B}(H)$ be a separably acting type* II_1 *factor without a cyclic vector. Then $\mathcal{M}$ has property $\boldsymbol{\Gamma}$ if and only if $\mathcal{M}$ is norming for $C^*(\mathcal{M}, \mathcal{M}')$.*

Proof. One direction is a consequence of Proposition 3.1. Conversely, let $\mathcal{M}$ be norming for $C^*(\mathcal{M}, \mathcal{M}')$. To get a contradiction, suppose that $\mathcal{M}$ is non-$\boldsymbol{\Gamma}$. Then $C^*(\mathcal{M}, \mathcal{M}')$ contains a non-zero compact operator [**3**, 2.1] and, since it is weakly dense in $\mathcal{B}(H)$, it contains all compact operators (see, for example, [**4**, I.10.4]). It follows that $\mathcal{M}$ norms $\mathcal{K}(H)$, so it has a cyclic vector, which is a contradiction. □

Remark 3.4. In the particular case when $\mathcal{M} = \mathcal{R}$, the hyperfinite type II_1 factor, there is another way to see that $\mathcal{R}$ norms $C^*(\mathcal{R}, \mathcal{R}')$. By [**7**], $C^*(\mathcal{R}, \mathcal{R}')$ is isomorphic to the spatial tensor product $\mathcal{R} \otimes_{\min} \mathcal{R}'$ and $\mathcal{R}$ is norming for $\mathcal{R} \bar{\otimes} \mathcal{B}(H)$ [**11**].

4. Free-product factors

Denote by $\boldsymbol{F}_n$ the free group on n generators and by $L(\boldsymbol{F}_n)$ the type II_1 factor generated by the left regular representation of $\boldsymbol{F}_n$. The free products in this section are reduced free products with respect to the trace. Recall from [**12**] that if $\mathcal{M}$ and $\mathcal{N}$ are type II_1 factors, then $\mathcal{M}' \cap (\mathcal{M} * \mathcal{N}) = \boldsymbol{C}$.

It was proved in [**11**] that the maximal abelian subalgebras of $\mathcal{M} = L(\boldsymbol{F}_2)$ corresponding to any one of the two generators are not λ-norming for $\mathcal{M}$. In this section we present examples of subfactors with trivial relative commutant which are not λ-norming.

Proposition 4.1. *For any type* II_1 *factor $\mathcal{M}$, $0 < \lambda \leqslant 1$, and $n \in \{1, 2, \dots\} \cup \{\infty\}$, $\mathcal{M}$ is not λ-norming for $\mathcal{M} * L(\boldsymbol{F}_n)$.*

Proof. Since every $L(\boldsymbol{F}_n)$ contains $L(\boldsymbol{F}_\infty)$ as a subfactor, it suffices to assume $n = \infty$. Dykema and Rădulescu [**6**, 1.1] proved that for any type II_1 factors $\mathcal{M}$ and $\mathcal{N}$, the factors

$$(\mathcal{M} \otimes M_n) * (\mathcal{N} \otimes M_n) \quad \text{and} \quad (\mathcal{M} * \mathcal{N} * L(\boldsymbol{F}_{n^2-1})) \otimes M_n$$

are isomorphic. In particular, $(\mathcal{M} \otimes M_n) * (L(\boldsymbol{F}_\infty) \otimes M_n)$ is isomorphic to

$$(\mathcal{M} * L(\boldsymbol{F}_\infty) * L(\boldsymbol{F}_{n^2-1})) \otimes M_n = (\mathcal{M} * L(\boldsymbol{F}_\infty)) \otimes M_n.$$

In other words, $\mathcal{M} \otimes M_n$ has a free complement, denoted by $\mathcal{A}$, relative to $(\mathcal{M} * L(\boldsymbol{F}_\infty)) \otimes M_n$. Consider projections $P \in \mathcal{A}$ and $Q \in \mathcal{M} \otimes M_n$ both of trace $1/n$. As a consequence of [**16**, 3.6.7],

$$\|PQ\| \leqslant \frac{4}{n} - \frac{4}{n^2} < \frac{4}{n}.$$

For n large enough this contradicts the remark following Theorem 2.4, hence $\mathcal{M}$ is not λ-norming for $\mathcal{M} * L(\boldsymbol{F}_n)$. □

An important problem in free probability is the existence of free complements: if $\mathcal{A} \subseteq \mathcal{B}$ is an inclusion of von Neumann algebras, does $\mathcal{B}$ contain a non-trivial projection free with respect to $\mathcal{A}$? For instance, if $\mathcal{M}$ and $\mathcal{N}$ are type II_1 factors, then no projection $p \in \mathcal{M} \bar{\otimes} \mathcal{N}$, $p \neq 0, I$, is free with respect to $\mathcal{M}$. To see this, suppose p was free with respect to $\mathcal{M}$ and let $\mathcal{A} = \{p\}''$ denote the two-dimensional algebra generated by p. Then $\mathcal{M} * \mathcal{A} = \mathcal{M} \bar{\otimes} \mathcal{B}$ for some non-trivial subalgebra $\mathcal{B} \subseteq \mathcal{N}$ [**8**]. We have $\mathcal{M}' \cap (\mathcal{M} * \mathcal{A}) = \boldsymbol{C}$ [**12**], while $\mathcal{M}' \cap (\mathcal{M} \bar{\otimes} \mathcal{B}) = \mathcal{B}$, which is a contradiction. We conclude with yet another example of absence of free complements.

Proposition 4.2. *Let*

$$\mathcal{N} = \mathcal{N}_0 \subseteq \mathcal{N}_1 \subseteq \cdots \subseteq \mathcal{N}_n \subseteq \cdots \subseteq \mathcal{B}(H)$$

be an increasing sequence of type II_1 *factors such that* $[\mathcal{N}_{i+1} : \mathcal{N}_i] < \infty$ *for all* $i \geqslant 0$ *and let* $\mathcal{M} = (\bigcup_{n \geqslant 1} \mathcal{N}_n)''$*. Then no projection* $p \in \mathcal{M}$, $p \neq 0, I$, *is free with respect to* $\mathcal{N}$.

Proof. To reach a contradiction, suppose that $p \in \mathcal{M}$ is free with respect to $\mathcal{N}$. By Corollary 2.6, there are projections $q_n \in \mathcal{N}$ of trace $\tau(q_n) = 1/n$ such that $\|pq_n\| > 1 - 1/n$. On the other hand, from [**16**, 3.6.7] we have, for $\tau(p) = t$,

$$\|pq_n\| \leqslant t + \frac{1}{n} - \frac{2t}{n} + \frac{2}{n}\sqrt{t(1-t)(n-1)} < t + \frac{2}{\sqrt{n}}.$$

Since $t < 1$, n large enough leads to a contradiction. □

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