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Note on a Theorem in Continued Fractions.

By Prof. STEGGALL.

Note on the Fundamental Inequality Theorems connected with e^x and x^m.

By Prof. GEORGE A. GIBSON.

The subject of this note is that dealt with in Mr Tweedie's paper in the *Proceedings*, vol. XVII., 33–37, and my only reason for bringing it before the Society is to call attention to a slightly different method of presenting the same order of ideas. The method is that adopted by Peano, *Lezioni di Analisi Infinitesimale*, vol. I., §23, but as the book is not readily accessible to teachers, there may be some interest in having the method reproduced in our *Proceedings*. I add one or two remarks.

Peano starts, as Mr Tweedie does, from the generalised arithmetico-geometrical mean, namely, that if a, b, m, n be any positive quantities and a not equal to b,

$$a^m b^n < \left(\frac{ma+nb}{m+n}\right)^{m+n}.$$

His procedure is as follows:—Let $a = 1 + 1/m$, $b = 1$ and we get

$$\left(1+\frac{1}{m}\right)^m < \left(1+\frac{1}{m+n}\right)^{m+n} \quad . \quad . \quad . \quad (1)$$

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Next let $a=1$, $b=1-1/n$ ($n>1$, so that b may be positive) and we get

$$\left(1-\frac{1}{n}\right)^n < \left(1-\frac{1}{m+n}\right)^{m+n},$$

that is,
$$\left(1-\frac{1}{n}\right)^{-n} > \left(1-\frac{1}{m+n}\right)^{-(m+n)} \quad - \quad - \quad - \quad (2)$$

Then, let $a=1+1/m$, $b=1-1/n$ and we get

$$\left(1+\frac{1}{m}\right)^m\left(1-\frac{1}{n}\right)^n < 1,$$

that is,
$$\left(1+\frac{1}{m}\right)^m < \left(1-\frac{1}{n}\right)^{-n} \quad - \quad - \quad - \quad - \quad (3)$$

Equation (1) shows that the expression $(1+1/z)^z$ increases as z increases, provided z is positive, while equation (2) shows that when z is negative, the expression decreases as z increases in absolute value; equation (3) proves that for any positive value of z, the expression is less than for any negative value of z (z being numerically greater than unity).

Finally, to show that when z increases indefinitely in absolute value the expression $(1+1/z)^z$ has the same limit whether z be positive or negative, put $m+1$ for n in equation (3); then

$$\left(1-\frac{1}{n}\right)^{-n} = \left(\frac{m}{m+1}\right)^{-(m+1)} = \left(1+\frac{1}{m}\right)^m\left(1+\frac{1}{m}\right).$$

Hence
$$\underset{n=\infty}{\text{Limit}}\left(1-\frac{1}{n}\right)^{-n} = \underset{m=\infty}{\text{Limit}}\left(1+\frac{1}{m}\right)^m \times \underset{m=\infty}{\text{Limit}}\left(1+\frac{1}{m}\right)$$

$$= \underset{m=\infty}{\text{Limit}}\left(1+\frac{1}{m}\right)^m;$$

so that
$$\left(1+\frac{1}{m}\right)^m < e < \left(1+\frac{1}{m}\right)^{m+1}.$$

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It is obvious that Peano's method may at once be extended to the limits for e^x whether x be positive or negative, it being remembered that $1+x/m$, $1-x/n$ must always be positive. Thus we have

$$\left(1+\frac{x}{m}\right)^m < e^x < \left(1-\frac{x}{n}\right)^{-n} \qquad - \quad - \quad - \quad - \quad (4)^*$$

or
$$\left(1+\frac{x}{m}\right)^m < e^x < \left(1+\frac{x}{m}\right)^m \left(1+\frac{x}{m}\right)^x,$$

by putting $m+x$ for n.

The standard inequalities for $e^{\pm x}$, $\log(1 \pm x)$ follow at once.

Again if x be numerically less than unity, equation (4) gives, by putting $m=1=n$,

$$x < e^x - 1 < \frac{x}{1-x},$$

and ∴
$$\underset{x=0}{\text{Limit}} \; \frac{e^x-1}{x} = 1.$$

With reference to the inequalities

$$1 + mx^{m-1}(x-1) \lesseqgtr x^m \lesseqgtr 1 + m(x-1),$$

it is perhaps worth being more carefully emphasized that when any one of the four has been proved generally the other three follow by simple substitutions. One of the inequalities is given at once by the inequality

$$a^m b^n < \left(\frac{ma+nb}{m+n}\right)^{m+n}.$$

Put $b=1$, $n=1-m$, so that m is a positive proper fraction, and we get

$$a^m < 1 + m(a-1) \qquad - \quad - \quad - \quad - \quad (\alpha)$$

Write $1/a$ for a and we get, after multiplying by a^m

$$a^m > 1 + ma^{m-1}(a-1) \qquad - \quad - \quad - \quad (\beta)$$

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* The reason for writing e^x as the function of x which lies between $(1+x/m)^m$ and $(1-x/n)^{-n}$ is that

$$\left(1+\frac{x}{m}\right)^m = \left\{\left(1+\frac{1}{\text{M}}\right)^{\text{M}}\right\}^x,$$

where $\text{M}=mx$. The inequalities hold whether x be positive or negative, but m, n must be positive.

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Again in (a) for m write $1/m$ so that m is now greater than unity, and we get

$$a^{\frac{1}{m}} < 1 + \frac{a-1}{m}.$$

For a write now a^m and we get

$$a^m > 1 + m(a-1) \quad - \quad - \quad - \quad - \quad (\gamma)$$

For a write $1/a$ and we get

$$a^m < 1 + ma^{m-1}(a-1) \quad - \quad - \quad - \quad (\delta)$$

So far the index is necessarily positive; but if m be any negative number, $1/(1-m)$ will be a positive proper fraction.
In (a) write $1/(1-m)$ for m and a^{1-m} for a and we get

$$a^m < 1 + ma^{m-1}(a-1),$$

which is (δ). If we now write $1/a$ for a we get (γ).

In these inequalities the important variable is the index. If we write x in place of m and compare the graphs of a^x and $1 + x(a-1)$ we see (i) that the graphs intersect at $x = 0$ and $x = 1$; (ii) that by (a) the ordinate of a^x is less than the corresponding ordinate of the line $1 + x(a-1)$, so long as x lies between 0 and 1; (iii) that by (γ) the ordinate of a^x is greater than the corresponding ordinate of the line $1 + x(a-1)$ for x lying outside the range from 0 to 1. The geometrical representation may aid the memory in retaining the inequalities, as the graph of the exponential curve is about as familiar as that of the straight line.

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