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EXTENSION OF A TIGHT SET FUNCTION WITH VALUES IN A LOCALLY CONVEX SPACE

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1. **Introduction.** The purpose of the paper is to extend a tight set function on a lattice $\mathscr{L}$ with values in a locally convex space of special type to a measure on the σ-ring generated by $\mathscr{L}$. This result generalizes the extension theorem of Thomas [12, p. 151], which in turn contains the extension theorems of Pauc [9, p. 710], Fox [4, p. 525] and J. J. Uhl, Jr. [14, Corollary 2].

By X will be denoted a Hausdorff locally convex space over the real field with the dual space X'.

Let us recall that X possesses the $B-P$ property if, for every sequence $\{x_n\}$ in X such that $\sum_{n=1}^{\infty} |\langle x_n, x'\rangle| < \infty$ for all $x' \in X'$, there exists $x \in X$ with $x = \sum_{n=1}^{\infty} x_n$ [6, p. 176]. It is clear that if X is weakly sequentially complete (in particular, semi-reflexive), then X possesses the property $B-P$. This property was called weak Σ-completeness by Thomas, who proved that every strongly separable dual of a normed space possesses the property $B-P$ [12, pp. 140–141]. It has been shown by Bessaga and Pelczyński [1, p. 160] that a Banach space possesses the $B-P$ property if and only if it does not contain an isomorphic copy of c_0, the Banach space of null sequences of reals with supremum norm. This result was generalized by Tumarkin [13, Theorem 4] for sequentially complete locally convex spaces. The afore-mentioned Banach spaces played an important role in the extension theorem of Gould [5].

By $\mathscr{L}$ will be denoted a lattice of subsets of a fixed non-empty set T with $\phi \in \mathscr{L}$.

Recall that a set function λ on $\mathscr{L}$ to an Abelian group G is modular if $\lambda(\phi)=0$ and $\lambda(A \cup B)+\lambda(A \cap B)=\lambda(A)+\lambda(B)$ for all $A, B \in \mathscr{L}$. We will apply the following extension theorem of Pettis [10, p. 189]: Every modular set function $\lambda:\mathscr{L}\to G$ extends uniquely to an additive set function on the ring $\mathscr{R}(\mathscr{L})$ generated by $\mathscr{L}$.

Let $\lambda : \mathscr{L}\to X$. Let us recall that

(i) λ is continuous at ϕ if $L_n \downarrow \phi$, $L_n \in \mathscr{L}$ imply $\lambda(L_n)\to 0$.

(ii) λ is strongly bounded if $\lambda(L_n)\to 0$ for any disjoint sequence $\{L_n\}$ in $\mathscr{L}$.

(iii) λ is tight if, for every neighbourhood V of 0 and $A, B \in \mathscr{L}$ with $A \supseteq B$, there exists $K \in \mathscr{L}$ such that $K \subseteq A-B$ and $(\lambda(A)-\lambda(B))-\lambda(L) \in V$ whenever $K \subseteq L \subseteq A-B$, $L \in \mathscr{L}$.

It is clear that if λ is tight, then λ is modular.

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2. **Extension theorem.** The following lemma generalizes the Theorem 1.8 of Diestel [2, p. 216]:

2.1 LEMMA. *If X possesses the $B-P$ property and $\lambda:\mathscr{L}\to X$ is a bounded modular set function, then λ is strongly bounded.*

Proof. Let $\{L_n\}$ be a disjoint sequence in $\mathscr{L}$ and let $x'\in X'$. Since $\lambda(\mathscr{L})\subseteq X$ is bounded, $\lambda(\mathscr{L})$ is $\sigma(X, X')$-bounded. So $\sup_{L\in\mathscr{L}}|\langle\lambda(L), x'\rangle|=M<\infty$. For all $n=1, 2, 3, \ldots$ we have

$$\begin{aligned}\textstyle\sum_{i=1}^n|\langle\lambda(L_i), x'\rangle| &= \textstyle\sum_i^+\langle\lambda(L_i), x'\rangle-\sum_i^-\langle\lambda(L_i), x'\rangle\\ &= \langle\lambda(\textstyle\bigcup_i^+ L_i), x'\rangle-\langle\lambda(\textstyle\bigcup_i^- L_i), x'\rangle\end{aligned}$$

where $\sum_i^+$ and $\bigcup_i^+(\sum_i^-$ and $\bigcup_i^-)$ are taken over those i for which $\langle\lambda(L_i), x'\rangle\geq 0(\langle\lambda(L_i), x'\rangle<0)$.
Thus

$$\sum_{i=1}^n|\langle\lambda(L_i), x'\rangle|\leq 2M \qquad \text{for all } n.$$

So

$$\sum_{n=1}^\infty|\langle\lambda(L_n), x'\rangle|<\infty.$$

Since X possesses the $B-P$ property, there exists $x\in X$ such that $x=\sum_{n=1}^\infty\lambda(L_n)$, so $\lambda(L_n)\to 0$.

2.2 LEMMA. *If X possesses the $B-P$ property and $\lambda:\mathscr{L}\to X$ is a bounded, tight and continuous at ϕ set function, then the Pettis extension $\bar\lambda$ on $\mathscr{R}(\mathscr{L})$ is a σ-additive strongly bounded set function.*

Proof. This follows from Lemma 2.1 and Theorem 1 of Lipecki [7, p. 107].

2.3. THEOREM. *Let X be a Hausdorff locally convex space over the real field which possesses the $B-P$ property, let $\mathscr{L}$ be a lattice of subsets of a set T with $\phi\in\mathscr{L}$ and let $\sigma(\mathscr{L})$ be the σ-ring generated by $\mathscr{L}$. Every bounded, tight and continuous at ϕ set function $\lambda:\mathscr{L}\to X$ extends uniquely to a σ-additive set function on $\sigma(\mathscr{L})$.*

Proof. Let $\bar\lambda$ be the Pettis extension of λ on the ring $\mathscr{R}=\mathscr{R}(\mathscr{L})$ generated by $\mathscr{L}$. By Lemma 2.2, $\bar\lambda$ is σ-additive and strongly bounded. For every $x'\in X'$ the scalar measure on $\mathscr{R}:E\to\langle\bar\lambda(E), x'\rangle$ is strongly bounded, so bounded. Thus it has a unique extension to a real measure, denoted $\bar\lambda_{x'}$, on the σ-ring $\sigma(\mathscr{R})$ generated by $\mathscr{R}$. It is clear that $\sigma(\mathscr{R})=\sigma(\mathscr{L})$.

From the uniqueness of $\bar\lambda_{x'}$ it follows that, for every $E\in\sigma(\mathscr{L})$, the real function $x'\to\bar\lambda_{x'}(E)$ is linear in $x'\in X'$, hence it is an element of X'^*, the algebraic dual of X'; denote it by $\lambda'(E)$.

Since every element x of X can be identified as the linear form on $X':x(x')=\langle x, x'\rangle$, $x'\in X'$, we are able to define the following subset of $\sigma(\mathscr{L}):\mathscr{F}=\{E:\lambda'(E)\in X\}$.

We will show that $\mathscr{R} \subseteq \mathscr{F}$ and that $\mathscr{F}$ is closed under the formation of proper differences and countable disjoint unions. In fact, let $E \in \mathscr{R}$. For every $x' \in X'$ we have $\langle \lambda'(E), x' \rangle = \lambda'(E)(x') = \bar{\lambda}_{x'}(E) = \langle \bar{\lambda}(E), x' \rangle$. Since X is Hausdorff, we have $\lambda'(E) = \bar{\lambda}(E) \in X$, so $E \in \mathscr{F}$. Let $A, B \in \mathscr{F}$ be such that $A \supseteq B$. For every $x' \in X'$ we have $\lambda'(A-B)(x') = \bar{\lambda}_{x'}(A) - \bar{\lambda}_{x'}(B) = (\lambda'(A) - \lambda'(B))(x')$, so

$$\lambda'(A-B) = \lambda'(A) - \lambda'(B) \in X$$

and therefore $A - B \in \mathscr{F}$. Now let $\{A_n\}$ be a disjoint sequence in $\mathscr{F}$ and let $A = \bigcup_{n=1}^{\infty} A_n$. For every $x' \in X'$, $\bar{\lambda}_{x'}(\sigma(\mathscr{L}))$ is bounded, so strongly bounded. Then, by the Theorem 2.3 of Rickart [11, p. 655] the serie $\sum_{n=1}^{\infty} \bar{\lambda}_{x'}(A_n)$ is unconditionally convergent, and therefore $\sum_{n=1}^{\infty} |\langle \lambda'(A_n), x' \rangle| = \sum_{n=1}^{\infty} |\bar{\lambda}_{x'}(A_n)| < \infty$. Since X posseses the $B-P$ property, there exists $x \in X$ such that $x = \sum_{n=1}^{\infty} \lambda'(A_n)$. Hence $x(x') = \sum_{n=1}^{\infty} \lambda'(A_n)(x') = \lambda'(A)(x')$ for every $x' \in X'$, and therefore $A \in \mathscr{F}$.

Since $\sigma(\mathscr{R})$ is the smallest set of subsets of T containing $\mathscr{R}$ and closed under 1a formation de proper differences and countable disjoint unions [12, pp. 151–152], we have $\mathscr{F} = \sigma(\mathscr{L})$, so $\lambda' : \sigma(\mathscr{L}) \to X$. Let $\{A_n\}$ be a disjoint sequence in $\sigma(\mathscr{L})$ and let $x' \in X'$. Then

$$\left\langle \lambda'\left(\bigcup_{n=1}^{\infty} A_n\right), x' \right\rangle = \bar{\lambda}_{x'}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \bar{\lambda}_{x'}(A_n) = \sum_{n=1}^{\infty} \langle \lambda'(A_n), x' \rangle$$

so λ' is weakly σ-additive. By the Theorem 1 of Métivier [8, p. 2993], λ' is σ-additive. It is clear that λ' extends λ.

To prove the uniqueness, let $\lambda'' : \sigma(\mathscr{L}) \to X$ be a second σ-additive set function extending λ. Then $\lambda' | \mathscr{L} = \lambda'' | \mathscr{L}$. Since every set in $\mathscr{R} = \mathscr{R}(\mathscr{L})$ is a finite disjoint union of proper differences of sets of $\mathscr{L}$, we have that $\lambda' \mid \mathscr{R} = \lambda'' \mid \mathscr{R}$. By the uniqueness of the scalar measure extensions $x' \circ \lambda' = x' \circ \lambda''$ for all $x' \in X'$. Let $E \in \sigma(\mathscr{L})$ Then, for every $x' \in X'$, $\langle \lambda'(E), x' \rangle = \langle \lambda''(E), x' \rangle$. Since X is Hausforff we obtain $\lambda'(E) = \lambda''(E)$.

Corallary. ([12, p. 151]). *Let X be a Hausdorff locally convex space over the real field which possesses the $B-P$ property, let $\mathscr{R}$ be a ring of subsets of a set T and let $\sigma(\mathscr{R})$ the σ-ring generated by $\mathscr{R}$. Every bounded weakly σ-additive set function $\lambda : \mathscr{R} \to X$ extends uniquely to a σ-additive set function on $\sigma(\mathscr{R})$.*

Proof. Since $\mathscr{R}$ is a ring and λ is additive, λ is tight. To prove that λ is continuous at ϕ, let $A_n \downarrow \phi$, $A_n \in \mathscr{R}$. Then, for every $x' \in X'$, we have $\lim_{n\to\infty} \langle \lambda(A_n), x' \rangle = 0$, so $\langle \lim_{n\to\infty} \lambda(A_n), x' \rangle = 0$. Since X is Hausdorff, $\lim_{n\to\infty} \lambda(A_n) = 0$. Therefore by the Theorem 2.3, λ extends uniquely to a σ-additive set function on $\sigma(\mathscr{R})$.

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