Probability in the Engineering and Informational Sciences (2023), **37**:3 737–739
doi:10.1017/S0269964822000158

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CORRIGENDUM

Correction to "On the distribution of winners' scores in a round-robin tournament"

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The normalized scores $s_1^*, s_2^*, \ldots, s_n^*$ are exchangeable random variables for the fixed n, i.e., n-exchangeable or finite exchangeable. Their distribution depends on n, and their correlation is a function of n. Therefore, if they are a segment of the infinite sequence $s_1^*, s_2^*, \ldots$, then they are not exchangeable, i.e., not infinite exchangeable and also not stationary. Accordingly, Berman's theorem [2] (Theorem 2.1 in the article) and Theorems 4.5.2. and 5.3.4. from [5], which hold for stationary sequences, cannot be used in the proofs of Results 2.1. and 2.2. However, from Theorem 1 presented and proved below, for $P(X_{ij} = 1/2) = p \in [1/3, 1)$ or $p = 0$, Results 2.1. and 2.2. follows, and therefore, so do the follow-up Corollaries 2.1. and 2.2.

Let $I_j^{(n)} = I(s_j^* > x_n(t))$, where we choose $x_n(t) = a_n t + b_n$, in which a_n and b_n are as defined in equation (1) in the article:

$$a_n = (2\log n)^{-1/2}, \quad b_n = (2\log n)^{1/2} - \tfrac{1}{2}(2\log n)^{-1/2}(\log\log n + \log 4\pi).$$

Set $S_n = I_1^{(n)} + I_2^{(n)} + \cdots + I_n^{(n)}$.

We prove the following result.

Theorem 1. *For $p = 0$ or $p \in [1/3, 1)$ and a fixed value of k, $\lim_{n\to\infty} P(S_n = k) = e^{-\lambda(t)}(\lambda(t)^k/k!)$, $\lambda(t) = e^{-t}$.*

For $p = 0$ or $p \in [1/3, 1)$, Results 2.1. and 2.2. follow from Theorem 1, since $P(s_{(n-j)}^* \le x_n) = P(S_n \le j)$, and therefore

$$\lim_{n\to\infty} P(s_{(n-j)}^* \le x_n) = \lim_{n\to\infty} P(S_n \le j) = e^{-e^{-t}} \sum_{k=0}^{j} \frac{e^{-tk}}{k!}.$$

Remark 1. It remains an open problem if Theorem 1 holds also for $p \in (0, 1/3)$.

Proof. (Theorem 1) The result follows from Assertions presented below. Set

$$\pi_i^{(n)} = P(I_i^{(n)} = 1), \quad W_n = \sum_{i=1}^{n} I_i^{(n)}, \quad \lambda_n = E(W_n) = \sum_{i=1}^{n} \pi_i^{(n)}.$$

Assertion 1.

$$d_{\mathrm{TV}}(L(W_n), \mathrm{Poi}(\lambda_n)) \le \frac{1 - e^{\lambda_n}}{\lambda_n}(\lambda_n - \mathrm{Var}(W_n)) = \frac{1 - e^{\lambda_n}}{\lambda_n}\left(\sum_{i=1}^{n}(\pi_i^{(n)})^2 - \sum_{i\ne j}\mathrm{Cov}(I_i^{(n)}, I_j^{(n)})\right), \quad \text{(A1)}$$

where $d_{\mathrm{TV}}(L(W_n), \mathrm{Poi}(\lambda_n))$ is the total variation distance between distributions of W_n and Poisson distribution with mean λ_n.

Assertion 2.

$$\pi_1^{(n)} = P(s_1^* > x_n(t)) \sim 1 - \Phi(x_n(t)), \tag{A2}$$

where $c_n \sim k_n$ means $\lim_{n\to\infty} c_n/k_n = 1$.

Assertion 3.

$$\lim_{n\to\infty} n\pi_1^{(n)} = \lim_{n\to\infty} nP(s_1^* > x_n(t)) = \lambda(t) = e^{-t}. \tag{A3}$$

Assertion 4.

$$\lim_{n\to\infty} n^2(P(s_1^* > x_n(t), s_2^* > x_n(t))) = \lambda(t)^2 = e^{-2t}. \tag{A4}$$

In our case, since $s_1^*, \dots, s_n^*$ are identically distributed, $\sum_{i=1}^n (\pi_i^{(n)})^2 = nP(s_1^* > x_n)P(s_1^* > x_n)$, and $\sum_{i\neq j} \mathrm{Cov}(I_i^{(n)}, I_j^{(n)}) = n(n-1)[P(s_1^* > x_n(t), s_2^* > x_n(t)) - P(s_1^* > x_n(t))P(s_2^* > x_n(t))]$. Hence, from (A2) and (A3) it follows that

$$\lim_{n\to\infty} \sum_{i=1}^n (\pi_i^{(n)})^2 = 0. \tag{F1}$$

and from (A3) and (A4) it follows that

$$\lim_{n\to\infty} \sum_{i\neq j} \mathrm{Cov}(I_i^{(n)}, I_j^{(n)}) = 0. \tag{F2}$$

Then, from (F1) and (F2) it follows that $\lim_{n\to\infty} d_{\mathrm{TV}}(L(W_n), \mathrm{Poi}(\lambda_n)) = 0$, and this completes the proof of Theorem 1. □

Proof. (*Assertion 1*). If $p = P(X_{ij}) = 0$ then X_{ij} has Bernoulli distribution which is log-concave; or if $p \geq 1/3$ then $2X_{ij}$ has log-concave distribution (i.e., for integers $u \geq 1$, $(p(u))^2 \geq p(u-1)p(u+1)$). Proposition 1 and Corollary 2 [6] hold in our model with $p = 0$ or $p \in [1/3, 1)$, and therefore $\sum_{i=1, i\neq j}^n I_i^{(n)} | I_j^{(n)} = 1$ is stochastically smaller than $\sum_{i=1, i\neq j}^n I_i^{(n)}$. Therefore, from the Corollary 2.C.2 [1] we obtain (A1). □

Proof. (*Assertion 2*). Follows from [4, pp. 552–553, Thms. 2 or 3]. □

Proof. (*Assertion 3*). Follows from *Assertion 2* combined with [3] result on p. 374 of his book. □

Proof. (*Assertion 4*). Recall that $s_1 = X_{12} + X_{13} + \cdots + X_{1n}$ and $s_2 = X_{21} + X_{23} + \cdots + X_{2n}$. Hence, condition on the event $X_{12} = k, k \in \{0, 1/2, 1\}$, s_1 and s_2 are independent. Let $s_{1'} = X_{13} + \cdots + X_{1n}$, $s_{2'} = X_{23} + \cdots + X_{2n}$ and denote by $s_{1'}^*, s_{2'}^*$ the corresponding normalized scores (zero expectation and unit variance). We have,

$$\begin{aligned}
&P(s_1^* > x_n(t), s_2^* > x_n(t) \mid X_{12} = k) = P(s_1^* > x_n(t) \mid X_{12} = k)P(s_2^* > x_n(t) \mid X_{12} = k) \\
&= P\left(s_{1'}^* > x_{n-1}(t)\frac{x_n(t)}{x_{n-1}(t)}\sqrt{\frac{n-1}{n-2}} - \frac{\sqrt{2}(k-1/2)}{\sqrt{n-2}}\right) \\
&\quad \times P\left(s_{2'}^* > x_{n-1}(t)\frac{x_n(t)}{x_{n-1}(t)}\sqrt{\frac{n-1}{n-2}} - \frac{\sqrt{2}((1-k)-1/2)}{\sqrt{n-2}}\right) \\
&\sim P(s_{1'}^* > x_{n-1}(t))P(s_{2'}^* > x_{n-1}(t)).
\end{aligned} \tag{F3}$$

Combining (F3) with the formula of total probability we obtain

$$P(s_1^* > x_n(t), s_2^* > x_n(t)) \sim P(s_{1'}^* > x_{n-1}(t))P(s_{2'}^* > x_{n-1}(t)),$$

and combining it with *Assertion 3* we obtain (A4). □

https://doi.org/10.1017/S0269964822000158 Published online by Cambridge University Press

Acknowledgement. I am grateful to Pavel Chigansky for pointing out my mistake.

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Cite this article: Malinovsky Y (2023). Correction to "On the distribution of winners' scores in a round-robin tournament". *Probability in the Engineering and Informational Sciences* **37**, 737–739. https://doi.org/10.1017/S0269964822000158