1 Some Preliminaries

1.1 Introduction to the Linear Theory of Sound Wave Motion

What we generally call sound wave motion is a form of unsteady motion of a fluid and is governed by the fundamental laws of conservation of mass, momentum and energy and the thermodynamic state equations pertinent to the fluid in question. These equations are classical and are given in Appendix A, both in integral and differential forms, together with their derivations. There are many books treating these equations in more detail and giving their solutions for specific fluid flow problems. In this book, we are concerned about their applications in duct acoustics. The dilemma here is that, since these equations are non-linear, they can be solved only numerically even for the simplest of duct systems in industry and this is very expensive in terms of computational resources and time. On the other hand, it is a heuristic fact that in most noise control problems, including fluid machinery duct-borne noise, it is adequately accurate to assume that sound wave motion remains within linear limits. This is a very convenient situation that is worth taking advantage of, because linear differential equations are very much easier to solve than non-linear ones and linearization of non-linear equations is a mathematically well-understood process having its roots in Taylor series expansions of continuous functions. But, passing from the basic non-linear equations to those governing linear sound wave motions is not trivial and involves some mathematical and conceptual difficulties. In particular, we are confronted with certain subtle issues when dealing with turbulent flows in this manner. In this section, we discuss how we go about these problems to develop one-dimensional and three-dimensional linear theories of sound wave motion that are used in this book for modeling of sound transmission in ducts and duct systems.

1.1.1 Linearization Hypothesis

In many problems of unsteady fluid flow, it is permissible to assume that properties of flow at a given point $\mathbf{x}$ at time t may be represented by an asymptotic expansion of the form $q = q_o + q' + q'' + \cdots$, where $q = q(t, \mathbf{x}) \in p, \rho, \mathbf{v}, T, c, \gamma, \ldots$ denotes any property of the fluid, for example, pressure (p), density (ρ), particle velocity ($\mathbf{v}$), absolute temperature (T), speed of sound (c), the ratio of specific heat coefficients (γ) and so on. The same symbol with subscript "o," like $q_o = q_o(\mathbf{x})$, denotes the steady (time-independent) part of a property and the corresponding part of the flow is called mean

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flow. The same symbol with a prime, like $q' = q'(t, \mathbf{x})$, denotes a fluctuating part of zero mean that is much small compared to q_o, that is, $q'/q_o = O(\varepsilon)$, except for the particle velocity which scales as $|\mathbf{v}'|/c_o = O(\varepsilon)$, where $\varepsilon << 1$.[1] Similarly, two primes, like in $q'' = q''(t, \mathbf{x})$, denotes a fluctuating part of zero mean that is much small compared to q', that is, $q''/q_o = O(\varepsilon^2)$, except for the particle velocity which scales as $|\mathbf{v}''|/c_o = O(\varepsilon^2)$, and so on. In this book, we allow $|\mathbf{v}_o|$ to be well in the subsonic range, that is, substantially smaller than c_o.

The fundamental non-linear equations of unsteady fluid motion are linearized by substituting in them $q = q_o + q' + q'' + \cdots$ and then invoking two hypotheses:

(i) mean flow satisfies the fundamental non-linear equations;
(ii) linear sound wave motion is given by the $O(\varepsilon)$ terms.

The latter hypothesis is largely based on our experience with the amplitudes of sound waves we measure in everyday life and engineering applications. For example, the threshold of pain of human hearing is about $p'_{rms} = 200$ Pa, which is only 0.2% of the atmospheric pressure, p_o, where the subscript "*rms*" denotes the root-mean-square value (see Section 1.2.4). Very close to a jet engine or inside the exhaust ports of an internal combustion engine p'_{rms}/p_o is only about 2–3%, which is about as large as it can be expected to get in many engineering applications.

The $O(\varepsilon)$ terms in expansions $q = q_o + q' + q'' + \cdots$ are usually referred to as the acoustic terms. As will be seen in later chapters, acoustic pressure and density in a perfect gas are related as $p'/p_o = \gamma_o \rho'/\rho_o$ under isentropic conditions. Consequently, acoustic density is also small to $O(\varepsilon)$. On the other hand, the particle velocity $\mathbf{v}' = \mathbf{e}v'$ of longitudinal wave motion in direction of the unit vector $\mathbf{e}$ scales with the acoustic pressure as $v' \sim p'/\rho_o c_o$ [1]. Therefore, $v'/c_o \sim \rho'/\rho_o$, showing that v'/c_o is also small to $O(\varepsilon)$. These orders of magnitudes set the stage for the linearization of the fundamental non-linear equations of unsteady fluid motion.

1.1.2 Partitioning Turbulent Fluctuations

A complication arises, however, if the flow is turbulent, because then time-dependent fluctuating structures of multiple scales (called eddies) are also convected with the mean flow and it becomes necessary to separate these from acoustic fluctuations. Turbulent fluctuations may be assumed to be of constant density, but, at the present time, the partitioning problem has no clear-cut solution. For this reason, here we take a pragmatic approach and look into the extent to which linearization may work without detailed modeling of turbulent fluctuations.

Turbulence is known as the most extensively studied, but the least understood, topic of fluid dynamics. However, it is a well-established empirical fact that flow in a duct turns from an orderly sheared form (called laminar) to a chaotic (turbulent) one at about $\mathrm{Re} = \rho_o U_o D/\mu_o \approx 2000$, where Re is the Reynolds number, D and U_o denote,

[1] The rate of growth of a function is called its order and denoted by the letter O (which is also called the Landau symbol). For example, if $f(x)$ is of $O(g(x))$, this means that f does not grow faster than g.

respectively, a characteristic dimension of the duct section and a characteristic velocity of the flow, and μ_o denotes the shear viscosity of the fluid. Based on Lighthill's theory of aerodynamically generated sound, we know that turbulence can generate sound waves and attenuate sound waves. Acoustic power radiated, W_V, say, from a compact region, V, of subsonic turbulence may be estimated qualitatively from Lighthill's theory, but this theory involves rather advanced ideas which could not be treated in this book fully. So, it suffices to state that, if $M = U_c/c_o << 1$, that is, the flow is subsonic in low Mach number range, W_V scales as $W_V \sim \rho_o \left(U_c^8/c_o^5\right)(V/L)$, where U_c and L denote the appropriate convection velocity and length scales of the turbulence [2]. This is the famous eight-power law for sound power radiated from free turbulence. Taking the volume V as L^3, this law can be written as $W_V \sim \rho_o U_c^8 L^2/c_o^5$. A quantity which is useful for further insight is the quotient of W_V and the mechanical power supplied to the fluid, since this quotient may be considered as the acoustic efficiency of turbulence [3]. Thus, correlating the mechanical power supplied to the fluid with $\left(\rho_o U_c^2/2\right)L^2 U_c$, we obtain $W_V/\rho_o U_c^3 L^2 \sim M^5$, which shows that the acoustic efficiency of turbulence is of $O\left(M^5\right)$. For subsonic low Mach number flows ($M^2 << 1$), this is small compared to the efficiency of the usual ducted primary fluid machinery sources. For example, it may similarly be shown that the corresponding acoustic efficiency of pressure loading on rotating blades as a sound source is of $O\left(M^3\right)$ (see Section 10.4), and that due to the unsteady mass injection is of $O(M)$(see Section 10.2), for the characteristic flow velocities and length scales involved in these processes [3].

Thus, the effect of turbulence as an acoustic source may be neglected in subsonic low Mach number flows dominated by primary fluid machinery sources. However, even if turbulence fluctuations are of secondary importance as such, they cause scattering of incident acoustic waves and absorb energy from the acoustic waves. Both of these mechanisms lead to attenuation of sound waves. Based on Lighthill's theory, Tack and Lambert give the following semi-empirical upper-bound estimate for the fraction of the incident acoustic energy scattered by isotropic turbulence per unit duct length, namely, $0.0032\, M_o^2(\omega L/c_o)^6/L$, where M_o denotes the Mach number of the free stream velocity, L denotes the size of eddies (which can be no larger than the duct section size), and ω denotes the radian frequency of the incident sound wave [4]. Thus, the effect of scattering may be neglected at relatively low frequencies and subsonic low Mach numbers. But absorption tends to be important at lower frequencies. It is attributed to the irreversible straining of eddies by the sound wave and is mainly confined in a duct to the turbulent boundary layer where the particle velocity gradients are large. The analysis of the phenomenon is difficult because it requires complete understanding of the turbulent boundary layer, which in turn requires accurate solutions of the non-linear equations of unsteady fluid motion including effects of viscosity and thermal conductivity and all length scales of turbulent and acoustic fluctuations. For this reason, we rely on the progress achieved by simplified models at subsonic low Mach numbers. The simplest model assumes that the sound field in a duct is not influenced by the presence of turbulence and vice versa. Then the problem reduces to the study of sound waves convected with a mean flow that is characterized by the velocity profile. This model, which is usually referred to as the quasi-laminar model, is adopted throughout this book. It is adequately accurate at relatively high frequencies, but underestimates attenuation at low frequencies. Howe, taking into account the frequency dependence of the straining of turbulence by sound waves, proposed a semi-empirical model which correlates satisfactorily with measurements also at low frequencies [5]. This model is described in Section 3.9.3.4, where it is also shown how it can be integrated with the quasi-laminar model.

1.1.3 Linearization of Inviscid Fluid Flow

Thus, unsteady motions of fluids may be linearized assuming a laminar mean flow or a quasi-laminar one with a subsonic low Mach number, if the mean flow is turbulent. The process is described here for inviscid fluids and differential forms of the fluid dynamic equations with no source terms; however, the corresponding equations with source and viscothermal terms, and their integral forms are linearized in exactly the same way.

From Equations (A.21) and (A.23), the conservation of mass and momentum in a source-free region of a duct containing an inviscid fluid can be expressed in differential forms as:

$$\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla\rho + \rho\nabla\cdot\mathbf{v} = 0 \tag{1.1}$$

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + \nabla p = 0, \tag{1.2}$$

respectively, where ∇ denotes the gradient operator and $\nabla\cdot$ denotes the divergence operator. To obtain the linearized forms of these equations, we substitute $p = p_o + p' + \cdots$, $\rho = \rho_o + \rho' + \cdots$ and $\mathbf{v} = \mathbf{v}_o + \mathbf{v}' + \cdots$. Equation (1.1) then becomes

$$\left(\frac{\partial \rho_o}{\partial t} + \mathbf{v}_o\cdot\nabla\rho_o + \rho_o\nabla\cdot\mathbf{v}_o\right) + \left(\frac{\partial \rho'}{\partial t} + \mathbf{v}_o\cdot\nabla\rho' + \rho_o\nabla\cdot\mathbf{v}' + \mathbf{v}'\cdot\nabla\rho_o + \rho'\nabla\cdot\mathbf{v}_o\right)$$
$$+ \left(\frac{\partial \rho''}{\partial t} + \mathbf{v}_o\cdot\nabla\rho'' + \mathbf{v}'\cdot\nabla\rho' + \mathbf{v}''\cdot\nabla\rho_o + \rho_o\nabla\cdot\mathbf{v}'' + \rho'\nabla\cdot\mathbf{v}' + \rho''\nabla\cdot\mathbf{v}_o\right) + O(\varepsilon^3) = 0 \tag{1.3}$$

In view of the first linearization hypothesis stated in Section 1.1.1, the first bracket vanishes identically, giving the following continuity equation for the mean flow, namely,

$$\nabla\cdot(\rho_o\mathbf{v}_o) = 0 \tag{1.4}$$

since mean flow is time independent.

The terms in the second bracket in Equation (1.3) contain only single-primed variables. These terms represent the $O(\varepsilon)$ terms. This may not be immediately evident from stipulations like $\rho'/\rho_o = O(\varepsilon)$ and $|\mathbf{v}'|/c_o = O(\varepsilon)$, which we have made in Section 1.1.1 about the order of magnitudes of the acoustic (single primed) variables. To see that the consideration of single-primed variables as $O(\varepsilon)$ terms is compatible with these stipulations, observe that we can always write identities like $\mathrm{L}q' = q_o\mathrm{L}(q'/q_o) + (q'/q_o)\mathrm{L}q_o{\sim}O(\varepsilon)$, for thermodynamic properties $q \in p, \rho, \ldots,$ where L denotes a linear partial differential operator, for example, $\mathrm{L} = \mathbf{v}_o.\nabla$, and $\nabla\cdot\mathbf{v}' = c_o\nabla\cdot(\mathbf{v}'/c_o) + (\mathbf{v}'/c_o)\cdot\nabla c_o{\sim}O(\varepsilon)$ for the particle velocity.

It can similarly be seen that double primed quantities represent $O(\varepsilon^2)$ terms. Then, the third bracket in Equation (1.3), which contains only double-primed quantities and products of single-primed quantities, is of $O(\varepsilon^2)$ and it is neglected by the linearization hypothesis as being small compared to the first-order terms in the second bracket. Thus, the linearized form of Equation (1.1) is given by the vanishing of the second bracket in Equation (1.2), that is,

$$\frac{\partial\rho'}{\partial t} + \mathbf{v}_o\cdot\nabla\rho' + \rho_o\nabla\cdot\mathbf{v}' + \mathbf{v}'\cdot\nabla\rho_o + \rho'\nabla\cdot\mathbf{v}_o = 0. \tag{1.5}$$

This is called the linearized (or acoustic) continuity equation.

Equation (1.2) is linearized similarly (here, for brevity, we give the terms up to $O(\varepsilon^2)$ only):

$$\left(\rho_o\frac{\partial\mathbf{v}_o}{\partial t} + \rho_o\mathbf{v}_o\cdot\nabla\mathbf{v}_o + \nabla p_o\right) + \left(\rho_o\left(\frac{\partial\mathbf{v}'}{\partial t} + \mathbf{v}_o\cdot\nabla\mathbf{v}' + \mathbf{v}'\cdot\nabla\mathbf{v}_o\right)\right.$$
$$\left. + \rho'\mathbf{v}_o\cdot\nabla\mathbf{v}_o + \rho'\frac{\partial\mathbf{v}_o}{\partial t} + \nabla p'\right) + O\left(\varepsilon^2\right) = 0 \tag{1.6}$$

Again, the first bracket vanishes identically, giving the momentum equation for the mean flow:

$$\rho_o\mathbf{v}_o\cdot\nabla\mathbf{v}_o + \nabla p_o = 0 \tag{1.7}$$

and the linearized form of Equation (1.2) follows as

$$\rho_o\left(\frac{\partial\mathbf{v}'}{\partial t} + \mathbf{v}_o\cdot\nabla\mathbf{v}' + \mathbf{v}'\cdot\nabla\mathbf{v}_o\right) + \rho'\mathbf{v}_o\cdot\nabla\mathbf{v}_o + \nabla p' = 0, \tag{1.8}$$

which is called linearized (or acoustic) momentum equation.

Sound wave motion in a duct containing an inviscid fluid is thus governed by Equations (1.5) and (1.8). Much of the material contained in this book is concerned with the solutions of these equations for the acoustic variables p', ρ' and $\mathbf{v}'$, given the fluid type and the mean flow properties. Therefore, one more scalar equation is required for closing the solution of the acoustic variables. For inviscid fluids, this equation comes from the conservation of energy for the unsteady motion. As shown in Appendix A.3.3, in a region of an inviscid fluid free of sources, the energy equation reduces to

$$\frac{\partial s}{\partial t} + \mathbf{v}\cdot\nabla s = 0, \tag{1.9}$$

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where s denotes the specific entropy of the fluid. This means that the specific entropy of a fluid particle remains constant (but that constant can be different for different particles), which in turn means that the flow is reversible, that is, adiabatic (no heat transfer is involved) and reversible. Since the specific entropy is not involved in the continuity and momentum equations, we use the thermodynamic state equation for the specific entropy, Equation (A.15), to write Equation (1.9) in terms of the pressure and particle velocity as:

$$\frac{\partial p}{\partial t} + \mathbf{v}\cdot\nabla p = c^2\left(\frac{\partial \rho}{\partial t} + \mathbf{v}\cdot\nabla \rho\right), \tag{1.10}$$

where the speed of sound, c, may be decomposed as $c = c_o + c' + \cdots$, like other fluid properties. Upon linearization of Equation (1.10), we obtain for the acoustic fluctuations:

$$\frac{\partial p'}{\partial t} + \mathbf{v}_o\cdot\nabla p' + \mathbf{v}'\cdot\nabla p_o = c_o^2\left(\frac{\partial \rho'}{\partial t} + \mathbf{v}_o\cdot\nabla \rho' + \mathbf{v}'\cdot\nabla \rho_o + 2c'\mathbf{M}_o\cdot\nabla \rho_o\right), \tag{1.11}$$

where $\mathbf{M}_o = \mathbf{v}_o/c_o$ denotes the local Mach number of the mean flow velocity. The corresponding equation for the mean flow is

$$\mathbf{v}_o\cdot\nabla p_o = c_o^2\mathbf{v}_o\cdot\nabla \rho_o. \tag{1.12}$$

A form of mean flow which satisfies this is $\nabla p_o = c_o^2\nabla\rho_o$, which implies that the mean specific entropy, s_o, is constant everywhere. This is called homentropic flow. Then Equation (1.11) may be simplified as

$$\frac{\partial p'}{\partial t} + \mathbf{v}_o\cdot\nabla p' = c_o^2\left(\frac{\partial \rho'}{\partial t} + \mathbf{v}_o\cdot\nabla \rho' + 2c'\mathbf{M}_o\cdot\nabla \rho_o\right). \tag{1.13}$$

For subsonic low Mach number mean flows, the term involving c' may be neglected as being small compared to the remaining terms in Equations (1.11) and (1.13). On this premise, Equation (1.13) is usually implemented tacitly as

$$\frac{\partial p'}{\partial t} + \mathbf{v}_o\cdot\nabla p' = c_o^2\left(\frac{\partial \rho'}{\partial t} + \mathbf{v}_o\cdot\nabla \rho'\right). \tag{1.14}$$

On the other hand, in many applications, the mean flow gradients are small and the assumptions $p_o \approx$ constant and $\rho_o \approx$ constant (or $T_o \approx$ constant) may be adequately accurate for practical purposes. Then Equation (1.12) is also satisfied and, since in view of the state postulate of thermodynamics (Appendix A.2) we may also assume that $c_o \approx$ constant, Equation (1.11) becomes simply

$$p' = c_o^2\rho'. \tag{1.15}$$

Thus, the acoustic continuity and momentum equations, Equations (1.5) and (1.8), may be closed by Equation (1.15) or other derivatives of Equation (1.11) for the determination of p', ρ' and $\mathbf{v}'$. Solutions of these equations are sometimes considered by stipulating a mean flow which is not completely compatible with the corresponding non-linear inviscid mean flow equations. For example, the mean flow is often assumed to be sheared with a non-uniform axial velocity profile, although the viscosity of the fluid is neglected in the acoustic equations. In this case, we may discard the incompatible mean flow equations, because the acoustic equations still provide an adequately accurate mathematical model (for example, a quasi-laminar model) for studying the effect of refraction in shear layers.

1.1.4 Evolution of Non-Linear Waves

For further justification of linearization and insight into the effect of the neglected $O(\varepsilon^n)$, $n \geq 2$, terms, it is useful to review briefly some implications of the fluid dynamic equations for one-dimensional flow. Putting $\mathbf{v} = \mathbf{e}v$ and $\nabla = \mathbf{e}\partial/\partial x$, Equations (1.1) and (1.2) become, $\partial\rho/\partial t + \partial(\rho v)/\partial x = 0$ and $\rho(\partial v/\partial t + v\partial v/\partial x) + \partial p/\partial x = 0$, respectively. Also, from Equation (A.15a), we have $\mathrm{d}p = c^2\mathrm{d}\rho$ for isentropic motion. It was shown by Riemann in 1860 that these equations are tantamount to the partial differential equation $\partial p/\partial t + (c+v)\partial p/\partial x = 0$ for flow in the positive x direction [1, 6]. This result shows that spatial forms of pressure waveforms are convected with velocity $c + v$ and that, to an observer moving with velocity $c + v$, a pressure waveform appears to be stationary in shape. This may be seen by applying the Galilean transformation $y = x - (c+v)t$ and $\tau = t$, giving $\partial p/\partial\tau = 0$, which implies the general solution to be a function of the form $f(x - (c+v)t)$. But, $c + v$ is a function of the pressure. Therefore, to a fixed observer, each point of a pressure waveform moves with different velocity and, consequently, the shape of a pressure waveform at time $t = 0$ is continuously distorted as it is convected with the flow. Then, if the speed of sound is an increasing function of the pressure (which is usually the case), points of the waveform with higher pressure move faster than those at lower pressure and portions of the waveform where the pressure is increasing become steeper with time, eventually causing the waveform to become multi-valued, which signifies occurrence of a shock, a moving plane of discontinuity in fluid properties, giving rise to finite amplitude waves. Here, we are concerned about the conditions for the occurrence of shock formation.

As in the linearization procedure, we introduce the decomposition $p = p_o + p'$ and so on for other variables, but now we neglect the mean flow $(v_o = 0)$ and allow the acoustic wave $p'(x,t) = p'(x - (c+v)t)$ to have finite amplitudes. Typically, let us assume that, initially, p' is a traveling sinusoidal acoustic pressure waveform of radian frequency ω and amplitude $\hat{p}$, such as $p' = \hat{p}\sin\omega(t - x/c)$. It may then be shown that for a perfect gas, the earliest value of x for which shock formation occurs is $x_{\max} \approx (p_o/\hat{p})(c_o/\omega)(2\gamma_o/(1+\gamma_o)[1]$. For example, at exhaust conditions of internal combustion engines, $x_{\max} = 1.2 \times 10^7/\hat{p}f$ meters, where f denotes the frequency in Hz. At 1000 Hz, this is $12000/\hat{p}$ meters, or 600 meters for exhaust noise pressure amplitudes of about 5% of the atmospheric pressure. Evolution of waveform distortions until the shock may be estimated from the Fubini-Ghiron solution $p' = \sum_{n=1}^{\infty}\hat{p}_n \sin(n\omega(t - x/c_o))$, where $\hat{p}_n/\hat{p} = 2\mathrm{J}_n(nx/x_{\max})$ and J_n denotes a Bessel function of order n. Amplitude of the fundamental harmonic $(n = 1)$ decreases

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with $x/x_{\max}$, but this is compensated with the growth of an infinite number of higher-order harmonic amplitudes.

In general, therefore, the distances needed for harmonic acoustic perturbations with amplitudes initially in the linear range to travel freely before they may be subjected to substantial distortion are unrealistically long compared to the actual distances involved in practical ductworks and frequencies of interest. Furthermore, the viscous and thermal losses, which are neglected in the foregoing considerations, tend to act in opposition to the non-linear wave distortion process [1].

1.2 Representation of Acoustic Waves in the Frequency Domain

In the previous section, the linear acoustic equations are given in the time domain (time appears explicitly). If we are interested in the evolution of the wave motion immediately after the source (for example, a fluid machine) is switched on, it is necessary to obtain solutions of these equations as a function of time for the appropriate initial conditions on the acoustic variables. However, transient wave motions which the initial conditions generate vanish rapidly due to damping and the source settles into a steady-state operation after a few cycles. In most practical applications of duct acoustics, it suffices for acoustic design purposes to consider the acoustic wave motion at such steady operating points of the source, as the damping, which is unavoidably present in real systems, is usually sufficient for preventing growth of resonances that may be excited during the transient stage. Then the initial conditions are not required, and the Fourier transform may be used for removing the explicit dependence of the acoustic equations on time. This approach is known as the transformation to the frequency domain and is used throughout the present book. In this section, we review the basic Fourier transform based frequency domain concepts which are invoked throughout the book.

1.2.1 Fourier Transform

The Fourier transform [7], is an integral operation which maps a function given in time domain, $h = h(t)$, say, to the function

$$h(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{i\omega t} dt, \tag{1.16}$$

where, i denotes the unit imaginary number, $i = \sqrt{-1}$, and ω denotes the Fourier transform variable which has the unit of radians per second, that is, $\omega = 2\pi f$, where f denotes frequency in Hertz (Hz). Thus, Equation (1.16) represents a mapping from time domain to frequency domain. It should be noted that we use (here and throughout the book) the same symbol for both time and frequency domain representations of a

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function. The domain distinction is usually clear from the argument of the function or the context of the discussion. The reader should be warned that, although Equation (1.16) is a commonly used definition of the Fourier transform, there is no fixed convention regarding the sign of exponent and the value of a constant non-dimensional multiplication factor in the integrand. This is worthwhile to keep in mind when comparing transformed equations given by different authors.

If $h(\omega)$ is the Fourier transform of $h(t)$, then the Fourier transform of $\breve{h}(t)$ is $\breve{h}(-\omega)$, where an inverted over-arc denotes the complex conjugate. Consequently, for real time-functions: $h(-\omega) = \breve{h}(\omega)$. Thus, the magnitude of the Fourier transform of an acoustic variable is symmetrical about the origin of the frequency axis, that is, $|h(-\omega)| = |h(\omega)|$. Physically, however, only the positive frequencies are meaningful. Negative frequencies occur as a mathematical artifact of the theory.

The Fourier transform of the partial time-derivatives in the linearized acoustic equations are taken by using the following property of the transformation: If $h(\omega)$ is the Fourier transform of $h(t)$, then

$$\int_{-\infty}^{\infty} \frac{\partial h}{\partial t} \mathrm{e}^{\mathrm{i}\omega t} \mathrm{d}t = \int_{-\infty}^{\infty} \left(\lim_{\Delta t \to 0} \frac{h(t + \Delta t)\mathrm{e}^{\mathrm{i}\omega t} - h(t)\mathrm{e}^{\mathrm{i}\omega t}}{\Delta t} \right) \mathrm{d}t = -\mathrm{i}\omega h(\omega), \tag{1.17}$$

which follows after taking the limit operation out of the integral and noting that the Fourier transform of $h(t + \tau)$ is $\mathrm{e}^{-\mathrm{i}\omega\tau} h(\omega)$. Thus, the acoustic equations given in the time domain may be transformed to the frequency domain simply by replacing the operator $\partial/\partial t$ by $-\mathrm{i}\omega$.

Fourier transforms of acoustic state variables $q' \in p', \rho', \mathbf{v}', \ldots$ are determined by solving the Fourier transformed forms of the acoustic continuity and momentum equations. The corresponding solutions in time domain can be recovered by using the formula

$$h(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(\omega) \mathrm{e}^{-\mathrm{i}\omega t} \mathrm{d}\omega, \tag{1.18}$$

which is called the inverse Fourier transform. This integral shows that, each point of $h(\omega)$ contributes nothing to $h(t)$, but indicates the relative weighting of each frequency component. So, the larger $|h(\omega)|$ is, the larger will be the contribution to $h(t)$ in a narrow frequency band $(\omega - \Delta\omega/2, \omega + \Delta\omega/2)$, which can be calculated by integrating the area under $h(\omega)$ in this band.

The Fourier transform and the inverse Fourier transform can be computed efficiently by using the numerical algorithm called the fast Fourier transform [7]. However, the inverse Fourier transformation is seldom necessary in practical applications, since the frequency domain solutions contain more useful information for acoustic design than their counterparts in the time domain. Indeed, not only is the acoustic response of flow duct systems frequency selective, but so is human hearing [1].

1.2.2 Periodic Functions

The above considerations hold for any continuous function of time. If $h(t)$ is periodic of fundamental period $T = 2\pi/\varpi$, where T is the smallest number which satisfies $h(t) = h(t+T)$, it can be represented by the exponential Fourier series

$$h(t) = \sum_{n=-\infty}^{\infty} h_n \mathrm{e}^{-\mathrm{i}n\varpi t} \tag{1.19}$$

with

$$h_n = \frac{1}{T}\int_{-T/2}^{T/2} h(t)\mathrm{e}^{\mathrm{i}n\varpi t}\mathrm{d}t. \tag{1.20}$$

It can be shown that $h_n = h_T(n\varpi)/T$, where $h_T(\omega)$ denotes the Fourier transform of the truncated (windowed) part of $h(t)$ in the interval $-T/2 \le t \le T/2$. For real periodic functions: $h_{-n} = \breve{h}_n$, since $h_T(-\omega) = \breve{h}_T(\omega)$ for real functions. Thus, a periodic function has all its amplitude components h_n at discrete frequencies, which are integer multiples of the fundamental frequency ϖ, and each of these has a definite contribution. The Fourier transform of Equation (1.19) is

$$h(\omega) = 2\pi \sum_{n=-\infty}^{\infty} h_n \delta(\omega - n\varpi), \tag{1.21}$$

since the Fourier transform of $\mathrm{e}^{-\mathrm{i}n\varpi t}$ is $\delta(\omega - n\varpi)$, where $\delta(x)$ denotes a unit impulse function at $x = 0$. Thus, the area (weight) of each impulse is 2π times the value of its corresponding coefficient in the complex Fourier series.

The simplest form of Equation (1.19) is a single frequency function $h(t) = h\mathrm{e}^{-\mathrm{i}\varpi t} + \breve{h}\mathrm{e}^{\mathrm{i}\varpi t} = \mathrm{Re}\{2h\mathrm{e}^{-\mathrm{i}\varpi t}\} = \mathrm{Re}\{\hat{h}\mathrm{e}^{-\mathrm{i}\varpi t}\}$, where ϖ denotes the radian frequency, $\hat{h}$ is called the complex amplitude and Re denotes the real part of a complex number. So, if the acoustic variables are assumed a priori to be single frequency functions of the form $\hat{h}\mathrm{e}^{-\mathrm{i}\varpi t}$, it suffices to take the real parts of the corresponding solutions of the acoustic equations. In fact, if we assume a priori that the time-dependence of the variables in acoustic equations is of the form $\mathrm{e}^{-\mathrm{i}\varpi t}$, we get, after the common factor $\mathrm{e}^{-\mathrm{i}\varpi t}$ is cancelled out, relationships between the complex amplitudes of the acoustic variables, which are in exactly of the same form as the Fourier transformed equations. For this reason, the transformation of the acoustic equations from time domain to frequency domain is often effected in the literature informally by assuming a priori that the acoustic variables have $\mathrm{e}^{-\mathrm{i}\varpi t}$ time-dependence. Then, the resulting acoustic equations yield relations between the complex amplitudes of the acoustic variables, but these equations may also be understood as the Fourier transformed acoustic equations, if the complex amplitudes are replaced by the corresponding Fourier transforms of the acoustic variables. Again, caution is needed when comparing results of different authors, because some authors prefer to assume $\mathrm{e}^{\mathrm{i}\varpi t}$ time-dependence.

Conversely, acoustic variables determined in frequency domain may be transformed back to time domain on single frequency basis, if they are known a priori to encompass only discrete frequencies. Then, for example, if $h(\omega)$ is a variable determined in frequency domain, the contribution of a typical single frequency component of frequency ϖ to $h(\omega)$ is $2\pi h(\varpi)\delta(\omega - \varpi)$. The inverse Fourier transform of this is $h e^{-i\varpi t}$, where $h = h(\varpi)$. But, in the Fourier transform $h(\omega)$, each single frequency component, such as $he^{-i\varpi t}$, occurs with its complex conjugate $\breve{h} e^{i\varpi t}$, since both positive and negative frequencies are included and $h(-\omega) = \breve{h}(\omega)$ for real functions. Therefore, a single frequency component of frequency ϖ of $h(\omega)$ contributes to $h(t)$ by $he^{-i\varpi t} + \breve{h} e^{i\varpi t}$, that is, by $2\mathrm{Re}\{he^{-i\varpi t}\}$. It should be noted that, the root-mean-square value of the single frequency function $2\mathrm{Re}\{he^{-i\varpi t}\}$ is $\sqrt{2}|h|$, which corresponds to a peak amplitude of $2|h|$, and that only positive frequencies are relevant.

1.2.3 Impulse Sampling

An impulse-sampled function, $h^{(s)}(t)$, of the continuous function $h(t)$ is defined as

$$h^{(s)}(t) = h(t) \sum_{k=-\infty}^{\infty} \delta(t - kT_s), \quad (1.22)$$

where T_s denotes the sampling period. Fourier transform of this sampled function is

$$h^{(s)}(\omega) = f_s \sum_{k=-\infty}^{\infty} h(\omega - k\omega_s), \quad (1.23)$$

where $\omega_s = 2\pi f_s$ and $f_s = 1/T_s$ is called sampling frequency. This result shows that, apart from the scaling factor f_s, the Fourier transform of an impulse-sampled function is a periodic repetition, of period T_s, of the Fourier transform of the continuous function, $h(\omega)$. Therefore, if we evaluate $h^{(s)}(\omega)$, we can see $h(\omega)$ in the frequency range $-\omega_s \le \omega \le \omega_s$, provided that

$$f_s \ge 2f_{\max}, \quad (1.24)$$

where $f_{\max}$ denotes the maximum frequency of interest in $h(\omega)$. This is known as the Nyquist sampling criterion.

If the Nyquist condition is not satisfied, the periodic repetitions of $h(\omega)$ will overlap and $h(\omega)$ itself will not be visible exactly at some frequencies lower than $f_{\max}$. In practice, overlap (also called aliasing) is unavoidable, because we must necessarily use functions in a finite time window and time-limited functions are not band limited in the frequency domain. On the other hand, a function in a finite time window is represented mathematically by the product of $h(t)$ with a rectangular gate function having the value of unity in the window and zero elsewhere. The Fourier transform of the product of two time functions is given by the convolution of the Fourier transforms of these functions. For this reason, when we take the Fourier transform of an impulse-sampled function $h^{(s)}(t)$ limited to a finite time window,

$h^{(s)}(\omega)$ convolves with the Fourier transform of the rectangular gate function. The latter is characterized by a main lobe and infinite number of side lobes, the amplitudes of which attenuate at the rate of 6 dB/octave, the amplitude of the first side lobe being 13.3 dB lower than that of the main lobe. Because of the convolution process, the side lobes can interfere with the computed Fourier transform even in frequency ranges of the Nyquist criterion. This interference, which is known as leakage, may be of some concern, as it can result in an inaccurate representation of the Fourier transform of the actual signal. Leakage is usually combated by using different window functions than the rectangular gate function, which aim to make the main lobe narrower and to generate side lobes which attenuate at large rates with frequency.

An efficient numerical procedure known as the fast Fourier transform is widely used for numerical calculation of the Fourier transform of continuous functions [7]. This is executed by impulse sampling the function at $N = 2^m$ equidistant points at times $t = 0, T_s, 2T_s, \ldots, (N-1)T_s$, in a window of length $T = (N-1)T_s$. The procedure yields the values of the Fourier transform at equidistant frequencies:

$$f = 0, \quad \mp\frac{f_s}{N}, \quad \mp2\frac{f_s}{N}, \ldots, \quad \mp\left(\frac{N}{2}-1\right)\frac{f_s}{N}, \quad \mp\frac{f_s}{2},$$

where $f = \omega/2\pi$. Thus, the Fourier transform is sampled at frequency intervals $\Delta f = f_s/N$ or, if N is large enough, $\Delta f = 1/T$. Accordingly, the time interval should be selected so as to obtain a desired frequency resolution. The fast Fourier transform is invertible; that is, if the Fourier transform of a function is known and is sampled at the foregoing $2N+1$ frequency points, the inverse fast Fourier transform yields the values of the function at times $t = 0, T_s, 2T_s, \ldots, (N-1)T_s$.

1.2.4 Power Spectral Density

Root-mean-square value of an acoustic variable is defined by the formula

$$h_{rms} = \sqrt{\frac{1}{T}\int_0^T h^2(t)\mathrm{d}t} = \sqrt{\langle h^2(t)\rangle}, \tag{1.25}$$

where T is called averaging time and the second equality defines a shorthand notation frequently used to denote a time average. If $h(t)$ is a random function, this definition tacitly assumes that it is stationary and ergodic [7]. Since acoustic variables have, by definition, zero mean parts, the root-mean-square value can be interpreted as temporal standard deviation. The following Parseval's formula links the root-mean-square value of a variable with its Fourier transform:

$$\int_0^T h^2(t)\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |h_T(\omega)|^2\mathrm{d}\omega, \tag{1.26}$$

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where $h_T(\omega)$ denotes the Fourier transform of the truncated (windowed) part of the function in the interval $0 \le t \le T$. Therefore, h_{rms} can be expressed as

$$h_{rms}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_h(\omega) \mathrm{d}\omega, \tag{1.27}$$

where

$$S_h(\omega) = \frac{|h_T(\omega)|^2}{T} \tag{1.28}$$

is called power spectral density of $h(t)$. In contrast to $h(\omega)$, $S_h(\omega)$ cannot be inverted back to $h(t)$, since it contains no phase information.

The integral in Equation (1.27) is often evaluated by splitting it into contiguous frequency bands. Then, for real time signals, it can be expressed as

$$h_{rms}^2 = \sum_{n=1}^{\infty} h_{rms}^2(\bar{\omega}_n), \tag{1.29}$$

where

$$h_{rms}^2(\bar{\omega}_n) = \int_{\omega_n}^{\omega_{n+1}} |h(\omega)|^2 S_x(\omega) \mathrm{d}\omega \tag{1.30}$$

and $\omega_n \le \omega \le \omega_{n+1}$ denotes the width of band n and $\bar{\omega}_n$ is called the center frequency. In practice, usually two types of frequency bands are used, namely, constant and proportional. The bandwidth of constant bands does not vary with frequency and the center frequency may be defined by the arithmetic mean $\bar{\omega}_n = (\omega_n + \omega_{n+1})/2$. The most common proportional bands are known as $1/N$ octave bands. The bandwidths of $1/N$ octave bands are defined as $\omega_{n+1}/\omega_n = 2^{1/N}$ and the center frequency of band n is given by the geometric mean $\bar{\omega}_n = \sqrt{\omega_n \omega_{n+1}}$. Thus, a given frequency band can be covered by $1/N$ octave bands contiguously by selecting the center frequencies appropriately.

The power spectral density of a periodic function consists of impulse functions located at frequencies $n\varpi$, $n = 0, \mp 1, \mp 2, \ldots$:

$$S_h(\omega) = 2\pi \sum_{n=-\infty}^{\infty} |h_n|^2 \delta(\omega - n\varpi). \tag{1.31}$$

Then, the root-mean-square value of a periodic signal is given simply by

$$h_{rms}^2 = \sum_{n=-\infty}^{\infty} |h_n|^2, \tag{1.32}$$

which is Parseval's formula for periodic functions.

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1.3 Representation of Waves in the Wavenumber Domain

1.3.1 Spatial Fourier Transform

Many important problems of duct acoustics are concerned about sound propagation in ducts which are axially uniform and homogeneous. In such cases, the linear acoustic equations can also be Fourier transformed with respect to the duct axis. Let $h(\omega, x_1, x_2, x_3)$ denote the Fourier transform of function $h(t, x_1, x_2, x_3)$, which is also function of the Cartesian coordinates x_1, x_2, x_3, where x_1 denotes the duct axis. The spatial Fourier transform of h with respect to x_1 is defined similarly as the temporal Fourier transform:

$$h(\omega, \kappa, x_2, x_3) = \int_{-\infty}^{\infty} h(\omega, x_1, x_2, x_3)\,\mathrm{e}^{-\mathrm{i}\kappa x_1}\mathrm{d}x_1, \tag{1.33}$$

where the transform variable κ is called the axial wavenumber. As an example, consider the application of this transform to the temporal Fourier transform of Equation (1.5) for a uniform and homogeneous duct with $\rho_o =$ constant, $\mathbf{v}_o = \mathbf{e}_1 v_{o1} + \mathbf{e}_2 v_{o2} + \mathbf{e}_3 v_{o3}$, where v_{o1}, v_{o2}, v_{o3} denote components of $\mathbf{v}_o$ in x_1, x_2, x_3 directions, respectively. Assume for simplicity that v_{o1}, v_{o2}, v_{o3} are constants. Then, upon applying the spatial counterparts of Equations (1.16) and (1.17), Equation (1.5) transforms to:

$$\mathrm{i}(-\omega + \kappa v_{o1})\rho' + v_{o2}\frac{\partial \rho'}{\partial x_2} + v_{o3}\frac{\partial \rho'}{\partial x_3} + \rho_o\left(\mathrm{i}\kappa v_1' + \frac{\partial v_2'}{\partial x_2} + \frac{\partial v_3'}{\partial x_3}\right) = 0. \tag{1.34}$$

Inspection of this result shows that acoustic equations can be transformed to the wavenumber domain simply by replacing the operator $\partial/\partial x_1$ by $\mathrm{i}\kappa$. Likewise, the spatial Fourier transform may also be taken by assuming a priori that acoustic variables have $\mathrm{e}^{\mathrm{i}\kappa x_1}$ dependence (or $\mathrm{e}^{-\mathrm{i}\kappa x_1}$ dependence, if time dependence is assumed to be $\mathrm{e}^{\mathrm{i}\varpi t}$).

Spatial Fourier transformed acoustic equations will have one less spatial derivative than the only temporal Fourier transformed equations, which can be convenient when getting solutions. As will be seen in later chapters, solutions of the acoustic equations for uniform ducts are feasible for ω and κ satisfying $\Delta(\omega, \kappa) = 0$. This equation is called the dispersion equation and its actual form depends on the duct and its acoustic boundary conditions. Roots of the dispersion equation give the axial wavenumbers as functions of ω, and each wavenumber corresponds to a wave mode in the duct. The time-domain characteristics of wave modes can be determined by taking first the inverse spatial Fourier transform

$$h(\omega, x_1, x_2, x_3) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(\omega, \kappa, x_2, x_3)\,\mathrm{e}^{\mathrm{i}\kappa x_1}\mathrm{d}\kappa \tag{1.35}$$

and then the inverse temporal Fourier transform.

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1.3.2 Briggs' Criterion

By studying the properties of the temporal and spatial Fourier transforms, Briggs has shown that the information about the directions of wave propagation can be obtained without explicit evaluation of inverse Fourier transforms [8]. First, the wavenumbers are computed for a fixed (real) ω and then for $\omega + \mathrm{i}\chi$, where χ is an arbitrarily large positive number. Briggs' criterion states that: if an axial wavenumber κ originally (for ω) having a positive imaginary part, acquires a negative imaginary part for $\omega + \mathrm{i}\chi$ as $\chi \to +\infty$, then the corresponding wave mode propagates in the $+x_1$ direction. Conversely, if an axial wavenumber κ originally (for ω) having a negative imaginary part, acquires a positive imaginary part for $\omega + \mathrm{i}\chi$, $\chi \to +\infty$, then the corresponding wave mode propagates in the $-x_1$ direction.

For real wavenumbers, Briggs' criterion can be stated as: if $\mathrm{Re}\{V_g\} > 0$, then propagation is in the $+x_1$ direction, and $\mathrm{Re}\{V_g\} < 0$ implies propagation in $-x_1$ direction. Here,

$$V_g = \frac{\mathrm{d}\omega}{\mathrm{d}\kappa} \tag{1.36}$$

and is called the group velocity.

If a complex axial wavenumber propagating in the $+x$ direction has a positive imaginary part, the corresponding wave mode decays in the direction of propagation (since $\mathrm{e}^{\mathrm{i}\kappa x_1}$ dependence is being assumed); but, if it has a negative imaginary part, it amplifies in the direction of propagation. Similarly, if a complex axial wavenumber propagating in the $-x_1$ direction has a negative imaginary part, it amplifies in the direction of propagation; but, if it has positive imaginary part, it decays in the direction of propagation.

A wave mode which decays in the direction of propagation is stable and called evanescent. Amplification of a wave mode in the direction of propagation is known as convective instability, which means that the wave amplitudes increase with distance traveled but remain finite at fixed locations. Instability can also be of the absolute type; that is, wave amplitudes blow up at every point in time at every location. A necessary condition for absolute instability is the occurrence of an axial wavenumber of multiplicity two [8].

1.4 Intensity and Power of Sound Waves

The term acoustic energy is usually understood as part of the fluid's total energy associated with the presence of a sound wave. In an inviscid (ideal) fluid, this is purely mechanical energy and consists of the sum of kinetic and potential (compression strain) energies associated with the sound wave motion [1, 6]. But in a Newtonian fluid with non-negligible viscosity, the presence of a sound wave is also associated with some thermal energy dissipated due to friction. In practice, however, the physically relevant metric for sound energy is the mechanical power it transmits when sound

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waves reach a reception point such as a microphone or human ear. In this book, we understand acoustic energy as this mechanical part of the fluid's total energy associated with the presence of a sound wave.

Consider a control volume V of surface Σ, which contains no internal energy sources and is free of external fields and shaft work. The integral form of the energy equation for the unsteady motion of a Newtonian fluid in this control volume is given by Equation (A.11a). Denoting the rate of viscous work done by the fluid in the control volume on the surroundings by $\dot{W}_V$, and the rate of heat loss from the control volume to the surroundings by conduction by $\dot{Q}_c$, Equation (A.11a) can be expressed as

$$-\frac{\partial}{\partial t}\int_V \rho\, \mathrm{e}\, \mathrm{d}V = \int_\Sigma h^o \rho\, \mathbf{v}\cdot\mathbf{n}\mathrm{d}\Sigma + \dot{Q}_c + \dot{W}_V \tag{1.37}$$

Here, the term which is of interest to us is the time-average of the integral on the right-hand side, that is, $\int_\Sigma \langle h^o(\rho\,\mathbf{v})\cdot\mathbf{n}\rangle \mathrm{d}\Sigma$, where angled brackets denote time-averaging. This integral gives the total time-averaged power of the fluid crossing normal to Σ. We assume that the momentum density, $\rho\mathbf{v}$, and the specific stagnation enthalpy, h^o, may be represented, as the intensive properties of the fluid, by asymptotic expansions $\rho\mathbf{v} = (\rho\mathbf{v})_o + (\rho\mathbf{v})' + \cdots$ and $h^o = (h^o)_o + (h^o)' + \cdots$, respectively, and hypothesize that the fluctuating parts of $O(\varepsilon)$ are solely due to the presence of sound waves (Section 1.1.1). This is a crucial assumption, because it means that all acoustic ramifications of turbulence fluctuations and losses incurring from flow-acoustic interactions are neglected in the subsequent analysis. Then the total time-averaged power associated with the presence of sound waves may be expressed as [9]:

$$W^o = \int_\Sigma \left\langle (h^o)'(\rho\,\mathbf{v})'\cdot\mathbf{n}\right\rangle \mathrm{d}\Sigma \tag{1.38}$$

because terms of $O(\varepsilon)$ have zero mean by definition and vanish on time-averaging. Upon evaluating the momentum density and the specific stagnation enthalpy to $O(\varepsilon)$ and using the linearized state equation $h' = T_o s' + p'/\rho_o$ (see Equation (A.16)), the foregoing equation may be expanded as:

$$W^o = \int_\Sigma \left\langle (T_o s' + \mathbf{v}_o\cdot\mathbf{v}' + p'/\rho_o)\ \ (\rho_o\mathbf{v}' + \mathbf{v}_o\rho')\cdot\mathbf{n}\right\rangle \mathrm{d}\Sigma \tag{1.39}$$

But this is not fully mechanical power; some part of it is thermal power dissipated by friction due to the viscosity of the fluid. The contributor of this part in the foregoing integral is the fluctuating entropy term, because an entropy increase corresponds to a change in the amount of heat energy per unit volume, which must equal the loss of mechanical energy per unit volume per unit time from the sound wave due to fluid viscosity. Then, discarding this term, the acoustic power (the mechanical part of the time-averaged power associated with the presence of sound waves) crossing normal to Σ can be expressed as

$$W = \int_{\Sigma} \langle \mathbf{N}'' \cdot \mathbf{n} \rangle \mathrm{d}\Sigma, \tag{1.40}$$

where

$$\mathbf{N}'' = (\rho_o \mathbf{v}' + \mathbf{v}_o \rho') \left(\mathbf{v}_o \cdot \mathbf{v}' + \frac{p'}{\rho_o} \right) \tag{1.41}$$

denotes the instantaneous local flux (rate of transport) of acoustic energy, which is called acoustic intensity. Thus, the time-averaged acoustic power transmitted in a duct normal to the duct sections can be calculated by using the formula

$$W = \int_{S} \langle \mathbf{N}'' \cdot \mathbf{n} \rangle \mathrm{d}S, \tag{1.42}$$

where S denotes the duct cross-sectional area. It should be noted that, the double prime syntax is used to denote the part of a fluctuating quantity that is of $O(\varepsilon^2)$. The true acoustic intensity is given by $\mathbf{N} = \mathbf{N}' + \mathbf{N}'' + O(\varepsilon^3)$; however, only the $O(\varepsilon^2)$ term is relevant for the calculation of the time-averaged acoustic power, since $\langle \mathbf{N} \rangle = \langle \mathbf{N}'' \rangle$.

In the literature, it is traditional to deduce expressions for the acoustic intensity from an energy continuity equation which is derived as a corollary to the acoustic continuity, momentum and energy equations and is of the general form

$$\frac{\partial \chi''}{\partial t} + \nabla \cdot \mathbf{N}'' = \psi'', \tag{1.43}$$

where χ'' represents the acoustic energy density and ψ'' is called the production term, the double prime syntax still denoting quantities small to the second order. This equation derives its rational from the fact that, upon application of the divergence theorem, Equation (A.1), for a fixed control volume of volume V of surface Σ, it turns into a statement of conservation of acoustic energy: $\mathbf{N}''$ crossing normal to the surface Σ, plus the rate of change of χ'' in V, is balanced by ψ'' in V. A discussion of the ramifications of this approach is given by Morfey, who also presents a general formulation including fluid viscosity and thermal conductivity and non-uniform mean flow [10]. This formulation partitions the density fluctuations that are non-acoustic, by hypothesizing that the effect of viscosity and thermal conductivity on density fluctuations is negligible. Then, the acoustic momentum density fluctuations $(\rho_o \mathbf{v}' + \mathbf{v}_o \rho')$ are evaluated by using Equation (1.15) for ρ'. Apart from this difference, the expression of Morfey's theory for $\mathbf{N}''$ is the same as Equation (1.41). The acoustic fields considered in this book do not contain non-acoustic density fluctuations by definition (Section 1.1.1), and Equation (1.41) is applicable.

An expression for χ'' corresponding to $\mathbf{N}''$ may be derived from Equations (1.5) and (1.8). Letting $\mathbf{m}' = \mathbf{v}_o \rho' + \rho_o \mathbf{v}'$, the former becomes $\partial \rho' / \partial t + \nabla \cdot \mathbf{m}' = 0$. Assuming homentropic mean flow (Section 1.1.2), Equation (1.8) can be expressed as $\partial \mathbf{v}' / \partial t + \nabla E' = s'(p_o / c_{vo}) \nabla (1/\rho_o)$, where $E' = \mathbf{v}_o \cdot \mathbf{v}' + p'/\rho_o$, $c_v \approx c_{vo}$ denotes the specific heat coefficient at constant volume and s' denotes specific entropy fluctuations,

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which enters as we use the linearized form of Equation (A.15a). Upon adding the dot product of the modified form of Equation (1.8) with $\mathbf{m}'$, to E' times the modified form of Equation (1.5), Equation (1.43) is obtained with $\mathbf{N}''$ given by Equation (1.41), $\psi'' = -\mathbf{m}' \cdot (s' p_o / c_{vo}) \nabla (1/\rho_o)$ and

$$\chi'' = \frac{1}{2}\rho_o \mathbf{v}' \cdot \mathbf{v}' + \rho' \mathbf{v}_o \cdot \mathbf{v}' + \int \frac{p' \mathrm{d}\rho'}{\rho_o}. \tag{1.44}$$

The third term on the right-hand side of this equation represents the specific potential (compression strain) energy of the fluid associated with the presence of acoustic waves [1]. The specific kinetic energy of the fluid is given by $(\rho_o + \rho')(\mathbf{v}_o + \mathbf{v}')^2/2$, the $O(\varepsilon^2)$ term of which is $\rho_o \mathbf{v}' \cdot \mathbf{v}'/2 + \rho' \mathbf{v}_o \cdot \mathbf{v}'$. Therefore, χ'' gives the specific acoustic energy (sum of specific kinetic and potential energies) associated with the presence of acoustic waves. If Equation (1.15) is used, the specific potential energy becomes $p'^2/2\rho_o c_o^2$ and Equation (1.44) reduces to the form given by Morfey [10], where all terms involving the specific entropy (and also the viscosity and thermal conductivity, which are neglected in the foregoing derivation) are also collected in the production term.

Equation (1.42) may be expressed explicitly as

$$W = \int_S \left\langle p' \mathbf{v}' \cdot \mathbf{n} + \rho_o \mathbf{v}_o \cdot \mathbf{v}' \mathbf{v}' \cdot \mathbf{n} + \frac{\mathbf{v}_o \cdot \mathbf{n}}{\rho_o} p' \rho' + \mathbf{v}_o \cdot \mathbf{n} \mathbf{v}_o \cdot \mathbf{v}' \rho' \right\rangle \mathrm{d}S. \tag{1.45}$$

Since this expression is in the time domain, it is desirable to have an equivalent expression which may be evaluated directly by using the solutions obtained in the frequency domain. The simplest way to derive this expression is to assume that the acoustic variables are single frequency functions of the form $h(t) = 2\mathrm{Re}\{h\mathrm{e}^{-\mathrm{i}\omega t}\}$ (see Section 1.2.2). Then, Equation (1.45) may be written as

$$\begin{aligned} W = 4 \int_S \frac{1}{T} \int_o^T \Big(& \mathrm{Re}\{p' \mathrm{e}^{-\mathrm{i}\omega t}\} \mathrm{Re}\{\mathbf{v}' \mathrm{e}^{-\mathrm{i}\omega t}\} \cdot \mathbf{n} + \rho_o \mathbf{v}_o \cdot \mathrm{Re}\{\mathbf{v}' \mathrm{e}^{-\mathrm{i}\omega t}\} \mathrm{Re}\{\mathbf{v}' \mathrm{e}^{-\mathrm{i}\omega t}\} \cdot \mathbf{n} \\ & + \frac{\mathbf{v}_o \cdot \mathbf{n}}{\rho_o} \mathrm{Re}\{p' \mathrm{e}^{-\mathrm{i}\omega t}\} \mathrm{Re}\{\rho' \mathrm{e}^{-\mathrm{i}\omega t}\} + \mathbf{v}_o \cdot \mathbf{n} \mathbf{v}_o \cdot \mathrm{Re}\{\mathbf{v}' \mathrm{e}^{-\mathrm{i}\omega t}\} \mathrm{Re}\{\rho' \mathrm{e}^{-\mathrm{i}\omega t}\} \Big) \mathrm{d}t \mathrm{d}S, \end{aligned} \tag{1.46}$$

where $T = 2\pi/\omega$. Upon integration we obtain:[2]

$$W = 2 \int_S \left(\mathrm{Re}\{\breve{p}' \mathbf{v}'\} \cdot \mathbf{n} + \rho_o \mathbf{v}_o \cdot \mathrm{Re}\{\breve{\mathbf{v}}' \mathbf{v}'\} \cdot \mathbf{n} + \frac{\mathbf{v}_o \cdot \mathbf{n}}{\rho_o} \mathrm{Re}\{\breve{p}' \rho'\} + \mathbf{v}_o \cdot \mathbf{n} \mathbf{v}_o \cdot \mathrm{Re}\{\breve{\mathbf{v}}' \rho'\} \right) \mathrm{d}S \tag{1.47}$$

or

$$W = 2\mathrm{Re} \int_S \left(\breve{p}' \mathbf{v}' \cdot \mathbf{n} + \rho_o \mathbf{v}_o \cdot \breve{\mathbf{v}}' \mathbf{v}' \cdot \mathbf{n} + \frac{\mathbf{v}_o \cdot \mathbf{n}}{\rho_o} \breve{p}' \rho' + \mathbf{v}_o \cdot \mathbf{n} \mathbf{v}_o \cdot \breve{\mathbf{v}}' \rho' \right) \mathrm{d}S \tag{1.48}$$

[2] Recall that $\mathrm{Re}\{ab\} = \mathrm{Re}\{a\}\mathrm{Re}\{b\} - \mathrm{Im}\{a\}\mathrm{Im}\{b\}$.

where an inverted over-arc denotes the complex conjugate. Counterparts of this expression are often given in the literature in terms of the complex amplitudes of the acoustic variables. Equation (1.48) can be converted to complex amplitudes, by replacing $h \in p', \rho', \ldots$ by $\hat{h}/2$, where a circumflex denotes the complex amplitude (Section 1.2.2).

Three-dimensional acoustic fields in ducts consist of a linear combination of, theoretically, an infinite number of wave modes, and acoustic variables may be defined as vectors with components representing the contributions of these modes. For operations with the vector variables, it is convenient to write Equation (1.48) in a form that implies matrix operations:

$$W = 2\mathrm{Re}\int_S \left(\left(\breve{p}'\right)^{\mathrm{T}}\mathbf{v}'\cdot\mathbf{n} + \rho_o\mathbf{v}_o\cdot\left(\breve{\mathbf{v}}'\right)^{\mathrm{T}}\mathbf{v}'\cdot\mathbf{n} + \frac{\mathbf{v}_o\cdot\mathbf{n}}{\rho_o}\left(\breve{p}'\right)^{\mathrm{T}}\rho' + \mathbf{v}_o\cdot\mathbf{n}\mathbf{v}_o\cdot\left(\breve{\mathbf{v}}'\right)^{\mathrm{T}}\rho'\right)\mathrm{d}S, \tag{1.49}$$

where the superscript "T" denotes matrix transposition. Note that this form is not compulsory and in each of the four terms in the large brackets, either one of the two acoustic variables may be complex conjugated and transposed, provided that the transposed variable is moved to the leading position.

1.5 Introduction to the Linear System View of Duct Acoustics

Acoustic models based on the linear theory of acoustic wave motion inherit all generic mathematical properties of linear dynamic systems. In the time domain, a single-input-single-output linear system is associated with the mathematical operation $y(t) = \mathrm{L}x(t)$, where $x = x(t)$ and $y = y(t)$ are, respectively, the input (excitation) and output (response) of the system and L denotes a linear mathematical operator (involving differentiation and/or integration). If the input and output are vectors of variables, the system is called multi-input-multi-output system. In this case, the mathematical operation is written in matrix format as $\mathbf{y}(t) = \mathbf{L}\mathbf{x}(t)$, where $\mathbf{x} = \mathbf{x}(t)$ and $\mathbf{y} = \mathbf{y}(t)$ are, respectively, the input and output vectors of the system and $\mathbf{L}$ denotes a matrix of linear mathematical operations. Most properties of single-input-single-output systems can be extended to multi-input-multi-output systems by changing to matrix notation.

An arbitrary input can be represented by the expansion:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau. \tag{1.50}$$

Here, $\delta(t)$ denotes an impulse function of unit strength placed at time $t = 0$, which is defined formally by the property:

$$\int_{-\infty}^{\infty} g(t)\delta(t)\mathrm{d}t = g(0), \tag{1.51}$$

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where $g = g(t)$ denotes an arbitrary function. Therefore, the foregoing expansion actually models the input $x(t)$ as the sum of an infinite number of impulses having strengths equal to the value of the input function at the time of their action.

Let the response of a linear system to a unit impulse input at time $t = \tau$ be denoted by $h(t, \tau) = \mathrm{L}\delta(t - \tau)$. Then,

$$y(t) = \mathrm{L}\left\{ \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\mathrm{d}\tau \right\} = \int_{-\infty}^{\infty} x(\tau)h(t,\tau)\mathrm{d}\tau. \tag{1.52}$$

In words, the system output is given by the sum of the outputs of the system to its individual responses to the constituent pulses of the actual input. This is called superposition property of linear systems.

A linear system is called time-invariant if $h(t, \tau) = h(t - \tau)$, that is, if the response of the system to a unit impulse at time $t = \tau$ is same as its response to the unit impulse at $t = 0$ translated to time $t = \tau$. Then, Equation (1.52) reduces to the convolution

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\mathrm{d}\tau, \tag{1.53}$$

which is also known as Duhamel's integral. This relationship acquires a more practical form when transformed to frequency domain. Taking the Fourier transform of the both sides of Equation (1.53), we obtain:

$$y(\omega) = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} h(t-\tau)\mathrm{e}^{\mathrm{i}\omega t}\mathrm{d}t \right) x(\tau)\mathrm{d}\tau = h(\omega) \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\varpi\tau} x(\tau)\mathrm{d}\tau, \tag{1.54}$$

that is,

$$y(\omega) = h(\omega)x(\omega), \tag{1.55}$$

where $h(\omega)$ is called the frequency response function of the system. Since Equations (1.53) and (1.55) are Fourier transforms of each other, they carry the same information; however, the latter is more widely used, because $h(\omega)$ is relatively simpler to obtain than the unit impulse response $h(t)$. The term frequency response function is usually abbreviated as FRF, but it should be noted that this terminology is normally employed when the input and output of the system are scalar variables. For a multi-input-multi-input system, Equation (1.55) becomes a matrix equation of the form:

$$\mathbf{y}(\omega) = \mathbf{h}(\omega)\mathbf{x}(\omega), \tag{1.56}$$

where $\mathbf{h}(\omega)$ may be called the frequency response matrix. In this book, however, we will usually refer to it as a wave transfer matrix, since the input and output are associated with a wave motion.

Frequency response functions and matrices are extensively used in duct acoustics as descriptors of generation, propagation and radiation of duct-borne sound waves. They are derived from solutions of the governing acoustic equations in frequency domain without being concerned about the actual input. But if we are interested in the

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output of the system, for example, the sound pressure radiated from the exhaust tailpipe of an engine, the knowledge of the actual input gains importance. It is clear from Equation (1.55) that we can determine the Fourier transform of the output only if we know the Fourier transform of the input.

In some applications, the input may be known not as a time function or its Fourier transform, but as root-mean-square values in frequency bands which cover the frequency range of interest contiguously. Then we can only estimate the root-mean-square values of the output in the same bands. Taking the modulus of Equation (1.55):

$$|y(\omega)|^2 = |h(\omega)|^2 |x(\omega)|^2. \tag{1.57}$$

Therefore, from Equation (1.22), the power spectral density of the input and output signals are related as

$$S_y(\omega) = |h(\omega)|^2 S_x(\omega). \tag{1.58}$$

Thus,

$$\int_{-\infty}^{\infty} S_y(\omega)\mathrm{d}\omega = \int_{-\infty}^{\omega} |h(\omega)|^2 S_x(\omega)\mathrm{d}\omega. \tag{1.59}$$

Assuming real time signals and splitting the integral over the bands on the positive frequency axis,

$$\sum_{i=1}^{\infty} \int_{\omega_i}^{\omega_{i+1}} S_y(\omega)\mathrm{d}\omega = \sum_{i=1}^{\infty} \int_{\omega_i}^{\omega_{i+1}} |h(\omega)|^2 S_x(\omega)\mathrm{d}\omega, \tag{1.60}$$

where $\omega_i \leq \omega \leq \omega_{i+1}$ denotes the width of band i. Thus, the root-mean-square value of the output in a typical frequency band is given by

$$y_{rms}^2(\bar{\omega}_i) = \int_{\omega_i}^{\omega_{i+1}} |h(\omega)|^2 S_x(\omega)\mathrm{d}\omega, \tag{1.61}$$

where $\bar{\omega}_i$ denotes the center frequency of band i. If the magnitude of the frequency response function remains sufficiently uniform over a band, the root-mean-square values of the input and output signals may be related in constant or proportional frequency bands (Section 1.2.4) as:

$$y_{rms}(\bar{\omega}_i) \approx |h(\bar{\omega}_i)| x_{rms}(\bar{\omega}_i). \tag{1.62}$$

Obviously, the narrower the bandwidth, the more accurate this description will be.

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