PROBLEMS FOR SOLUTION

P40. (Conjecture). If the edges of a convex polyhedron form a "cage" surrounding a sphere of unit radius, then these edges have a total length of at least $9\sqrt{3}$ (see Math. Rev. 20 (1959), Rev. 1950).

H. S. M. Coxeter

P41. Let P_1, P_2, P_3, P_4 be any four points in the plane, no three collinear. On P_iP_{i+1} construct a square with centre Q_i so that the triangles $Q_iP_iP_{i+1}$ all have the same "orientation" ($i = 1, 2, 3, 4$; $P_5 = P_1$). Show that the segments Q_1Q_3 and Q_2Q_4 have the same lengths, and the lines containing them are perpendicular.

W. A. J. Luxemburg

P42. Let $q_n = 1 + \Sigma_{r=1}^{n} \phi(r)$ where ϕ denotes the Euler totient function and let p_n be the n-th prime ($p_1 = 2$). Prove that $p_n = q_n$ for $n = 1, 2, 3, 4, 5, 6$ but for no other values of n.

L. Moser

P43. Let G be a group generated by P and Q, and let H be the cyclic subgroup generated by P. If P and Q satisfy only the relations $P^2PQ = Q^2$ and $Q^2PQ^{-4} = P^k$ for some k, then the index of H in G is 14.

N. S. Mendelsohn

SOLUTIONS

P7. Define f(n) by $n^{f(n)} \| n!$, i.e., $n^{f(n)} | n!$ and $n^{f(n)+1} \nmid n!$

https://doi.org/10.1017/S0008439500025625 Published online by Cambridge University Press

(a) Prove that $\limsup (f(n) \log n/(n \log \log n)) = 1$.

(b) Prove that if p is the greatest prime factor of n, then for almost all n,

$$f(n) = \Sigma_{i=1}^{\infty} [n/p^i].$$

(Almost all means all n excepting a sequence of density 0.)

P. Erdős

Solution by the proposer.

(a) I shall prove that for infinitely many n

(i) $f(n) > (1 - \varepsilon)(n \log \log n)/\log n$,

and for all $n > n_o(\varepsilon)$

(ii) $f(n) < (1 + \varepsilon)(n \log \log n)/\log n$.

Put $n! = \prod_{p \leq n} p^{\alpha_p}$, $\alpha_p = [n/p] + [n/p^2] + \ldots$

Clearly $\alpha_p < n/(p - 1)$. Thus, if $q^{\beta_q} \| n$ we have

(1) $$f(n) < \min_{q|n} n/(q - 1)\beta_q.$$

Hence, if $f(n) \geq (1 + \varepsilon)n(\log \log n)/\log n$, we have from (1) that for every $q|n$,

(2) $$1 \leq \beta_q < (\log n)/(1 + \varepsilon)(q - 1) \log \log n.$$

Now (2) implies that all prime factors of n are less than or equal to $1 + (\log n)/(1 + \varepsilon) \log \log n$. Hence, from (2),

$$n \leq \prod_{q < 1 + (\log n)/(1 + \varepsilon)\log \log n} q^{(\log n)/(q - 1)(1 + \varepsilon)\log \log n}$$

or

$$\log n \leq \frac{\log n}{(1 + \varepsilon)\log \log n} \Sigma_{q < 1 + (\log n)/(1 + \varepsilon)\log \log n} (\log q)/(q - 1) < \frac{\log n}{1 + \varepsilon}.$$

To prove (i), let t be large and put

$$\beta_q = [t/(q-1)], \quad m = \prod_{q \le t} q^{\beta_q}.$$

Then

$$\text{(3)} \qquad \log m = \Sigma_{t \le q}\, \beta_q \log q = t \log t + O(t).$$

Clearly

$$\text{(4)} \qquad f(m) = \min_{q \le t} \alpha_q / \beta_q \ge \min_{q \le t} \alpha_q (q-1)/t$$

where $q^{\alpha_q} || m!$.

Now

$$\text{(5)} \qquad \alpha_q = [m/q] + [m/q^2] + \ldots \ge \frac{m}{q-1} - \frac{\log m}{\log 2}.$$

Thus, from (3), (4), and (5),

$$\text{(6)} \qquad f(m) \ge \min_{q \le t} \left(\frac{m}{q-1} - \frac{\log m}{\log 2} \right) \frac{q-1}{t}$$

$$> \frac{m}{t} - 2 \log m.$$

Now (3) implies, by simple calculation, that

$$\text{(7)} \qquad t = (1 + o(1))(\log m)/\log \log m$$

and (6) and (7) imply that (1) holds for m.

(b) Let $p_{(n)}$ be the largest prime factor of n. We show that for almost all n

$$\text{(8)} \qquad f(n) = \Sigma_k [n/p_{(n)}^{\,k}].$$

Of course

$$p_{(n)}^{\Sigma[n/p_{(n)}^{\,k}]} \, || \, n!.$$

Thus, for all n, $f(n) \le \Sigma_k [n/p_{(n)}^{\,k}]$. Now we show that for every $\varepsilon > 0$ and $x > x(\varepsilon)$, (8) holds for all but εx integers $\le x$.

Let $A = A(\varepsilon)$ be large. If $q^\alpha | n$, $\alpha > 1$, $q^\alpha > A$, then n is divisible by a square $\geq A^{2/3}$. To see this, observe that if α is even nothing has to be proved; and if α is odd, then $\alpha \geq 2$ and $q^{\alpha - 1} \geq A^{(\alpha - 1)/\alpha} \geq A^{2/3}$. It follows that the number of integers $\leq x$ with $p^\alpha | n$, $\alpha > 1$, $p^\alpha > A$ is less than

$$\sum_{k \geq A^{2/3}} n/k^2 < n/A^{1/3} < \varepsilon n/2$$

for $A > A(\varepsilon)$. Thus we may restrict our attention to the integers n for which $p^\alpha | n$, $\alpha > 1$ implies $p^\alpha \leq A$. The number of integers $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_{(n)}^{(\alpha)_n}$ $(p_1 < p_2 < \ldots < p_{(n)} \leq A)$ in this class is clearly less than $2^A < \varepsilon x/2$ for sufficiently large x (since $p_i^{\alpha_i} \leq A$ and we have $\leq A$ choices for the $p_i^{\alpha_i}$). For the integers which do not belong to any of these classes (8) holds. To see this put

$$n = \prod_{i=1}^{(n)} p_i^{\alpha_i}, \quad p_i^{\alpha_i} || n, \quad p_i^{\alpha_i} \leq A \text{ for } \alpha_i > 1, \quad p_{(n)} > A.$$

If $\alpha_i = 1$, then $p_i^{f(n)} | n!$ (for $p_i^{\alpha_i f(n)} || n!$).

If $\alpha_i > 1$, then because $p_i^{\alpha_i} \leq A < p_{(n)}$, we have

$$\Sigma[n/p_i^{k}] > \alpha_i \Sigma[n/p_{(n)}^{k}]$$

so that $p_i^{\alpha_i f(n)} || n!$, and we are done.

P8. Let F be a field of characteristic 2; let F^* be its multiplicative group and let F^2 be the subfield of the squares of F. Assume $F^2 \neq F$. Show (i) F, F^2 and F^*/F^{2*} have the same cardinal number; (ii) there exist two fields F^o and F^∞ such that $F^\infty \subset F \subset F^o$, $(F^o)^2 = F^o$, $(F^\infty)^2 = F^\infty$ and such that every field G with $G = G^2$ and containing F (contained in F) contains a field isomorphic to F^o (is contained in F^∞).

P. Scherk.

Solution by the proposer. Let $F = \{\alpha, \beta, \ldots\}$ be a field of characteristic 2; thus $\alpha = -\alpha$. Put $F^2 = \{\alpha^2, \beta^2, \ldots\}$. Then

$$(1) \qquad \alpha \to \alpha^2$$

is a mapping of F onto F^2. Since $\alpha^2 = \beta^2$ implies $\alpha = \beta$, this mapping is 1-1. It maps products onto products and due to

$$(\alpha + \beta)^2 = \alpha^2 + \beta^2$$

also sums on sums. Thus F^2 is a subfield of F and (1) is an isomorphism of F onto F^2. In particular F and F^2 have the same cardinal number. Hence $F = F^2$ if F is finite. (However there are transcendental extensions F of finite fields of characteristic 2 such that $F \neq F^2$.)

For any F let F^* denote the multiplicative group of the elements $\neq 0$ of F. Suppose now $F^2 \neq F$. Thus F^{2*} will be a proper subgroup of F^*. Choose any $\delta \in F$, $\delta \notin F^2$. Then

$$(2) \qquad \alpha \to \delta + \alpha^2$$

will map F^* into itself. If $\delta + \alpha^2$ and $\delta + \beta^2$ lie in the same coset of F^{2*}, then $\delta + \alpha^2 = \gamma^2(\delta + \beta^2)$ for some γ. If $\gamma \neq 1$, then

$$\delta = \frac{\alpha^2 + \gamma^2\beta^2}{1 + \gamma^2} = \left(\frac{\alpha + \gamma\beta}{1 + \gamma}\right)^2 \in F^2.$$

Thus $\gamma = 1$, $\alpha^2 = \beta^2$ and $\alpha = \beta$. Hence (2) maps different elements of F^* into different cosets. In particular, the cardinal number of F^*/F^{2*} is not smaller than that of F^*. Hence they are equal.

Iterating the construction of F^2, we obtain a sequence of subspaces

$$(3) \qquad F \supset F^2 \supset F^4 \supset \ldots \supset F^\infty = \bigcap_1^\infty F^{2n}.$$

[Being the intersection of fields, F^∞ certainly is a field.]

Let $\alpha \in F^\infty$. Thus $\alpha \in F^{2n}$ for all $n > 0$. Hence there is a $\beta_n \in F^{2n-1}$ such that $\alpha = \beta_n{}^2$. Let $n < m$. Then $\beta_m{}^2 = \beta_n{}^2 = \alpha$ and both β_m and β_n lie in F^{2n-1}. The solution of $\xi^2 = \alpha$ being unique in F^{2n-1}, we have $\beta_m = \beta_n = \beta$, say. Thus $\beta \in F^{2n-1}$ for all n, i.e. $\beta \in F^\infty$. We have $\alpha = \beta^2$. Hence the restriction of (1) to F^∞ becomes an automorphism, and $(F^\infty)^2 = F^\infty$.

Obviously, two elements α and β of F^* lie in the same coset of F^{2*} if and only if α^2 and β^2 lie in the same coset of F^{4*} in F^{2*}. Thus (1) induces an isomorphism of F^*/F^{2*} onto F^{2*}/F^{4*}. More generally, all the factor groups $(F^{2n-1})^*/(F^{2n})^*$ will be isomorphic. Hence the fields (3) must be mutually distinct if $F \neq F^2$.

If H is a subfield of F, H^2 will be a subfield of F^2, H^{2n} will be one of F^{2n} and $H^\infty \subset F^\infty$. In particular $H = H^2$ implies $H = H^\infty \subset F^\infty$.

We can also proceed in the opposite direction by adjoining to the field F the square roots of all of its elements. This leads to a new field $F^{\frac{1}{2}} \supset F$ satisfying $(F^{\frac{1}{2}})^2 = F$. We then can define $F^{2^{-n}}$ by induction. The field

$$F^o = \bigcup_{n=1}^{\infty} F^{2^{-n}}$$

satisfies

$$F \subset F^{\frac{1}{2}} \subset F^{\frac{1}{4}} \subset \ldots \subset F^o = (F^o)^2.$$

If F is a subfield of G, then $F^{2^{-n}} \subset G^{2^{-n}}$ for all n and $F^o \subset G^o$. In particular $G = G^2$ implies $F^o \subset G^o = G$.

P13. Angular measure in a Minkowski plane is sometimes defined proportional to the area of the corresponding sector of the unit circle U, sometimes proportional to the arc length of U. Determine all Minkowski metrics where the two measures are proportional.

H. Helfenstein.

Solution by the proposer. Let $r = f(\phi)$ be the polar equation of U in an associated Euclidean metric. If we assume continuous differentiability of $f(\phi)$ (the proof can be modified to avoid this) and denote by ψ the angle between the radius vector and the tangent of U we arrive at the following identity in θ

$$\int_0^\theta \frac{[f^2(\phi) + f'^2(\phi)]^{\frac{1}{2}} d\phi}{f(\phi + \psi)} = k \int_0^\theta f^2(\phi) d\phi,$$

where k is the factor of proportionality. Differentiating with respect to θ and replacing $f'(\theta)$ by $f(\theta)\cot\psi$ we obtain

$kf(\theta)f(\theta + \psi)\sin\psi = \pm 1$, which is the conjugate diameter condition for Radon curves. Hence the desired Minkowski metrics are those with symmetric perpendicularity.