R. Ware
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HASSE PRINCIPLES AND THE u-INVARIANT OVER FORMALLY REAL FIELDS

ROGER WARE

0. Introduction

In this paper we investigate the connection between the u-invariant, $u(F)$, of a formally real field F as defined by Elman and Lam [2] and certain Hasse Principles studied by Elman, Lam and Prestel in [3].

In section 2 the notion of an effective diagonalization of a quadratic form is introduced and in section 3 it is shown that if F is a field having at most a finite number of orderings such that every form over F has an effective diagonalization (which happens, for example, if F is any field having at most one ordering) then the finiteness of the u-invariant is equivalent to the Hasse Principle H_n holding for all n larger than some fixed integer m.

In section 4 we present two generalizations of a theorem of Kneser which states that if F is a non-formally real field then $u(F) \leq q$, where q denotes the number of distinct square classes of F. If F is a formally real field such that every form over F can be effectively diagonalized then it is shown that $u(F) \leq t$ where t is the number of distinct square classes of *totally positive* elements of F and H_n is satisfied for all $n > \dfrac{1}{2}q$.

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1. Notations and terminology

The terminology and notations will primarily follow [2,3,6]. All fields F will have characteristic different from two, $\dot{F}$ denotes the multiplicative group of $F, \dot{F}^2$ the subgroup of non-zero squares, and $\Sigma\dot{F}^2$ the

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subgroup consisting of all sums of squares (= totally positive elements). Isometries of quadratic forms over F will be written as $\cong$, $\phi \perp \psi$ and $\phi \otimes \psi$ will denote, respectively, the orthogonal sum and tensor product of two forms ϕ and ψ, and for any natural number m the form $\phi \perp \phi \perp \cdots \perp \phi$ (m times) will be denoted by $m\phi$. We will write $\phi = \langle a_1, a_2, \cdots, a_n\rangle$ to mean ϕ has an orthogonal basis $e_1, e_2, \cdots, e_n$ with $\phi(e_i) = a_i \in \dot{F}$. The *Witt ring* of non-singular quadratic forms over F will be denoted by $W(F)$ and its torsion subgroup by $W_t(F)$. The *u-invariant* of F is defined to be $u(F) = \max\{\dim \phi\}$ where ϕ ranges over all anisotropic forms in $W_t(F)$ [2].

If F is a formally real field then any ordering $<$ on F induces a ring homomorphism $\sigma_<: W(F) \to Z$ via $\sigma_<(\phi) = \sum_i \sigma_<(a_i)$, where $\phi = \langle a_1, \cdots, a_n\rangle$ and $\sigma_<(a_i) = 1$ if $0 < a_i$, $\sigma_<(a_i) = -1$ if $a_i < 0$. If ϕ is a form over F, $\sigma_<(\phi)$ is called *the signature of ϕ relative to the ordering $<$*. From [7, Satz 22] it follows that $W_t(F)$ consists precisely of those forms which have signature zero relative to all orderings on F. A form ϕ is called *totally indefinite* (or locally isotropic) over F if $|\sigma_<(\phi)| < \dim \phi$ for all orderings $<$ on F. Thus a form ϕ is totally indefinite if and only if ϕ is isotropic over all real closures $F_<$ of F as $<$ runs through the orderings of F. The formally real field F satisfies the *Hasse Principle* H_n (for some $n \geq 2$) if every totally indefinite form of dimension n over F is isotropic [3].

We denote by $X = X(F)$ the topological space of orderings on F [1,5]. The space X is compact, Hausdorff, and totally disconnected with a subbase of the topology given by the sets $W(a) = \{< \text{ in } X \mid a < 0\}$, $a \in \dot{F}$. We say F (or X) satisfies the *Strong Approximation Property* (SAP) if given any two disjoint closed subsets U, V of X there exists an element a in F which is positive at the orderings in U and negative at the orderings in V.

2. Effective diagonalization of quadratic forms

A form $\phi = \langle a_1, a_2, \cdots, a_n\rangle$ over a formally real field F is said to be *effectively diagonalized* if $W(a_i) \subset W(a_{i+1})$, $i = 1, 2, \cdots, n-1$. The field F is said to satisfy *ED* if every form over F can be effectively diagonalized.

LEMMA 2.1. *Suppose F is a formally real field and ϕ is a form which* *can be effectively diagonalized. Then*

(i) *If* ϕ *is totally indefinite then we can write* $\phi = \beta \perp \phi'$ *where* $\beta = \langle a, b\rangle$ *is a binary form with* a *totally positive and* b *totally negative,*

(ii) *If* ϕ *is totally indefinite then there exists an integer* $m \geq 1$ *such that* $m\phi$ *is isotropic (i.e.* ϕ *is weakly isotropic in the sense of* [3,8]).

(iii) *If* $\phi \in W_t(F)$ *then* $\phi = \beta_1 \perp \cdots \perp \beta_n$ *where* $\beta_i = \langle a_i, b_i\rangle \in W_t(F)$ *with* a_i *totally positive and* b_i *totally negative. In particular,* ϕ *is strongly balanced in the sense of* [7].

(iv) *If* $\phi \in W_t(F)$ *with* $\dim \phi = 2n$ *then* $\phi = \phi_1 \perp \phi_2$ *with* $\dim \phi_i = n$, $i = 1, 2$, *and where* ϕ_1 *has signature* n *and* ϕ_2 *has signature* $-n$ *relative to all orderings on* F.

Proof. (i) Write $\phi = \langle a_1, a_2, \cdots, a_k\rangle$ with $W(a_i) \subset W(a_{i+1})$ for all i. Since ϕ is totally indefinite $W(a_1)$ must be empty and $W(a_k) = X$. Thus a_1 is totally positive and a_k is totally negative so we can take $\beta = \langle a_1, a_k\rangle$.

(ii) Write $\phi = \beta \perp \phi'$ with $\beta = \langle a, b\rangle \in W_t(F)$. Choose $m \geq 1$ so that $m\beta = 0$ in $W(F)$. Then $m\phi$ is isotropic.

(iii) Write $\phi = \langle a_1, a_2, \cdots, a_k\rangle$ with $W(a_i) \subset W(a_{i+1})$ for all i. Since F is formally real and $\phi \in W_t(F)$ it follows that $k = 2n$ is even, $a_1, \cdots, a_n$ are totally positive and $a_{n+1}, \cdots, a_k$ are totally negative. Hence we can take $b_i = a_{n+i}$ for $i = 1, 2, \cdots, n$.

(iv) follows immediately from (iii).

COROLLARY 2.2. *If* F *is a formally real field satisfying ED then* F *satisfies SAP.*

Proof. This is a consequence of Lemma 2.1 (ii), [3, Th. C], and [8, Satz 3.1] (see also [9, Th. 3.1]).

EXAMPLES. (i) If F has a unique ordering then F satisfies ED.

(ii) Let $F = \boldsymbol{Q}((t))$ be the field of formal power series over $\boldsymbol{Q}$. As observed by Elman, Lam, and Prestel [3], the form $\langle t, -2t\rangle \in W_t(F)$ does not represent a totally negative element and consequently cannot be effectively diagonalized. Thus F does not satisfy ED. Since F has only two orderings, F does satisfy SAP. Thus SAP does not imply ED.

However, we do have the following

PROPOSITION 2.3. *A formally real field* F *satisfies SAP if and only* *if for any form* ϕ *over* F *there exists an effectively diagonalized form* $\psi = \langle b_1, b_2, \cdots, b_n \rangle$, $n = \dim \phi$, *such that* $\phi - \psi \in W_t(F)$.

Proof. ($\Rightarrow$) As in [9, Th. 3.1] we let $Y_k = \{< \text{ in } X \mid \sigma_<(\phi) = -n + 2k\}$, $k = 0, 1, \cdots, n$. Then the family $\{Y_k \mid k = 0, 1, \cdots, n\}$ is a partition of X and each Y_k is an open and closed subset of X. Since F satisfies SAP, there exist elements $b_1, b_2, \cdots, b_{n+1}$ in $\dot{F}$ such that $W(b_i) = Y_0 \cup Y_1 \cup \cdots \cup Y_{i-1}, i = 1, 2, \cdots, n+1$. Then $W(b_i) \subset W(b_{i+1})$ for all i and one readily checks that $\sigma_<(\langle b_1, b_2, \cdots, b_n \rangle) = \sigma_<(\phi)$ for all orderings $<$ in X. Hence $\phi - \langle b_1, b_2, \cdots, b_n \rangle$ lies in $W_t(F)$.
($\Leftarrow$) By [3, Th. C] and [8, Satz 3.1] it is enough to show that if ϕ is totally indefinite then there exists $m \geq 1$ such that $m\phi$ is isotropic. Let $\psi = \langle b_1, b_2, \cdots, b_n \rangle$, $n = \dim \phi$, be an effectively diagonalized form with $\phi - \psi \in W_t(F)$. Then there exists an integer $r \geq 1$ such that $r\phi \cong r\psi$. Since ϕ is totally indefinite, this implies ψ is also totally indefinite so by Lemma 2.1 (ii) there exists an integer $s \geq 1$ such that $s\psi$ is isotropic. Hence if $m = rs$ then $m\phi$ is isotropic.

THEOREM 2.4. *For a formally real field* F *the following statements are equivalent*:

(i) *F satisfies ED.*

(ii) *If* ϕ *is a form over* F *which represents* 1 *over all real closures of* F *then* ϕ *represents a totally positive element of* F.

Proof. (i) $\Rightarrow$ (ii). Write $\phi = \langle a_1, a_2, \cdots, a_n \rangle$ with $W(a_i) \subset W(a_{i+1})$. Since ϕ represents 1 over all real closures it follows that $W(a_1) = \phi$, i.e. a_1 is totally positive.
(ii) $\Rightarrow$ (i). We first show that any totally indefinite form over F is weakly isotropic and hence, in view of [3,8], F satisfies SAP. If ϕ is totally indefinite then ϕ represents 1 over all real closures and hence we can write $\phi = \langle a \rangle \perp \phi_1$ where a is totally positive element of F. But then ϕ_1 represents -1 over all real closures so ϕ_1 represent a totally negative element b in $\dot{F}$. Since $\langle a, b \rangle \in W_t(F)$ it follows that $\phi = \langle a, b \rangle \perp \psi$ is weakly isotropic.

Now let ψ be any form over F. Since F satisfies SAP there exists b in $\dot{F}$ such that $W(b) = \{< \in X \mid \sigma_<(\psi) = -\dim \psi\}$. If $W(b)$ is empty then ψ represents 1 over all real closures and hence represents a totally positive element. In this case the proof is finished by induction on $\dim \psi$. Hence we can assume that $W(b)$ is non empty. Now $W(b) \subset W(c)$ for all elements $c \neq 0$ represented by ψ and $\psi \perp \langle -b \rangle$ represents 1 over all real closures. Thus $\psi \perp \langle -b \rangle$ represents a totally positive element d. Since $-b$ is not totally positive we can write $d = a - bx^2$ where $a \neq 0$ is represented by ψ. Then $W(a) \subset W(b)$ so that $W(a) \subset W(c)$ for all c in $\dot{F}$ represented by ψ. Thus induction on $\dim \psi$ completes the proof.

COROLLARY 2.5. *If F is a formally real field satisfying some Hasse Principle H_n with $n \geq 4$ then F satisfies ED.*

Proof. Let ϕ be a form over F which represents 1 over all real closure of F. Then $\phi \perp n\langle -1 \rangle$ is totally indefinite whence isotropic. Thus there exists $x_1, \cdots, x_n$ in F such that ϕ represents the totally positive element $x_1^2 + \cdots + x_n^2 \in \dot{F}$.

COROLLARY 2.6 (cf. [1, Th. 5.3]). *For a formally real pythagorean field F the following statements are equivalent*:

(i) *F satisfies SAP.*

(ii) *F satisfies ED.*

(iii) *F satisfies H_n for all $n \geq 2$.*

Proof. The equivalence of (i) and (ii) is a consequence of Proposition 2.3 and the equivalence of (ii) and (iii) follows from Lemma 2.1 (i) and Corollary 2.5.

3. Hasse principles and the *u*-invariant

Any non-formally real field vacuously satisfies ED since $X = X(F)$ is empty but need not satisfy H_n for any n. In fact, for F non-formally real, F satisfies H_n for some $n \geq 2$ if and only if $u = u(F)$ is finite. For formally real fields we have

THEOREM 3.1. *Let F be a formally real field having at most a finite number of orderings. Then the following statements are equivalent*:

(i) *F satisfies H_n for some $n \geq 4$.*

(ii) *F satisfies ED and $u(F) < \infty$.*

Before proving Theorem 3.1 we introduce some terminology. A quadratic form ϕ over F will be called *totally positive* if every non zero element of F represented by ϕ is totally positive. Thus ϕ is totally positive if and only if $\phi = \langle a_1, \cdots, a_n \rangle$ with $a_i \in \Sigma \dot{F}^2, i = 1, \cdots, n$, if and only if $\sigma_<(\phi) = \dim \phi$ for all orderings $<$ of F. Denote by h the exponent of $W_t(F)$. h is called the *height of* F and (when finite) $h = 2^m$ where $m \geq 0$ is the smallest integer such that every totally positive element of F is a sum of 2^m squares in F [6, p. 311]. It follows immediately that if $u(F)$ is finite then h is finite and $h \leq u(F)$.

The proof of Theorem 3.1 will use the following lemma:

LEMMA 3.2. *Suppose* F *is a field with* $u = u(F) < \infty$. *If* ϕ *is a totally positive form over* F *with* $\dim \phi > 4^m(u+1)$ *for some* $m \geq 0$ *then there exists* a *in* $\Sigma \dot{F}^2$ *such that* $\phi = 2^{m+1}\langle a \rangle \perp \psi$.

Proof. We proceed by induction on m. If $m = 0$ then $\dim \phi > u + 1$ so there exists an integer n with $u + 1 \leq 2n \leq \dim \phi$. Write $\phi = \langle a_1, \cdots, a_n, b_1, \cdots, b_n \rangle \perp \phi'$. Then $\langle a_1, \cdots, a_n, -b_1, \cdots, -b_n \rangle \in W_t(F)$ and has dimension larger than u. Hence $\langle a_1, \cdots, a_n \rangle$ and $\langle b_1, \cdots, b_n \rangle$ represent a common element $a \in \Sigma \dot{F}^2$. Thus $\phi = 2\langle a \rangle \perp \psi$.

Now assume $m > 0$ and choose ϕ_1 of biggest dimension such that $\phi = 2\phi_1 \perp \phi_2$. Then the foregoing argument shows that $\dim \phi_2 \leq u + 1$. Hence $\dim \phi_1 > \frac{1}{2}(4^m - 1)(u + 1)$. But $m > 0$ implies that $\frac{1}{2}(4^m - 1) > 4^{m-1}$ so $\dim \phi_1 > 4^{m-1}(u + 1)$. Hence by the induction hypothesis there exists a in $\Sigma \dot{F}^2$ such that $\phi_1 = 2^m\langle a \rangle \perp \psi_1$. But then $\phi = 2^{m+1}\langle a \rangle \perp \psi$ where $\psi = 2\psi_1 \perp \phi_2$.

Proof of Theorem 3.1. (i) $\Rightarrow$ (ii). This follows from Corollary 2.5 and the fact that if H_n holds for some $n \geq 2$ then $u(F) < n$.

(ii) $\Rightarrow$ (i). Let $s < \infty$ be the number of orderings on F. Since $u = u(F)$ is finite the height h of F is also finite (with $h \leq u$) so we can write $h = 2^m$ for some integer $m \geq 0$. We now assert that if $n > (s+1)\left(\frac{h}{2}\right)^2 \cdot (u + 1)$ then H_n holds. To see this let ϕ be a totally indefinite form over F with $\dim \phi > (s+1)\left(\frac{h}{2}\right)^2(u+1)$. Since F satisfies ED we can find elements a_{ij} in F, $1 \leq i \leq k$, $1 \leq j \leq n_i$, such that for each i, $W(a_{i1}) = \cdots = W(a_{in_i})$, $W(a_{i1}) \subsetneqq W(a_{i+1,1})$, and $\phi = \phi_1 \perp \phi_2 \perp \cdots \perp \phi_k$ where $\phi_i = \langle a_{i1}, a_{i2}, \cdots, a_{in_i} \rangle$. Then by choosing orderings in $W(a_{i+1,1}) - W(a_{i1})$, $i = 1, 2, \cdots, k-1$ we see that $s \geq k - 1$. Hence $\dim \phi = n_1 + n_2 + \cdots + n_k > (s+1)\left(\frac{h}{2}\right)^2(u+1) \geq k\left(\frac{h}{2}\right)^2(u+1)$. Thus there must exist some i

with $n_i > \left(\frac{h}{2}\right)^2(u+1) = 4^{m-1}(u+1)$. Now $W(a_{i1}) = \cdots = W(a_{in_i})$ so the form $\langle a_{i1}\rangle\phi_i = \langle a_{i1}\rangle \otimes \phi_i$ is totally positive and hence by Lemma 3.2, $\langle a_{i1}\rangle\phi_i = 2^m\langle a\rangle \perp \psi$ for some a in $\Sigma\dot{F}^2$. Hence $\langle a_{i1}\rangle\phi = 2^m\langle a\rangle \perp \phi'$ for some subform ϕ'. Let $\phi' = \langle b_1, b_2, \cdots, b_r\rangle$ be an effective diagonalization of ϕ'. Then $\langle a_{i1}\rangle\phi = 2^m\langle a\rangle \perp \langle b_1, b_2, \cdots, b_r\rangle$ is an effective diagonalization. Since ϕ is totally indefinite so is $\langle a_{i1}\rangle\phi$ so b_r must be totally negative. But $h = 2^m$ implies that $2^m\langle a\rangle$ represents all totally positive elements of F. Thus $\langle a_{i1}\rangle\phi$ is isotropic whence ϕ is also isotropic.

Remark. For many fields the bound $\left(n > (s+1)\left(\frac{h}{2}\right)^2(u+1)\right)$ obtained in the proof of Theorem 3.1 is not very precise. In the case that $F = Q$, the proof shows that H_n holds for all $n > 40$ while it is well known that $n \geq 5$ suffices. Moreover, there exist fields having an infinite number of orderings (for example, the pythagorean closure of Q) which satisfy the equivalent conditions of the theorem.

COROLLARY 3.3. *Let F be a field having a unique ordering. Then $u(F) < \infty$ if and only if F satisfies H_n for some $n \geq 2$. In this case, F satisfies H_n for all $n > \frac{1}{2}h^2(u+1)$.*

Proof. A field having a unique ordering satisfies ED.

EXAMPLE. If $F = Q((t))$ then F has exactly two orderings and $u(F) = 8$ but as observed in [3], F fails to satisfy H_n for any $n \geq 2$.

4. Kneser's Theorem

In this section we present two more generalizations (cf. [2, Th. 2.4, Cor. 2.5, and Th. 3.1]) of Kneser's Theorem which states that if F is a non-formally real field and $q = |\dot{F}/\dot{F}^2|$ then $u(F) \leq q$. For this purpose we introduce the following notation. For a form ϕ over F, let $D(\phi) = \{a \in \dot{F}/\dot{F}^2 \mid a \text{ is represented by } \phi\}$.

LEMMA 4.1. *Let F be a field and ϕ a totally positive form over F. If $D(\phi) \neq \Sigma\dot{F}^2/\dot{F}^2$ then for any a in $\Sigma\dot{F}^2, D(\phi \perp \langle a\rangle) \neq D(\phi)$.*

Proof. If $D(\phi \perp \langle a\rangle) = D(\phi)$ then for any integer $n \geq 1, D(\phi \perp n\langle a\rangle) = D(\phi)$. Now if $b \in \Sigma\dot{F}^2$ then ab is a sum of k squares in F for some $k \geq 1$ which implies that b is represented by the form $k\langle a\rangle$. Hence

$b \in D(\phi \perp k\langle a\rangle) = D(\phi)$, contrary to assumption.

THEOREM 4.2. *If F is a formally real field satisfying ED then* $u(F) \leq |\Sigma\dot{F}^2/\dot{F}^2|$.

Proof. Let $t = |\Sigma\dot{F}^2/\dot{F}^2|$. It is enough to show that if $\phi \in W_t(F)$ with $\dim \phi \geq t + 2$ then ϕ is isotropic. Since F is formally real and satisfies ED we can write $\phi = \langle a_1, \cdots, a_m, b_1, \cdots, b_m\rangle$ where $a_i \in \Sigma\dot{F}^2$, $b_i \in -\Sigma\dot{F}^2$, $i = 1, \cdots, m$, and $m \geq \frac{t+2}{2}$. Then by Lemma 4.1, $|D(\langle a_1, \cdots, a_m\rangle)| > \frac{t}{2}$ and $|D(\langle -b_1, \cdots, -b_m\rangle)| > \frac{t}{2}$. Thus there exists $a \in D(\langle a_1, \cdots, a_m\rangle) \cap D(\langle -b_1, \cdots, -b_m\rangle)$. But then $-a \in D(\langle b_1, \cdots, b_m\rangle)$, whence ϕ is isotropic.

EXAMPLE. The hypothesis that F satisfies ED is needed here since if we let F_0 be a formally real field having square classes $\{\pm 1, \pm 2\}$ (such fields exist by [4, p. 302]) and let $F = F_0((t))$ then $u(F) = 4$ but $t = |\Sigma\dot{F}^2/\dot{F}^2| = 2$.

COROLLARY 4.3. *Let F be a formally real field satisfying ED. If $q = |\dot{F}/\dot{F}^2| < \infty$ then $u(F) \leq 2^{-s}q$ where s is the number of distinct orderings of F.*

Proof. Since F satisfies ED, F also satisfies SAP so it follows from (the proof of) Example 4.10 (iii) in [5] that $|\dot{F}/\Sigma\dot{F}^2| = 2^s$. Hence $q = |\dot{F}/\dot{F}^2| = |\dot{F}/\Sigma\dot{F}^2|\,|\Sigma\dot{F}^2/\dot{F}^2| = 2^s\,|\Sigma\dot{F}^2/\dot{F}^2|$.

THEOREM 4.4. *Let F be a formally real field which satisfies ED and suppose $q < \infty$. Write $q = 2^s t$ where $t = |\Sigma\dot{F}^2/\dot{F}^2|$ and s is the number of orderings on F. Then F satisfies H_n for all $n > s(t-1)+1$. In particular, H_n holds for all $n \geq \frac{q}{2} + 1$.*

Proof. Let ϕ be a totally indefinite form over F and write $\phi = \langle a_{11}, \cdots, a_{1n_1}, a_{21}, \cdots, a_{2n_2}, \cdots, a_{k1}, \cdots, a_{kn_k}\rangle$ where, for $i = 1, 2, \cdots, k$, $W(a_{i1}) = \cdots = W(a_{in_i})$ and $W(a_{i1}) \subsetneqq W(a_{i+1,1})$. Then $n_1 + n_2 + \cdots + n_k = \dim \phi$ and $k \leq s + 1$. If ϕ is anisotropic then by Lemma 4.1, $n_1 + n_k \leq t$ since otherwise $D(\langle a_{11}, \cdots, a_{1n_1}\rangle)$ and $D(\langle -a_{k1}, \cdots, -a_{kn_k}\rangle)$ would have an element in common. Moreover, by replacing ϕ by $\langle a_{i1}\rangle\phi$ and using effective diagonalization (as in the proof of Theorem 3.1) we see that

$n_i \leq t-1$ for $i = 2, \cdots, k-1$. Hence $\dim \phi = n_1 + n_2 + \cdots + n_k \leq t + (k-2)(t-1) \leq t + (s-1)(t-1) = s(t-1) + 1$. Thus if $\dim \phi > s(t-1)+1$ then ϕ is isotropic. For the last statement, note that $\frac{q}{2} + 1 = 2^{s-1}t + 1 > s(t-1) + 1$.

COROLLARY 4.5. *Let F be a field having a unique ordering. If $q < \infty$ then H_n holds for all $n > \frac{q}{2}$.*

COROLLARY 4.6. *Let F be a formally real field satisfying ED. If F has more than one ordering then H_n holds for all $n \geq \frac{q}{2}$.*

Proof. If $s \geq 2$ then $\frac{q}{2} = 2^{s-1}t > s(t-1) + 1$.

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Department of Mathematics
Pennsylvania State University