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Abstract

Weitzenböck formulas are an important tool in relating local differential geometry to global topological properties by means of the so-called Bochner method. In this article we give a unified treatment of the construction of all possible Weitzenböck formulas for all irreducible, non-symmetric holonomy groups. We explicitly construct a basis of the space of Weitzenböck formulas. This classification allows us to find customized Weitzenböck formulas for applications such as eigenvalue estimates or Betti number estimates.

1. Introduction

Weitzenböck formulas are an important tool for linking differential geometry and topology of compact Riemannian manifolds. They feature prominently in the Bochner method, where they are used to prove the vanishing of Betti numbers under suitable curvature assumptions or the non-existence of metrics of positive scalar curvature on spin manifolds with non-vanishing $\hat{A}$-genus. Moreover, they are used to prove eigenvalue estimates for Laplace and Dirac type operators. In these applications one tries to find a (positive) linear combination of hermitean squares D^*D of first-order differential operators D, which sums to an expression in the curvature only. In this approach one needs only to consider special first-order differential operators D known as generalized gradients or Stein–Weiss operators, which are defined as projections of a covariant derivative ∇. Examples of generalized gradients include the exterior derivative d and its adjoint d^* and the Dirac and twistor operator in spin geometry.

In this article we present a classification of all possible linear combinations of hermitean squares D^*D of generalized gradients D, which sum to pure curvature expressions, if the underlying connection is the Levi-Civita connection ∇ of an irreducible non-symmetric Riemannian manifold M. We describe a recursive procedure to calculate a generating set of Weitzenböck formulas. Full details are given in the cases of reduced holonomy $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$. Our approach can also be used for the reduced holonomies $\mathbf{U}(n)$, $\mathbf{SU}(n)$, $\mathbf{Sp}(n)$ and $\mathbf{Sp}(n) \cdot \mathbf{Sp}(1)$. These cases slightly differ from the first three cases, since either the holonomy algebra is not simple or the complexified holonomy representation is not irreducible (or both). However, after suitable modifications again a complete classification of Weitzenböck formulas is possible.

In order to describe the setup of the article in more detail we recall that every representation $G \longrightarrow \operatorname{Aut} V$ of the holonomy group G of a Riemannian manifold (M, g) on a complex vector space V defines a complex vector bundle VM on M with a covariant derivative induced from the Levi-Civita connection, in particular the complexified holonomy representation T of G defines the complexified tangent bundle TM. The generalized gradients on VM are the parallel

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first-order differential operators T_ε defined as the projection of $\nabla : \Gamma(VM) \longrightarrow \Gamma(TM \otimes VM)$ to the parallel subbundles $V_\varepsilon M \subset TM \otimes VM$ arising from a decomposition $T \otimes V = \oplus_\varepsilon V_\varepsilon$ into irreducible sub-spaces. It will be convenient in this article to call every (finite) linear combination $\sum_\varepsilon c_\varepsilon T_\varepsilon^* T_\varepsilon$ of hermitean squares of generalized gradients a *Weitzenböck formula*.

Our first important observation is that the space $\mathfrak{W}(V)$ of all Weitzenböck formulas on a vector bundle VM can be identified with the vector space $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ and thus is an algebra, which is commutative for irreducible representations V. Moreover, it is easy to see that the algebra $\mathfrak{W}(V)$ has a canonical involution, the twist $\tau : \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$, such that a Weitzenböck formula reduces to a pure curvature expression if and only if it is an eigenvector of τ of eigenvalue -1. Of course, there are interesting Weitzenböck formulas, which are eigenvectors of τ for the eigenvalue $+1$; perhaps the most prominent example is the connection Laplacian $\nabla^*\nabla$. The classical examples of Weitzenböck formulas such as the original Weitzenböck formula,

$$\Delta = dd^* + d^*d = \nabla^*\nabla + q(R),$$

or the Lichnerowicz–Weitzenböck formula in spin geometry,

$$D^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4},$$

reduce in this setting to the statements that $\Delta - \nabla^*\nabla$ and $D^2 - \nabla^*\nabla$ respectively are eigenvectors of τ for the eigenvalue -1 and thus pure curvature expressions.

Starting with the connection Laplacian $\nabla^*\nabla$, corresponding to $\mathbf{1} \in \mathrm{End}_{\mathfrak{g}}(T \otimes V)$, we will describe a recursion procedure to construct a basis of the space $\mathfrak{W}(V_\lambda)$ of Weitzenböck formulas on an irreducible vector bundle $V_\lambda M$ on M such that the base vectors are eigenvectors of τ with alternating eigenvalues ± 1. Interestingly, this recursive procedure makes essential use of a second fundamental Weitzenböck formula $B \in \mathfrak{W}(V)$, the so-called *conformal weight operator*, which was considered for the first time in the work of Gauduchon on conformal geometry [Gau91].

Eventually we obtain a sequence of B-polynomials $p_i(B)$ such that $p_{2i}(B)$ is in the $(+1)$- and $p_{2i-1}(B)$ is in the (-1)-eigenspace of τ. If b_ε are the B-eigenvalues on $V_\varepsilon \subset T \otimes V$, then the coefficient of $T_\varepsilon^* T_\varepsilon$ in the Weitzenböck formula corresponding to $p_i(B)$ is given by $p_i(b_\varepsilon)$. An interesting feature appears for holonomy $\mathbf{G}_2$ and $\mathbf{Spin}(7)$. Here we have the decomposition $\mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}\, V) \cong \mathrm{Hom}_g(\mathfrak{g}, \mathrm{End}\, V) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^\perp, \mathrm{End}\, V)$ and because of the holonomy reduction any Weitzenböck formula in the second summand has a zero curvature term.

Finally, we would like to mention that the problem of finding all possible Weitzenböck formulas is also considered in the work of Homma (e.g. in [Hom06]). He gives a solution in the case of Riemannian, Kählerian and hyper-Kählerian manifolds. Even if there are some similarities in the results, it seems fair to say that our method is completely different. In particular, we describe an recursive procedure for obtaining the coefficients of Weitzenböck formulas. The main difference is of course that we give a unified approach including the case of exceptional holonomies.

2. The holonomy representation

For the rest of this article we will essentially restrict ourselves to irreducible *non-symmetric holonomy algebras* $\mathfrak{g}$, i.e. holonomy algebras of non-symmetric and irreducible Riemannian manifolds. Moreover, $\mathfrak{g}$ will always denote the *complex* Lie algebra obtained as the complexification of the real holonomy algebra $\mathfrak{g}_{\mathbb{R}}$. Most of the statements easily generalize to holonomy algebras $\mathfrak{g}$ with no symmetric irreducible factor in their local de Rham decomposition,

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which could be called properly non-symmetric holonomy algebras. Some of the concepts introduced are certainly interesting for symmetric holonomy algebras as well. Turning to irreducible non-symmetric holonomy algebras $\mathfrak{g}$ leaves us with seven different cases:

	Algebra $\mathfrak{g}_{\mathbb{R}}$	Holonomy representation $T_{\mathbb{R}}$	T
General Riemannian	$\mathfrak{so}_n$	Defining representation $\mathbb{R}^n$	T
Kähler	$\mathfrak{u}_n \cong i\mathbb{R} \oplus \mathfrak{su}_n$	Defining representation $\mathbb{C}^n$	$\bar{E} \oplus E$
Calabi–Yau	$\mathfrak{su}_n$	Defining representation $\mathbb{C}^n$	$\bar{E} \oplus E$
Quaternionic Kähler	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$	Representation $\mathbb{H}^1 \otimes_{\mathbb{H}} \mathbb{H}^n$	$H \otimes E$
Hyper-Kähler	$\mathfrak{sp}(n)$	Defining representation $\mathbb{H}^n$	$\mathbb{C}^2 \otimes E$
Exceptional $\mathbf{G}_2$	$\mathfrak{g}_2$	Standard representation $\mathbb{R}^7$	[7]
Exceptional $\mathbf{Spin}(7)$	$\mathfrak{spin}_7$	Spinor representation $\mathbb{R}^8$	[8]

(2.1)

as follows from a theorem of Berger, where T denotes the complexified holonomy representation $T := T_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ endowed with the $\mathbb{C}$-bilinear extension $\langle \cdot, \cdot \rangle$ of the scalar product. For simplicity, we will work with the complexified holonomy representation T and the complexified holonomy algebra $\mathfrak{g} := \mathfrak{g}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ throughout as well as with irreducible complex representations V_λ of $\mathfrak{g}$ of highest weight λ. The notation such as E or H in (2.1) fixes the nomenclature for particularly important representations in special holonomy: in the Kähler and Calabi–Yau cases E and $\bar{E}$ refer to the spaces of $(1, 0)$- and $(0, 1)$-vectors in T respectively, while [7] and [8] are the standard seven-dimensional representation of $\mathbf{G}_2$ and eight-dimensional spinor representation of $\mathbf{Spin}(7)$ respectively. In passing we note that the complexified holonomy representation T is not isotypical in the Kähler and Calabi–Yau cases and this is precisely the reason why these two cases differ significantly from the rest.

In order to understand Weitzenböck formulas or parallel second-order differential operators it is a good idea to start with parallel first-order differential operators, usually called generalized gradients or Stein–Weiss operators. Their representation theoretic background is the decomposition of tensor products $T \otimes V$ of the holonomy representation T with an arbitrary complex representation V. The general case immediately reduces to studying irreducible representations $V = V_\lambda$ of highest weight λ. In this section we will see that the isotypical components of $T \otimes V_\lambda$ are always irreducible for a properly non-symmetric holonomy algebra $\mathfrak{g}$ and isomorphic to irreducible representations $V_{\lambda+\varepsilon}$ of highest weight $\lambda + \varepsilon$ for some weight ε of the holonomy representation T. Thus the decomposition of $T \otimes V_\lambda$ is completely described by the subset of relevant weights ε.

Definition 2.1 (Relevant weights). A weight ε of the holonomy representation T is said to be relevant for an irreducible representation V_λ of highest weight λ if the irreducible representation $V_{\lambda+\varepsilon}$ of highest weight $\lambda + \varepsilon$ occurs in the tensor product $T \otimes V_\lambda$. We will write $\varepsilon \subset \lambda$ for a relevant weight ε for a given irreducible representation V_λ.

Lemma 2.2 (Characterization of relevant weights). *Consider the holonomy representation T of an irreducible non-symmetric holonomy algebra $\mathfrak{g}$ and an irreducible representation V_λ of highest weight λ. The decomposition of the tensor product $T \otimes V_\lambda$ is multiplicity free in the sense that all irreducible subspaces are pairwise non-isomorphic. The complete decomposition of $T \otimes V_\lambda$ is thus the sum*

$$T \otimes V_\lambda \cong \bigoplus_{\varepsilon \subset \lambda} V_{\lambda+\varepsilon}$$

over all relevant weights ε. A weight $\varepsilon \neq 0$ is relevant if and only if $\lambda + \varepsilon$ is dominant. The zero weight $\varepsilon = 0$ only occurs for the holonomy algebras $\mathfrak{so}_n$ with n odd and $\mathfrak{g}_2$. In the $\mathfrak{so}$-case it is relevant if $\lambda - \lambda_\Sigma$ is still dominant, where λ_Σ is the highest weight of the spinor representation of $\mathfrak{so}_n$. In the $\mathfrak{g}_2$-case it is relevant if $\lambda - \lambda_T$ is still relevant, where λ_T is the standard representation of $\mathfrak{g}_2$.

Proof. The proof is essentially an exercise in Weyl's character formula (cf. [Feg76]). □

A particular consequence of Lemma 2.2 is that for sufficiently complicated representations V_λ all weights ε of the holonomy representation T are relevant. Indeed, we will see below that this is the case if the entries of the highest weight vector λ are large enough (we give explicit lower bounds). With this motivation we will call a highest weight λ *generic* if $\lambda + \varepsilon$ is dominant for all weights ε of the holonomy representation T. A simple consideration shows that λ is generic if and only if $\lambda - \rho$ is dominant, where ρ is the half sum of positive roots or equivalently the sum of fundamental weights, unless we consider odd-dimensional generic holonomy $\mathfrak{g} = \mathfrak{so}_{2r+1}$ or $\mathfrak{g} = \mathfrak{g}_2$. In the latter holonomies the generic weights λ must have $\lambda - \rho - \lambda_\Sigma$ or $\lambda - \rho - \lambda_T$ dominant respectively. In any case, the number of relevant weights for the representation V_λ,

$$N(G, \lambda) := \sharp\{\varepsilon \mid \varepsilon \text{ is relevant for } \lambda\} \leqslant \dim T,$$

is bounded above by $\dim T$ with equality if and only if λ is generic. In particular, there are at most $\dim T$ summands in the decomposition of $T \otimes V_\lambda$ into irreducibles, with exactly one copy of $V_{\lambda+\varepsilon}$ for every relevant weight ε.

On the other hand, the number $N(G, \lambda)$ of irreducible summands in the decomposition of $T \otimes V_\lambda$ agrees with the dimension of the algebra $\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda)$ of $\mathfrak{g}$-invariant endomorphisms of $T \otimes V_\lambda$, because all isotypical components are irreducible by Lemma 2.2. In the next section we will study the identification $\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V_\lambda)$ extensively, which allows us to break up $\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda)$ into interesting subspaces called Weitzenböck classes, whose dimension can be calculated in the following way.

LEMMA 2.3 (Dimension of Weitzenböck classes). *Let us call the space $W^{\mathfrak{t}} := \mathrm{Hom}_{\mathfrak{t}}(\mathbb{R}, W) \subset W$ of elements of a G-representation W invariant under a fixed Cartan subalgebra $\mathfrak{t} \subset \mathfrak{g}$ the zero weight space of W. The dimension of the zero weight space provides an upper bound*

$$\dim \mathrm{Hom}_{\mathfrak{g}}(W, \mathrm{End}\, V_\lambda) \leqslant \dim W^{\mathfrak{t}}$$

for the dimension of the space $\mathrm{Hom}_{\mathfrak{g}}(W, \mathrm{End}\, V_\lambda)$ for an irreducible representation V_λ. For sufficiently dominant highest weights λ (depending on the weights of W) this upper bound is sharp.

The lemma follows again from the Weyl character formula, but it is also an elementary consequence of Kostant's Theorem 6.3 formulated below. We will mainly use Lemma 2.3 for the subspaces W_α occurring in the decomposition $T \otimes T = \oplus W_\alpha$ into irreducible summands. In the case of the holonomy algebras $\mathfrak{so}_n$, $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ we have the decomposition $T \otimes T = \mathbb{C} \oplus \mathrm{Sym}^2_0 T \oplus \mathfrak{g} \oplus \mathfrak{g}^\perp$ and the following dimensions of the zero weight spaces.

	$\dim T$	$\dim[\mathbb{C}]^{\mathfrak{t}}$	$\dim[\mathrm{Sym}^2_0 T]^{\mathfrak{t}}$	$\dim[\mathfrak{g}]^{\mathfrak{t}}$	$\dim[\mathfrak{g}^\perp]^{\mathfrak{t}}$
$\mathfrak{so}_n$	n	1	$\left\lfloor \frac{n-1}{2} \right\rfloor$	$\left\lfloor \frac{n}{2} \right\rfloor$	—
$\mathfrak{g}_2$	7	1	3	2	1
$\mathfrak{spin}_7$	8	1	3	3	1

(2.2)

Note, in particular, that the dimensions of the zero weight spaces sum up to $\dim T$.

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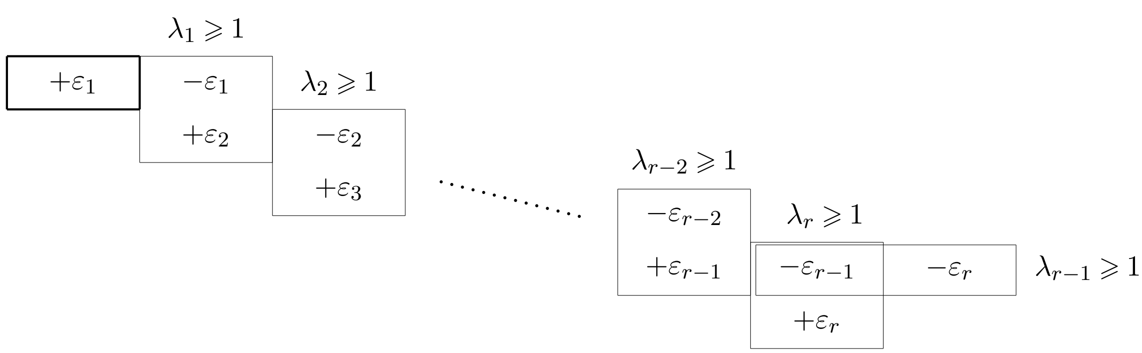
Although complete the decision criterion given in Lemma 2.2 is not particularly straightforward in general. At the end of this section we want to give a graphic interpretation of this decision criterion for all irreducible holonomy groups in order to simplify the task of finding the relevant weights. For a fixed holonomy algebra $\mathfrak{g}$ the information necessary in this graphic algorithm is encoded in a single diagram featuring the weights of the holonomy representation T and labeled boxes. A weight ε is relevant for an irreducible representation V_λ if and only if the highest weight $\lambda = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r$ of V_λ, written as a linear combination of fundamental weights $\omega_1, \ldots, \omega_r$, satisfies all inequalities labeling the boxes containing ε. The notation introduced for the weights of the holonomy representation T and the fundamental weights will be used throughout this article.

To begin with let us consider even-dimensional Riemannian geometry with generic holonomy $\mathfrak{g} = \mathfrak{so}_{2r}$, $r \geqslant 2$. In this case the holonomy representation T is the defining representation, with the weights $\pm\varepsilon_1, \pm\varepsilon_2, \ldots, \pm\varepsilon_r$, such that $\varepsilon_1, \ldots, \varepsilon_r$ forms an orthonormal basis for a suitable scalar product $\langle \cdot, \cdot \rangle$ on the dual $\mathfrak{t}^*$ of the maximal torus. The ordering of weights can be chosen in such a way that the fundamental weights $\omega_1, \ldots, \omega_r$ are given by

$$\begin{array}{llll}
\omega_1 & = \varepsilon_1 & \pm\varepsilon_1 & = \pm\,\omega_1 \\
\omega_2 & = \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = \pm(\omega_2 - \omega_1) \\
\vdots & \quad\vdots & \vdots & \quad\vdots \\
\omega_{r-2} & = \varepsilon_1 + \cdots + \varepsilon_{r-2} & \pm\varepsilon_{r-2} & = \pm(\omega_{r-2} - \omega_{r-3}) \\
\omega_{r-1} & = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r) & \pm\varepsilon_{r-1} & = \pm(\omega_{r-1} + \omega_r - \omega_{r-2}) \\
\omega_r & = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} - \varepsilon_r) & \pm\varepsilon_r & = \pm(\omega_{r-1} - \omega_r).
\end{array}$$

Every dominant integral weight of $\mathfrak{so}_{2r}$ can be written as $\lambda = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r$ with natural numbers $\lambda_1, \ldots, \lambda_r \geqslant 0$ and the criterion of Lemma 2.2 is then as follows.



A weight ε of the holonomy representation T of $\mathfrak{so}_{2r}$ is relevant for the irreducible representation V_λ if and only if λ satisfies all the conditions labeling the boxes containing ε. Say the weights $-\varepsilon_1$ and $+\varepsilon_2$ are relevant for all irreducible representations V_λ with $\lambda_1 \geqslant 1$, whereas $-\varepsilon_{r-1}$ is relevant for V_λ if and only if $\lambda_{r-1} \geqslant 1$ *and* $\lambda_r \geqslant 1$.

Odd-dimensional Riemannian geometry $\mathfrak{g} = \mathfrak{so}_{2r+1}$, $r \geqslant 1$, with generic holonomy is of course closely related to $\mathfrak{g} = \mathfrak{so}_{2r}$. The weights of the holonomy representation T are $\pm\varepsilon_1, \pm\varepsilon_2, \ldots, \pm\varepsilon_r$ and the zero weight. Again $\varepsilon_1, \ldots, \varepsilon_r$ form an orthonormal basis for a suitable scalar product $\langle \cdot, \cdot \rangle$ on the dual $\mathfrak{t}^*$ of the maximal torus. With a suitable choice of ordering of weights the

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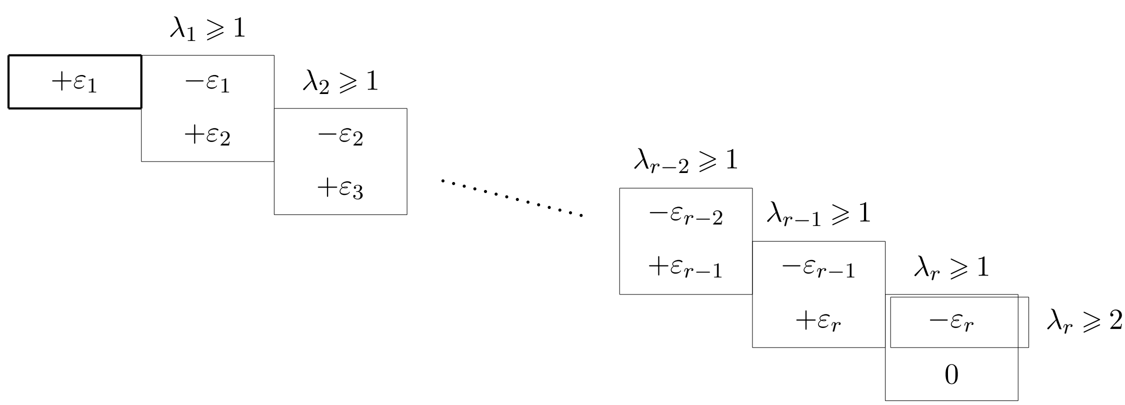
fundamental weights $\omega_1, \ldots, \omega_r$ and the weights of T relate via

$$\begin{array}{llll} \omega_1 & = \varepsilon_1 & \pm\varepsilon_1 & = \pm\,\omega_1 \\ \omega_2 & = \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = \pm(\omega_2 - \omega_1) \\ \vdots & \quad \vdots & \vdots & \quad \vdots \\ \omega_{r-1} & = \varepsilon_1 + \cdots + \varepsilon_{r-1} & \pm\varepsilon_{r-2} & = \pm(\omega_{r-1} - \omega_{r-2}) \\ \omega_r & = \frac{1}{2}(\varepsilon_1 + \cdots + \varepsilon_{r-1} + \varepsilon_r) & \pm\varepsilon_r & = \pm(2\omega_r - \omega_{r-1}). \end{array}$$

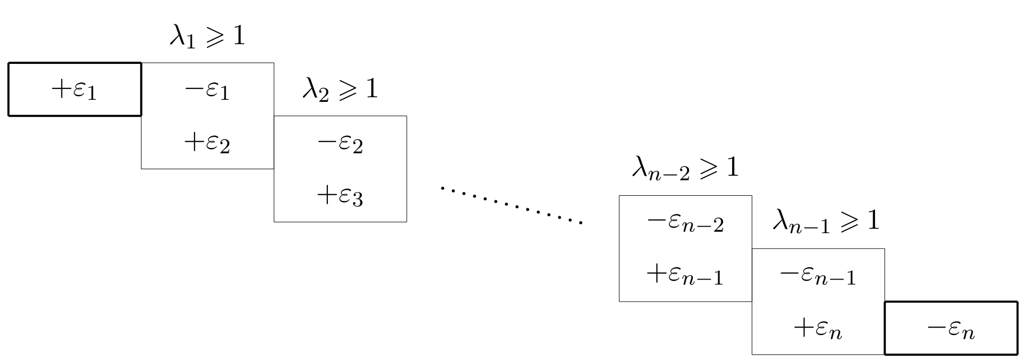
Writing a dominant integral weight $\lambda = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r$ as a linear combination of fundamental weights with integers $\lambda_1, \ldots, \lambda_r \geqslant 0$ the criterion of Lemma 2.2 is then as follows.



Turning from the Riemannian case to the Kähler case $\mathfrak{g} = \mathfrak{u}_n$, we have the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ of the defining standard representation $T = E \oplus \bar{E}$ and we observe that the weights $\varepsilon_1, \ldots, \varepsilon_n$ form an orthonormal basis for an invariant scalar product on the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t} \subset \mathfrak{u}_n$, but they become linearly dependent when projected to the dual of a maximal torus of the ideal $\mathfrak{su}_n \subset \mathfrak{u}_n$. In any case, the fundamental weights and the weights of T relate as

$$\begin{array}{llll} \omega_1 & = \varepsilon_1 & \pm\varepsilon_1 & = \pm\,\omega_1 \\ \omega_2 & = \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 & = \pm(\omega_2 - \omega_1) \\ \vdots & \quad \vdots & \vdots & \quad \vdots \\ \omega_n & = \varepsilon_1 + \cdots + \varepsilon_n & \pm\varepsilon_n & = \pm(\omega_n - \omega_{n-1}) \end{array}$$

and the criterion of Lemma 2.2 is then as follows.



The quaternionic Kähler case is more complicated, because the condition of being relevant has to be checked for both ideals $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$ of $\mathfrak{g}$. For a single ideal, however, the condition

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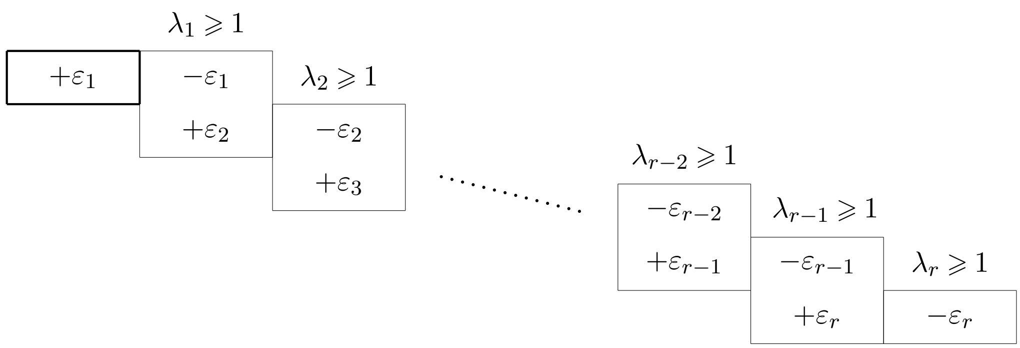
becomes simple again. We denote by $\pm\varepsilon_1, \ldots, \pm\varepsilon_n$ the weights of E and fix a suitable scalar product $\langle\cdot,\cdot\rangle$ on the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t} \subset \mathfrak{sp}(r)$ for $r = 1$ or $r = n$, such that $\varepsilon_1, \ldots, \varepsilon_r$ forms an orthonormal basis of $\mathfrak{t}^*$. The weights ε_i relate to the fundamental weights ω_j by the formulas

$$\begin{array}{ll} \omega_1 = \varepsilon_1 & \pm\varepsilon_1 = \pm\,\omega_1 \\ \omega_2 = \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 = \pm(\omega_2 - \omega_1) \\ \vdots \qquad \vdots & \vdots \qquad \vdots \\ \omega_r = \varepsilon_1 + \cdots + \varepsilon_r & \pm\varepsilon_r = \pm(\omega_r - \omega_{r-1}). \end{array}$$

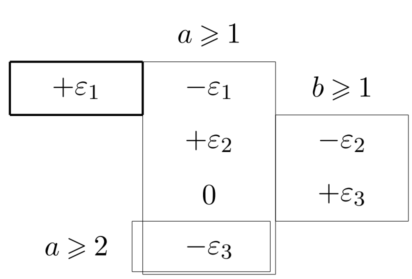
The graphical interpretation of Lemma 2.2 in the case $\mathfrak{g} = \mathfrak{sp}(n)$ is given by the following diagram.



Finally, we consider the two exceptional cases $\mathfrak{g}_2$ and $\mathfrak{spin}_7$. Recall that the group $\mathbf{G}_2$ is the group of automorphisms of the octonions $\mathbb{O}$ as an algebra over $\mathbb{R}$. In this sense the holonomy representation $T_{\mathbb{R}}$ is the defining representation $\operatorname{Im}\mathbb{O}$ of $\mathfrak{g}_2$ with complexification $T = [7]$. There are too many weights of the holonomy representation to be orthonormal for any scalar product on the dual $\mathfrak{t}^*$ of a fixed maximal torus $\mathfrak{t} \subset \mathfrak{g}_2$, but at least we can choose an ordering of weights for $\mathfrak{t}^*$ so that the weights of T become totally ordered $+\varepsilon_1 > +\varepsilon_2 > +\varepsilon_3 > 0 > -\varepsilon_3 > -\varepsilon_2 > -\varepsilon_1$. In this notation we have

$$\begin{array}{ll} \omega_1 = \varepsilon_1 & \pm\varepsilon_1 = \pm\,\omega_1 \\ \omega_2 = \varepsilon_1 + \varepsilon_2 & \pm\varepsilon_2 = \pm(\omega_2 - \omega_1) \\ & \pm\varepsilon_3 = \mp(\omega_2 - 2\omega_1). \end{array}$$

The scalar product of choice on $\mathfrak{t}^*$ is specified by $\langle\varepsilon_1,\varepsilon_1\rangle = 1 = \langle\varepsilon_2,\varepsilon_2\rangle$ and $\langle\varepsilon_1,\varepsilon_2\rangle = \frac{1}{2}$. Writing a dominant integral weight as $\lambda = a\omega_1 + b\omega_2$, $a, b \geqslant 0$, we read Lemma 2.2 as follows.



The holonomy representation of the holonomy algebra $\mathfrak{g} = \mathfrak{spin}_7$ is the eight-dimensional spinor representation $T = [8]$. It is convenient to write the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_4$ of T and the fundamental weights ω_1, ω_2 and ω_3 in terms of the weights $\pm\eta_1, \pm\eta_2, \pm\eta_3, 0$ of the seven-dimensional defining representation of $\mathfrak{spin}_7$, which form an orthonormal basis for a suitable

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scalar product on the dual $\mathfrak{t}^*$ of the maximal torus. With this proviso the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_4$ of T and the fundamental weights ω_1, ω_2 and ω_3 can be written as

$$\begin{aligned}
\omega_1 &= \eta_1 & \pm\varepsilon_1 &= \pm\tfrac{1}{2}(\eta_1+\eta_2+\eta_3) = \pm\,\omega_3\\
\omega_2 &= \eta_1+\eta_2 & \pm\varepsilon_2 &= \pm\tfrac{1}{2}(\eta_1+\eta_2-\eta_3) = \pm(\omega_2-\omega_3)\\
\omega_3 &= \tfrac{1}{2}(\eta_1+\eta_2+\eta_3) & \pm\varepsilon_3 &= \pm\tfrac{1}{2}(\eta_1-\eta_2+\eta_3) = \pm(\omega_3-\omega_2+\omega_1)\\
& & \pm\varepsilon_4 &= \pm\tfrac{1}{2}(\eta_1-\eta_2-\eta_3) = \pm(\omega_1-\omega_3),
\end{aligned}$$

and Lemma 2.2 for a dominant integral weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ translates into the following diagram.

		$c \geqslant 1$	
	$\boxed{+\varepsilon_1}$	$-\varepsilon_1$	$b \geqslant 1$
		$+\varepsilon_2$	$-\varepsilon_2$
$a \geqslant 1$	$-\varepsilon_4$	$-\varepsilon_3$	$+\varepsilon_3$
		$+\varepsilon_4$	

3. The space $\mathfrak{W}(V)$ of Weitzenböck formulas

In this section we will introduce generalized gradients, Weitzenböck formulas and the space of Weitzenböck formulas with its different realizations. Then we will define the conformal weight operator, which in many cases generates all possible Weitzenböck formulas. Finally, we define the classifying endomorphism and study the corresponding eigenspace decomposition.

3.1 Weitzenböck formulas

We consider parallel second-order differential operators P on sections of a vector bundle VM over a Riemannian manifold M with holonomy group $G \subset \mathbf{SO}(n)$. By definition, these are differential operators, which up to first-order differential operators can always be written as the composition

$$\Gamma(VM) \xrightarrow{\nabla^2} \Gamma(T^*M \otimes T^*M \otimes VM) \xrightarrow{\cong} \Gamma(TM \otimes TM \otimes VM) \xrightarrow{F} \Gamma(VM)$$

where F is a parallel section of the vector bundle $\mathrm{Hom}(TM \otimes TM \otimes VM, VM)$ corresponding to a G-equivariant homomorphism $F \in \mathrm{Hom}_G(T \otimes T \otimes V, V)$. A particularly important example is the connection Laplacian $\nabla^*\nabla$ which arises from the linear map $a \otimes b \otimes \psi \longmapsto -\langle a, b\rangle\psi$. Note that we are only considering reduced holonomy groups G, which are connected by definition, so that G-equivariance is equivalent to $\mathfrak{g}$-equivariance. Taking advantage of this fact we describe other parallel differential operators by means of the following identifications of spaces of invariant homomorphisms:

$$\mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V).$$

Of course, the identification $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ is the usual tensor shuffling $F(a \otimes b \otimes v) = F_{a\otimes b}v$ for all $a, b \in T$ and $v \in V$. The second important identification $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V)$ depends on the existence of a $\mathfrak{g}$-invariant scalar product on T or the musical isomorphism $T \cong T^*$ via a summation

$$F(b \otimes v) = \sum_\mu t_\mu \otimes F(t_\mu \otimes b \otimes v) \qquad F(a \otimes b \otimes v) = (\langle a, \cdot\rangle \lrcorner \otimes \mathrm{id})F(b \otimes v)$$

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over an orthonormal basis $\{t_\mu\}$. Under this identification the identity of $T \otimes V$ becomes the homomorphism $a \otimes b \otimes \psi \longmapsto \langle a, b\rangle \psi$ corresponding to the connection Laplacian $-\nabla^*\nabla$. The composition of endomorphisms turns $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ and thus $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ into an algebra; for $F, \tilde{F} \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ the resulting algebra structure reads

$$(F \circ \tilde{F})_{a\otimes b} = \sum_\mu F_{a\otimes t_\mu} \circ \tilde{F}_{t_\mu \otimes b}. \tag{3.3}$$

Last but not least we note that the invariance condition for $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ is equivalent to the identity $[X, F_{a\otimes b}] = F_{Xa\otimes b} + F_{a\otimes Xb}$ for all $X \in \mathfrak{g}$ and $a, b \in T$.

Assuming that $V = V_\lambda$ is irreducible of highest weight λ we know from Lemma 2.2 that the isotypical components of $T \otimes V_\lambda$ are irreducible for non-symmetric holonomy groups. The algebra $\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda)$ is thus commutative and spanned by the pairwise orthogonal idempotents pr_ε projecting onto the irreducible subspaces $V_{\lambda+\varepsilon}$ of $T \otimes V_\lambda$. In order to describe the corresponding second-order differential operators we introduce first-order differential operators T_ε known as Stein–Weiss operators or generalized gradients by

$$T_\varepsilon :\ \Gamma(V_\lambda M) \longrightarrow \Gamma(V_{\lambda+\varepsilon} M), \quad \psi \longmapsto \mathrm{pr}_\varepsilon(\nabla\psi).$$

A typical example of a Stein–Weiss operator is the twistor operator of spin geometry, which projects the covariant derivative of a spinor onto the kernel of the Clifford multiplication. Straightforward calculations show that the second-order differential operator associated to the idempotent pr_ε is the composition of T_ε with its formal adjoint operator $T_\varepsilon^* : \Gamma(V_{\lambda+\varepsilon} M) \longrightarrow \Gamma(V_\lambda M)$, i.e. $\mathrm{pr}_\varepsilon(\nabla^2) = -T_\varepsilon^* T_\varepsilon$; cf. [Sem06]. In consequence, we can write the second-order differential operator $F(\nabla^2)$ associated to $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, V_\lambda)$ as a linear combination of the squares of Stein–Weiss operators:

$$F(\nabla^2) = -\sum_\varepsilon f_\varepsilon T_\varepsilon^* T_\varepsilon. \tag{3.4}$$

In fact, with $\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V_\lambda)$ being spanned by the idempotents pr_ε, every $F \in \mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda)$ expands as $F = \sum_\varepsilon f_\varepsilon\, \mathrm{pr}_\varepsilon$ with coefficients f_ε determined by $F|_{V_{\lambda+\varepsilon}} = f_\varepsilon\, \mathrm{id}$. A particular instance of (3.4) is the identity $\nabla^*\nabla = \sum_\varepsilon T_\varepsilon^* T_\varepsilon$ associated to the expansion $\mathrm{id}_{T\otimes V_\lambda} = \sum_\varepsilon \mathrm{pr}_\varepsilon$. Motivated by this and other well-known identities of second-order differential operators of the form (3.4), we will in general call all elements $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V)$ Weitzenböck formulas.

DEFINITION 3.1 (Space of Weitzenböck formulas on VM). The Weitzenböck formulas on a vector bundle VM correspond bijectively to vectors in

$$\mathfrak{W}(V) := \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V).$$

Of course, we are mainly interested in Weitzenböck formulas inducing differential operators of zeroth order or equivalently 'pure curvature terms'. Clearly a Weitzenböck formula given by an invariant homomorphism $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V)$ skew-symmetric in its two T-arguments will induce a pure curvature term $F(\nabla^2)$, because we can then reshuffle the summation in the calculation:

$$F(\nabla^2 v) = \frac{1}{2}\sum_{\mu\nu} F(t_\mu \otimes t_\nu \otimes (\nabla^2_{t_\mu, t_\nu} - \nabla^2_{t_\nu, t_\mu})v) = \frac{1}{2}\sum_{\mu\nu} F_{t_\mu\otimes t_\nu} R_{t_\mu, t_\nu} v. \tag{3.5}$$

Here and in the following we will denote by $\{t_\nu\}$ an orthonormal basis of T and also a local orthonormal basis of the tangent bundle. Conversely the principal symbol of the differential operator $F(\nabla^2)$ is easily computed to be $\sigma_{F(\nabla^2)}(\xi)v = F_{\xi^\flat\otimes\xi^\flat}v$ for every cotangent vector ξ and every $v \in VM$. Hence the principal symbol vanishes identically exactly for the skew-symmetric Weitzenböck formulas. Weitzenböck formulas F leading to a pure curvature term $F(\nabla^2)$ are thus completely characterized by being eigenvectors of eigenvalue -1 for the involution

$$\tau :\ \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V), \quad F \longmapsto \tau(F)$$

defined in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V, V)$ as precomposition with the twist $\tau :\ T \otimes T \otimes V \longrightarrow T \otimes T \otimes V, a \otimes b \otimes v \longmapsto b \otimes a \otimes v$, which reads in the description $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ as $\tau(F)_{a\otimes b} = F_{b\otimes a}$. In other words, a Weitzenböck formula F will reduce to a pure curvature term if and only if $\tau(F) := F \circ \tau = -F$.

Considering the space of Weitzenböck formulas $\mathfrak{W}(V)$ as the algebra $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ we can introduce additional structures on it: the unit $\mathbf{1} := \mathrm{id}_{T\otimes V} \in \mathfrak{W}(V)$, the scalar product

$$\langle F, \tilde{F} \rangle := \frac{1}{\dim V}\, \mathrm{tr}_{T\otimes V}(F\tilde{F}) \quad F, \tilde{F} \in \mathfrak{W}(V)$$

satisfying $\langle FG, \tilde{F} \rangle = \langle F, G\tilde{F} \rangle$ and the trace $\mathrm{tr}\, F := \langle F, \mathbf{1} \rangle$. Clearly the trace of the unit is given by $\mathrm{tr}\, \mathbf{1} = \dim T$. The definition of the trace can be rewritten in the form

$$\mathrm{tr}\, F = \frac{1}{\dim V}\, \mathrm{tr}_V \Big(v \longmapsto \sum_\mu F_{t_\mu\otimes t_\mu} v \Big)$$

so that the trace is invariant under the twist τ. A slightly more complicated argument using (3.3) shows that the scalar product is invariant under the twist, too. In particular, the eigenspaces for τ for the eigenvalues ± 1 are orthogonal and all eigenvectors in the (-1)-eigenspace of τ have vanishing trace. From the definition of the trace we obtain that the trace of an element $F = \sum f_\varepsilon\, \mathrm{pr}_\varepsilon$ of $\mathfrak{W}(V_\lambda)$ in the irreducible case is given by

$$\mathrm{tr}\, F = \sum_\varepsilon f_\varepsilon \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}; \tag{3.6}$$

in particular, the idempotents pr_ε form an orthogonal basis of $\mathfrak{W}(V_\lambda)$:

$$\langle \mathrm{pr}_\varepsilon, \mathrm{pr}_{\tilde{\varepsilon}} \rangle = \delta_{\varepsilon\tilde{\varepsilon}} \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}.$$

A different way to interpret the trace is to note that for every Weitzenböck formula $F \in \mathfrak{W}(V)$ considered as an equivariant homomorphism $F :\ T \otimes T \longrightarrow \mathrm{End} V$ the trace endomorphism $\sum_\mu F_{t_\mu\otimes t_\mu} \in \mathrm{End}_{\mathfrak{g}} V$ is invariant under the action of $\mathfrak{g}$. For an irreducible representation V_λ it is thus by Schur's lemma a multiple of the identity id_{V_λ} and the definition above can be rewritten as $\sum_\mu F_{t_\mu\otimes t_\mu} = (\mathrm{tr}\, F)\, \mathrm{id}_{V_\lambda}$.

3.2 The conformal weight operator

In order to study the fine structure of the algebra $\mathfrak{W}(V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V)$ of Weitzenböck formulas it is convenient to introduce the *conformal weight operator* $B \in \mathfrak{W}(V)$ of the holonomy algebra $\mathfrak{g}$ and its variations $B^{\mathfrak{h}} \in \mathfrak{W}(V)$ associated to the non-trivial ideals $\mathfrak{h} \subset \mathfrak{g}$ of $\mathfrak{g}$. All these conformal weight operators commute and the commutative subalgebra of $\mathfrak{W}(V)$ generated by them in the irreducible case $V = V_\lambda$ is actually all of $\mathfrak{W}(V)$ for generic highest weight λ. In this subsection we work out some direct consequences of the description of Weitzenböck formulas as polynomials in the conformal weight operators.

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Recall that the scalar product $\langle\cdot,\cdot\rangle$ on T induces a scalar product on all exterior powers $\Lambda^k T$ of T via Gram's determinant. Using this scalar product on $\Lambda^2 T$ we can identify the adjoint representation $\mathfrak{so}\,T$ of $\mathbf{SO}(n)$ with $\Lambda^2 T$ through $\langle X, a\wedge b\rangle = \langle Xa, b\rangle$ and hence think of the holonomy algebra $\mathfrak{g}\subset\mathfrak{so}\,T$ as a subspace of the euclidean vector space $\Lambda^2 T$.

Definition 3.2 (Conformal weight operator). Consider an ideal $\mathfrak{h}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$ in the real holonomy algebra. Its complexification $\mathfrak{h} := \mathfrak{h}_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$ is an ideal in $\mathfrak{g}$ and a regular subspace $\mathfrak{h}\subset\mathfrak{g}\subset\Lambda^2 T$ in $\Lambda^2 T$ with associated orthogonal projection $\mathrm{pr}_{\mathfrak{h}}:\ \Lambda^2 T\longrightarrow\mathfrak{h}$. The conformal weight operator $B^{\mathfrak{h}}\in\mathfrak{W}(V)$ is defined by

$$B^{\mathfrak{h}}_{a\otimes b}v := \mathrm{pr}_{\mathfrak{h}}(a\wedge b)v$$

in the interpretation of Weitzenböck formulas as linear maps $T\otimes T\longrightarrow\mathrm{End}\,V$. Under the identification $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T,\mathrm{End}\,V) = \mathrm{End}_{\mathfrak{g}}(T\otimes V)$ the conformal weight operator $B^{\mathfrak{h}}$ becomes the following sum over an orthonormal basis $\{t_\mu\}$ of T:

$$B^{\mathfrak{h}}(b\otimes v) = \sum_\mu t_\mu\otimes\mathrm{pr}_{\mathfrak{h}}(t_\mu\wedge b)v.$$

The notation $B := B^{\mathfrak{g}}$ will be used for the conformal weight operator of the algebra $\mathfrak{g}$.

Most of the irreducible non-symmetric holonomy algebras $\mathfrak{g}$ are simple and hence there is only one ideal $\mathfrak{h}=\mathfrak{g}$ and only one weight operator $B = B^{\mathfrak{g}}$ (cf. Table (2.1)). The exceptions are the Kähler geometry $\mathfrak{g}_{\mathbb{R}} = i\mathbb{R}\oplus\mathfrak{su}_n$ with a one-dimensional center in dimension $2n$ and two commuting weight operators $B^{i\mathbb{R}}$ and $B^{\mathfrak{su}}$, and the quaternionic Kähler geometry $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ in dimension $4n$, $n\geqslant 2$ with two commuting weight operators B^H and B^E.

Lemma 3.3 (Fegan's lemma [Feg76]). *The conformal weight operator $B^{\mathfrak{h}}\in\mathfrak{W}(V)$ of an ideal $\mathfrak{h}\subset\mathfrak{g}\subset\Lambda^2 T$ can be written as*

$$B^{\mathfrak{h}} = -\sum_\alpha X_\alpha\otimes X_\alpha\in\mathrm{End}_{\mathfrak{g}}(T\otimes V)$$

where $\{X_\alpha\}$ is an orthonormal basis of $\mathfrak{h}$ for the scalar product on $\Lambda^2 T$ induced from T.

Proof. Let $\{t_\mu\}$ and $\{X_\alpha\}$ be orthonormal bases of T and $\mathfrak{h}$ respectively. Using the characterization $\langle X, a\wedge b\rangle = \langle Xa, b\rangle$ of the identification $\mathfrak{so}\,T = \Lambda^2 T$ we find that

$$\begin{aligned}B^{\mathfrak{h}}(b\otimes v) &= \sum_\mu t_\mu\otimes\mathrm{pr}_{\mathfrak{h}}(t_\mu\wedge b)v\\ &= \sum_{\mu\alpha} t_\mu\otimes\langle X_\alpha, t_\mu\wedge b\rangle X_\alpha v = -\sum_\alpha X_\alpha b\otimes X_\alpha v.\end{aligned}$$ □

A particularly nice consequence of Fegan's lemma is that the conformal weight operators $B^{\mathfrak{h}}$ and $B^{\tilde{\mathfrak{h}}}$ associated to two ideals $\mathfrak{h},\tilde{\mathfrak{h}}\subset\mathfrak{g}$ always commute. In fact, two disjoint ideals $\mathfrak{h}\cap\tilde{\mathfrak{h}} = \{0\}$ of $\mathfrak{g}$ commute by definition and the general case follows easily. Hence the algebra structure on $\mathfrak{W}(V)$ allows us to use the evaluation homomorphism

$$\Phi:\ \mathbb{C}[\{B^{\mathfrak{h}}\mid\mathfrak{h}\text{ irreducible ideal of }\mathfrak{g}\}]\longrightarrow\mathfrak{W}(V) \qquad (3.7)$$

from the polynomial algebra on abstract symbols $\{B^{\mathfrak{h}}\}$ to the algebra $\mathrm{End}_{\mathfrak{g}}(T\otimes V)$ for studying the fine structure of the space $\mathfrak{W}(V)$ of Weitzenböck formulas.

In order to turn Fegan's lemma into an effective formula for the eigenvalues of the conformal weight operator $B^{\mathfrak{h}}$ of an ideal $\mathfrak{h}\subset\mathfrak{g}$ we need to calculate the Casimir operator in the normalization given by the scalar product on $\Lambda^2 T$. Recall that the Casimir operator is an element in the center of the universal enveloping algebra $\mathcal{U}\mathfrak{h}$ of $\mathfrak{h}$ (for more details cf. § 6). It is defined as the sum $\mathrm{Cas} := \sum_\alpha X_\alpha^2 \in \mathcal{U}\mathfrak{h}$ over an orthonormal basis $\{X_\alpha\}$ of $\mathfrak{h}$ and is thus determined only up to a constant, depending on the chosen scalar product. Usually it is much more convenient to calculate the Casimir Cas with respect to a scalar product of choice and later normalize it to the Casimir Cas^{Λ^2} defined with vectors X_α orthonormal for the invariant scalar product induced from $\Lambda^2 T$.

For a given irreducible ideal $\mathfrak{h} \subset \mathfrak{g}$ in an irreducible holonomy algebra $\mathfrak{g}$ the Casimir operator Cas for $\mathfrak{h}$ is real, symmetric and $\mathfrak{g}$-invariant. Although the holonomy representation T of $\mathfrak{g}$ may not be irreducible itself, it is the complexification of the irreducible real representation $T_\mathbb{R}$ so that we can still conclude that Cas acts as the scalar multiple $\mathrm{c}_T\,\mathrm{id}_T$ of the identity on T. The Casimir eigenvalue $\mathrm{c}_{V_\lambda}^{\Lambda^2}$ of the Casimir operator Cas^{Λ^2} acting on a general irreducible representation V_λ of $\mathfrak{g}$ of highest weight λ can then be calculated from the Casimir eigenvalues c_{V_λ} and c_T of any other Casimir Cas using the normalization formula

$$\mathrm{c}_{V_\lambda}^{\Lambda^2} = -2\frac{\dim \mathfrak{h}}{\dim T}\,\frac{\mathrm{c}_{V_\lambda}}{\mathrm{c}_T}. \tag{3.8}$$

Here the ambiguity in the choice of the normalization cancels out in the quotient $(\mathrm{c}_{V_\lambda}/\mathrm{c}_T)$. In fact, the normalization (3.8) is readily checked for the holonomy representation $V = T$,

$$\mathrm{tr}_T\,\mathrm{Cas}^{\Lambda^2} = \dim T\cdot \mathrm{c}_T^{\Lambda^2} = \sum_\alpha \mathrm{tr}_T\,X_\alpha^2 = -2\dim\mathfrak{h},$$

since the scalar product induced from T on $\Lambda^2 T$ satisfies $\langle X, Y\rangle = \frac{1}{2}\,\mathrm{tr}_T\,XY$ (cf. [Sem06]). □

COROLLARY 3.4 (Explicit formula for conformal weights). *Consider the tensor product $T\otimes V_\lambda = \bigoplus_{\varepsilon\subset\lambda} V_{\lambda+\varepsilon}$ of the holonomy representation T with the irreducible representation V_λ of highest weight λ. For an ideal $\mathfrak{h}\subset\mathfrak{g}$ let $\varepsilon_{\max}$ be the highest weight of T and ρ be the half sum of positive weights of $\mathfrak{h}$ in the dual $\mathfrak{t}^*$ of a maximal torus $\mathfrak{t}$. With respect to the basis $\{\mathrm{pr}_\varepsilon\}$ of idempotents the conformal weight operator $B^\mathfrak{h}$ of the ideal $\mathfrak{h}$ can be expanded $B^\mathfrak{h} = \sum_{\varepsilon\subset\lambda} b_\varepsilon^\mathfrak{h}\,\mathrm{pr}_\varepsilon$ with conformal weights*

$$b_\varepsilon^\mathfrak{h} = 2\frac{\dim\mathfrak{h}}{\dim T}\frac{\langle\lambda+\rho,\varepsilon\rangle - \langle\rho,\varepsilon_{\max}\rangle + \frac{1}{2}(|\varepsilon|^2 - |\varepsilon_{\max}|^2)}{\langle\varepsilon_{\max}+2\rho,\varepsilon_{\max}\rangle},$$

where $\langle\cdot,\cdot\rangle$ is an arbitrary scalar product on $\mathfrak{t}^*$ invariant under the Weyl group of $\mathfrak{h}$.

Proof. According to Lemma 3.3 the conformal weight operator can be written as a difference $B^\mathfrak{h} = -\frac{1}{2}(\mathrm{Cas}^{\Lambda^2} - \mathrm{Cas}^{\Lambda^2}\otimes\mathrm{id} - \mathrm{id}\otimes\mathrm{Cas}^{\Lambda^2})$ of properly normalized Casimir operators. In particular, its restriction to the irreducible summand $V_{\lambda+\varepsilon}\subset T\otimes V_\lambda$ acts by multiplication with the B-eigenvalue $b_\varepsilon^\mathfrak{h} := -\frac{1}{2}(\mathrm{c}_{V_{\lambda+\varepsilon}}^{\Lambda^2} - \mathrm{c}_T^{\Lambda^2} - \mathrm{c}_{V_\lambda}^{\Lambda^2})$. The B-eigenvalues or *conformal weights* $b_\varepsilon^\mathfrak{h}$ can thus be calculated using Freudenthal's formula $\mathrm{c}_{V_\lambda} = \langle\lambda+2\rho,\lambda\rangle$ for the Casimir eigenvalues of irreducible representations V_λ and the normalization (3.8). □

It is clear from the definition that the conformal weight operator $B^\mathfrak{h}\in\mathfrak{W}(V)$ of an ideal $\mathfrak{h}\subset\mathfrak{g}$ of the holonomy algebra $\mathfrak{g}$ is in the (-1)-eigenspace of the involution τ and thus induces a pure curvature term $B^\mathfrak{h}(\nabla^2)$ on every vector bundle VM associated to the holonomy reduction of M. Explicitly we can describe this curvature term using an orthonormal basis $\{X_\alpha\}$ of the ideal $\mathfrak{h}$ for the scalar product induced on $\mathfrak{h}\subset\Lambda^2 T$. Namely the curvature operator

$$R:\Lambda^2 TM \longrightarrow \mathfrak{g}M\subset\Lambda^2 TM,\ a\wedge b\longmapsto R_{a,b},$$

associated to the Riemannian curvature tensor R of M allows us to write down a well-defined global section

$$q^{\mathfrak{h}}(R) := \sum_{\alpha} X_\alpha R(X_\alpha) \in \Gamma(\mathcal{U}^{\leqslant 2}\mathfrak{g}M) \tag{3.9}$$

of the universal enveloping algebra bundle associated to the holonomy reduction. Fixing a representation $G \longrightarrow \mathrm{Aut}(V)$ of the holonomy group the section $q^{\mathfrak{h}}(R)$ in turn induces an endomorphism on the vector bundle VM associated to V and the holonomy reduction.

A particularly important and well-known example of a Weitzenböck formula is the classical formula of Weitzenböck for the Laplace operator $\Delta = d^*d + dd^*$ acting on differential forms, i.e.

$$\Delta = \nabla^*\nabla + q(R). \tag{3.10}$$

The curvature term in this formula is precisely the curvature endomorphism for the full holonomy algebra $\mathfrak{g}$, in particular $q(R) = \mathrm{Ric}$ on the bundle of 1-forms on M. We recall that the curvature term in the Weitzenböck formula (3.10) is known to be the Casimir operator of the holonomy algebra $\mathfrak{g}$ on a symmetric space $M = \tilde{G}/G$; more precisely, for every ideal $\mathfrak{h} \subset \mathfrak{g}$ the curvature term $q^{\mathfrak{h}}(R)$ acts as the Casimir operator of the ideal $\mathfrak{h}$ on every homogeneous vector bundle VM over a symmetric space M (cf. [SW02]).

It is now easy to check that the curvature endomorphism $q(R)^{\mathfrak{h}}$ is indeed the curvature term defined by the conformal weight operator $B^{\mathfrak{h}}$.

Lemma 3.5. $B^{\mathfrak{h}}(\nabla^2) = q^{\mathfrak{h}}(R)$.

Proof. Expanding the second covariant derivative $\nabla^2\psi = \sum t_\mu \otimes t_\nu \otimes \nabla^2_{t_\mu,t_\nu}\psi$ of the section ψ with an orthonormal basis $\{t_\mu\}$ of T and using the same resummation as in the derivation of (3.5), we find for an orthonormal basis $\{X_\alpha\}$ of the ideal $\mathfrak{h}$,

$$\begin{aligned} B^{\mathfrak{h}}(\nabla^2\psi) &= \frac{1}{2}\sum_{\mu\nu} \mathrm{pr}_{\mathfrak{h}}(t_\mu \wedge t_\nu) R^V_{t_\mu,t_\nu}\psi \\ &= \frac{1}{2}\sum_{\alpha\mu\nu} \langle t_\mu \wedge t_\nu, X_\alpha\rangle X_\alpha R^V_{t_\mu,t_\nu}\psi = q^{\mathfrak{h}}(R). \end{aligned}$$ □

On the other hand, Corollary 3.4 tells us how to write the conformal weight operator $B^{\mathfrak{h}}$ in terms of the basis $\{\mathrm{pr}_\varepsilon\}$ of projections onto the irreducible summands $V_{\lambda+\varepsilon} \subset T \otimes V_\lambda$. Using the identification of $B^{\mathfrak{h}}(\nabla^2)$ with the universal curvature terms $q^{\mathfrak{h}}(R)$ proved above we obtain the first general examples of Weitzenböck formulas.

Proposition 3.6 (Universal Weitzenböck formula). *Consider a Riemannian manifold M of dimension n with holonomy group $G \subset \mathbf{SO}(n)$ and the vector bundle $V_\lambda M$ over M associated to the holonomy reduction of M and the irreducible representation V_λ of G of highest weight λ. In terms of the Stein–Weiss operators*

$$T_\varepsilon : \Gamma(V_\lambda M) \longrightarrow \Gamma(V_{\lambda+\varepsilon}M)$$

arising from the decomposition $T \otimes V_\lambda = \bigoplus_{\varepsilon\subset\lambda} V_{\lambda+\varepsilon}$ the action of the curvature endomorphisms $q^{\mathfrak{h}}(R)$ can be written as

$$q^{\mathfrak{h}}(R) = -\sum_{\varepsilon\subset\lambda} b^{\mathfrak{h}}_\varepsilon T^*_\varepsilon T_\varepsilon,$$

where the $b^{\mathfrak{h}}_\varepsilon$ are the eigenvalues of the conformal weight operator $B^{\mathfrak{h}} \in \mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda)$.

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As a direct consequence of Proposition 3.6 and the classical Weitzenböck formula (3.10) for the Laplace operator $\Delta = dd^* + d^*d$ on the bundle of differential forms we obtain

$$\Delta = \sum_{\varepsilon \subset \lambda} (1 - b_\varepsilon) T_\varepsilon^* T_\varepsilon.$$

In the case of the Riemannian holonomy group $G = \mathbf{SO}(n)$ the universal Weitzenböck formula stated in Proposition 3.6 was considered in [Gau91] for the first time. The definition of the conformal weight operator and its expression in terms of the Casimir is taken from the same article. The conformal weight operator B has been used for other purposes as well, see for example [CGH00]. Similar results can be found in [Hom04].

Considering B as an element of the algebra $\mathfrak{W}(V)$, all powers of B are $\mathfrak{g}$-invariant endomorphisms. In the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ these powers read

$$B^k_{a \otimes b} = \sum_{\mu_1, \ldots, \mu_{k-1}} \mathrm{pr}_{\mathfrak{g}}(a \wedge t_{\mu_1}) \mathrm{pr}_{\mathfrak{g}}(t_{\mu_1} \wedge t_{\mu_2}) \cdots \ \mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-2}} \wedge t_{\mu_{k-1}}) \mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-1}} \wedge b). \tag{3.11}$$

Recall now that in the irreducible case the trace endomorphism $\sum F_{t_\mu \otimes t_\mu} = (\mathrm{tr}\, F)\, \mathrm{id}_{V_\lambda}$ of an element $F \in \mathfrak{W}(V_\lambda)$ is a multiple of the identity of V_λ. Evidently the traces of the powers B^k of B correspond to the action of the elements

$$\mathrm{Cas}^{[k]} := \sum_{\mu_0, \ldots, \mu_{k-1}} \mathrm{pr}_{\mathfrak{g}}(t_{\mu_0} \wedge t_{\mu_1}) \mathrm{pr}_{\mathfrak{g}}(t_{\mu_1} \wedge t_{\mu_2}) \cdots \mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-2}} \wedge t_{\mu_{k-1}}) \mathrm{pr}_{\mathfrak{g}}(t_{\mu_{k-1}} \wedge t_{\mu_0}) \tag{3.12}$$

of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ on V. The elements $\mathrm{Cas}^{[k]}$, $k \geqslant 2$, all belong to the center of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ and are called *higher Casimirs* since $\mathrm{Cas}^{[2]} = -2\mathrm{Cas}^{\Lambda^2}$ (cf. [CGH00]). A straightforward calculation shows that

$$\mathrm{Cas}^{[k]} = \mathrm{tr}(B^k)\, \mathrm{id}_{V_\lambda} = \left(\sum_{\varepsilon} b_\varepsilon^k \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda} \right) \mathrm{id}_{V_\lambda}, \tag{3.13}$$

for an irreducible representation $V = V_\lambda$, where we use equation (3.6) for computing the trace of $B^k = \sum b_\varepsilon^k\, \mathrm{pr}_\varepsilon$. Note that using the Weyl dimension formula (3.13) enables us to explicitly calculate the action of the higher Casimirs. For the algebras $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ all eigenvalues of the higher Casimirs $\mathrm{Cas}^{[k]}$ are given in Appendix A.

As an example we consider the equation $\mathrm{Cas}^{[3]} = -\frac{1}{2} \mathrm{c}_{\mathfrak{g}}^{\Lambda^2} \mathrm{Cas}^{\Lambda^2}$, which follows from the recursion formula of Corollary 4.2 or by direct calculation. Indeed, $B^2 - \frac{1}{4} \mathrm{c}_{\mathfrak{g}}^{\Lambda^2} B$ is an eigenvector of the involution τ for the eigenvalue $+1$. Thus it is orthogonal to the eigenvector B for the eigenvalue -1 and so

$$0 = \langle B^2 - \tfrac{1}{4} \mathrm{c}_{\mathfrak{g}}^{\Lambda^2} B, B \rangle = \mathrm{tr}(B^3) - \tfrac{1}{4} \mathrm{c}_{\mathfrak{g}}^{\Lambda^2}\, \mathrm{tr}(B^2) = \mathrm{tr}(B^3) + \tfrac{1}{2} \mathrm{c}_{\mathfrak{g}}^{\Lambda^2} \mathrm{Cas}^{\Lambda^2}.$$

From a slightly more general point of view the evaluation at the conformal weight operator defines an algebra homomorphism $\Phi : \mathbb{C}[B] \longrightarrow \mathrm{End}_{\mathfrak{g}}(T \otimes V)$, whose kernel is generated by the minimal polynomial of B as an endomorphism on $T \otimes V$. With B being diagonalizable its minimal polynomial is the product $\min(B) = \prod_{b \in \{b_\varepsilon\}} (B - b)$ over all *different* conformal weights. In consequence, the injective algebra homomorphism

$$\Phi : \ \mathbb{C}[B] / \langle \min(B) \rangle \longrightarrow \mathrm{End}_{\mathfrak{g}}(T \otimes V)$$

is an isomorphism as soon as all conformal weights are pairwise different. Indeed, the dimension of $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ is the number $N(G, \lambda)$ of relevant weights or the number of conformal weights counted with multiplicity, while the number of different conformal weights determines the degree of $\min(B)$ and so the dimension of $\mathbb{C}[B] / \langle \min(B) \rangle$.

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In § 4.2 we compute the B-eigenvalues in the cases $\mathfrak{g} = \mathfrak{so}_n$, $\mathfrak{g}_2$ and $\mathfrak{spin}_7$. It follows that they are pairwise different unless the highest weight λ belongs to one of the following two exceptional families. The first for $\mathfrak{g} = \mathfrak{so}_{2r}$ and a representation of highest weight $\lambda = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r$, with $\lambda_r = \lambda_{r-1}$, which is equivalent to $b_{\varepsilon_r} = b_{-\varepsilon_r}$. The second for $\mathfrak{g} = \mathfrak{spin}_7$ and a representation with highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ with $c = 2a + 1$, which is equivalent to $b_{-\varepsilon_4} = b_{\varepsilon_4}$. In these cases the degree of the minimal polynomial is reduced by one and hence the image of Φ has codimension one. Thus, we have proved the following.

Proposition 3.7 (Structure of the algebra of Weitzenböck formulas). *Let G be one of the holonomy groups $\mathbf{SO}_n$, $\mathbf{G}_2$ or $\mathbf{Spin}(7)$ of non-symmetric manifolds. If V_λ is irreducible, then Φ is an isomorphism*

$$\Phi : \ \mathbb{C}[B]/\langle \min(B) \rangle \xrightarrow{\cong} \mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda),$$

with the only exception being the cases $G = \mathbf{SO}_{2r}$ and a highest weight λ with $\lambda_{r-1} = \lambda_r$, or $G = \mathbf{Spin}(7)$ and a highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ with $c = 2a + 1$. In both cases the homomorphism Φ is not surjective and its image has codimension one.

3.3 The classifying endomorphism

The decomposition of the space $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ of Weitzenböck formulas into the (± 1)-eigenspaces of the involution τ can be written as

$$\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V) \cong \mathrm{Hom}_{\mathfrak{g}}(\Lambda^2 T, \mathrm{End}\, V) \oplus \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2 T, \mathrm{End}\, V).$$

However, in general we have a further splitting of $T \otimes T$ leading to a further decomposition of the τ-eigenspaces. The aim of the present subsection is to introduce an endomorphism K on $\mathfrak{W}(V)$ whose eigenspaces correspond to this finer decomposition.

Definition 3.8 (The classifying endomorphism). The classifying endomorphism $K^{\mathfrak{h}}$ of an ideal $\mathfrak{h}_{\mathbb{R}} \subset \mathfrak{g}_{\mathbb{R}}$ of the real holonomy algebra $\mathfrak{g}_{\mathbb{R}}$ is the endomorphism $K^{\mathfrak{h}} : \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$ on the space of Weitzenböck formulas defined in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ by the formula

$$K^{\mathfrak{h}}(F)_{a \otimes b} v := -\sum_{\alpha} F_{X_\alpha a \otimes X_\alpha b} v$$

where $\{X_\alpha\}$ is an orthonormal basis for the scalar product induced on the ideal $\mathfrak{h} \subset \Lambda^2 T$. As before we denote the classifying endomorphism of the ideal $\mathfrak{g}$ simply by $K := K^{\mathfrak{g}}$.

Note that for every $\mathfrak{g}$-equivariant map $F : T \otimes T \longrightarrow \mathrm{End}\, V$ the map $K^{\mathfrak{h}}(F) : T \otimes T \longrightarrow \mathrm{End}\, V$ is again $\mathfrak{g}$-equivariant, because we sum over an orthonormal basis $\{X_\alpha\}$ of the ideal $\mathfrak{h} \subset \mathfrak{g}$ for a $\mathfrak{g}$-invariant scalar product. In the interpretation $\mathfrak{W}(V) = \mathrm{End}_{\mathfrak{g}}(T \otimes V)$ the definition of $K^{\mathfrak{h}}$ reads

$$K^{\mathfrak{h}}(F)(b \otimes v) = -\sum_{\mu\alpha} t_\mu \otimes F_{X_\alpha t_\mu \otimes X_\alpha b} v = \sum_{\mu\alpha} X_\alpha t_\mu \otimes F_{t_\mu \otimes X_\alpha b} v$$

or more succinctly

$$K^{\mathfrak{h}}(F) = \sum_{\alpha} (X_\alpha \otimes \mathrm{id}) F (X_\alpha \otimes \mathrm{id}). \tag{3.14}$$

In consequence, the classifying endomorphisms $K^{\mathfrak{h}}$ and $K^{\tilde{\mathfrak{h}}}$ for two ideals $\mathfrak{h}, \tilde{\mathfrak{h}} \subset \mathfrak{g}$ commute on the space $\mathfrak{W}(V)$ of Weitzenböck formulas similar to the conformal weight operators. The classifying endomorphisms can also be used to find an explicit form of the matrix corresponding to the twist $\tau : \ \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$ in the basis of $\mathfrak{W}(V)$ given by the orthogonal idempotents pr_ε.

In all cases considered below, i.e. for the holonomies $\mathfrak{so}_n$, $\mathfrak{g}_2$ and $\mathfrak{spin}_7$, we have only one ideal and thus only one classifying endomorphism $K = K^{\mathfrak{g}}$.

Lemma 3.9 (Eigenvalues of the classifying endomorphism). *Consider the decomposition of the tensor product $T \otimes T = \bigoplus_\alpha W_\alpha$ into irreducible summands. The classifying endomorphisms $K^{\mathfrak{h}}$ are diagonalizable on $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ with eigenspaces $\mathrm{Hom}_{\mathfrak{g}}(W_\alpha, \mathrm{End}\, V) \subset \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ and with eigenvalues*

$$\kappa_{W_\alpha} = \tfrac{1}{2}\mathrm{c}^{\Lambda^2}_{W_\alpha} - \mathrm{c}^{\Lambda^2}_T .$$

In particular, the classifying endomorphisms $K^{\mathfrak{h}}$ act as $K^{\mathfrak{h}}(F) = \sum_\alpha \kappa_{W_\alpha} F|_{W_\alpha}$ on the space of Weitzenböck formulas $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$.

Proof. For a given ideal $\mathfrak{h}$ it follows immediately from the definition of $K^{\mathfrak{h}}$ that it acts by precomposition with the map $-\sum X_\alpha \otimes X_\alpha$ in the interpretation $\mathfrak{W}(V) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ of the space of Weitzenböck formulas. The argument used in the proof of Corollary 3.4 shows that $K^{\mathfrak{h}}$ is actually a difference of Casimir operators leading to the stated formula for its eigenspaces and eigenvalues. □

In the case of the holonomies $\mathfrak{so}_n$, $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ we have $T \otimes T = \mathbb{C} \oplus \mathrm{Sym}^2_0 T \oplus \mathfrak{g} \oplus \mathfrak{g}^\perp$ and using Lemma 3.9 we find the following K-eigenvalues.

	$\kappa_{\mathbb{C}}$	$\kappa_{\mathrm{Sym}^2_0 T}$	$\kappa_{\mathfrak{g}}$	$\kappa_{\mathfrak{g}^\perp}$
$\mathfrak{so}_n$	$-(n-1)$	1	-1	—
$\mathfrak{g}_2$	-4	$\frac{2}{3}$	0	-2
$\mathfrak{spin}_7$	$-\frac{21}{4}$	$\frac{3}{4}$	$-\frac{1}{4}$	$-\frac{9}{4}$

(3.15)

Note that all these K-eigenvalues are different and consequently the twist τ is a polynomial in the classifying endomorphism K, for example for $\mathfrak{g} = \mathfrak{so}(n)$ we find

$$K^{\mathfrak{g}}(F) = \tau(F) - \mathrm{tr}(F)\mathbf{1}.$$

Moreover, a given invariant homomorphism $F \in \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ is an eigenvector of K if and only if F is different from zero on precisely one summand $W_\alpha \subset T \otimes T$, i.e. $\mathbf{1}$ and B are clearly K-eigenvectors.

Lemma 3.10 (Properties of the classifying endomorphism). *The classifying endomorphism $K : \mathfrak{W}(V) \longrightarrow \mathfrak{W}(V)$ is a symmetric endomorphism commuting with the twist map τ on the space $\mathfrak{W}(V)$ of Weitzenböck formulas equipped with the scalar product $\langle F, \tilde{F}\rangle := (1/\dim V)\, \mathrm{tr}_{T\otimes V}\, F\tilde{F}$. The special endomorphisms $\mathbf{1}$ and B for the same ideal $\mathfrak{h} \subset \mathfrak{g}$ are K-eigenvectors:*

$$K(\mathbf{1}) = \mathrm{c}^{\Lambda^2}_T \mathbf{1} \quad K(B) = (\mathrm{c}^{\Lambda^2}_T - \tfrac{1}{2}\mathrm{c}^{\Lambda^2}_{\mathfrak{h}})B.$$

Proof. The symmetry of K is a trivial consequence of (3.14) in the form

$$\langle K(F), \tilde{F}\rangle = \frac{1}{\dim V}\sum_\nu \mathrm{tr}_{T\otimes V}((X_\nu \otimes \mathrm{id})F(X_\nu \otimes \mathrm{id})\tilde{F})$$

and the cyclic invariance of the trace, moreover K commutes with τ by definition. Coming to the explicit determination of $K(\mathbf{1})$ and $K(B)$ we observe that the unit of $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ becomes the equivariant map $\mathbf{1}(a \otimes b) = \langle a, b\rangle\, \mathrm{id}_V$ in $\mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\, V)$ and so

$$(K\mathbf{1})(a \otimes b) = -\sum_\nu \langle X_\nu a, X_\nu b\rangle\, \mathrm{id}_V = \sum_\nu \langle a, X_\nu^2 b\rangle\, \mathrm{id}_V = \mathrm{c}^{\Lambda^2}_T \mathbf{1}(a \otimes b).$$

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The conformal weight operator B considered as an element of $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}\,V)$ lives by definition in the eigenspace $\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}, \mathrm{End}\,V)$ for the eigenvalue $-\frac{1}{2}\mathrm{c}_{\mathfrak{g}}^{\Lambda^2}+\mathrm{c}_T^{\Lambda^2}$ of K, where $\mathrm{c}_{\mathfrak{g}}^{\Lambda^2}$ is the Casimir eigenvalue of the adjoint representation. □

On a manifold with holonomy algebra $\mathfrak{g}\subset\mathfrak{so}_n\cong\Lambda^2T$, the Riemannian curvature tensor, takes values in $\mathfrak{g}$, i.e. it can be considered as an element of $\mathrm{Sym}^2\mathfrak{g}$. This fact has the following important consequence.

Proposition 3.11 (Bochner identities). *Suppose $F\in\mathfrak{W}(V)$ is an invariant homomorphism $T\otimes T\longrightarrow\mathrm{End}\,V$, which factors through the projection onto the orthogonal complement $\mathfrak{g}^\perp\subset\Lambda^2T\subset T\otimes T$ of the holonomy algebra $\mathfrak{g}\subset\Lambda^2T$. Then the curvature expression $F(\nabla^2)$ vanishes for any curvature tensor R.*

Note that by Schur's lemma an invariant homomorphism $F: T\otimes T\longrightarrow\mathrm{End}\,V$ which factors through $\mathfrak{g}^\perp\subset\Lambda^2T$ is different from zero only on the summand $\mathfrak{g}^\perp$ of $T\otimes T$ and thus automatically satisfies $\tau F=-F$, i.e. defines a pure curvature Weitzenböck formula. We will call such a Weitzenböck formula a *Bochner identity*.

Writing the corresponding invariant homomorphism F in terms of the basis $\{\mathrm{pr}_\varepsilon\}$ as a linear combination $F=\sum f_\varepsilon\,\mathrm{pr}_\varepsilon$ we get the following explicit form of the Bochner identity:

$$\sum_\varepsilon f_\varepsilon T_\varepsilon^* T_\varepsilon = 0.$$

The Bochner identities of $\mathbf{G}_2$- and $\mathbf{Spin}(7)$-holonomies correspond to eigenvectors of the classifying endomorphism K for the eigenvalues -2 and $-\frac{9}{4}$ respectively. Since the zero weight space of $\mathfrak{g}^\perp$ is in both cases one-dimensional, it follows from Lemma 2.3 that

$$\dim\mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g}^\perp, \mathrm{End}\,V_\lambda)\leqslant 1, \tag{3.16}$$

i.e. there is at most one Bochner identity. Moreover, the K-eigenvector $\mathbf{1}\in\mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda)$ spans the K-eigenspace $\mathrm{Hom}_{\mathfrak{g}}(\mathbb{C}, \mathrm{End}\,V_\lambda)\cong\mathbb{C}$. Finally, we note that because the zero weight space of $\mathfrak{g}$ itself is the fixed Cartan subalgebra $\mathfrak{t}\subset\mathfrak{g}$, an application of Lemma 2.3 results in the estimates

$$\dim\mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2, \mathrm{End}\,V_\lambda)\leqslant 2, \qquad \dim\mathrm{Hom}_{\mathfrak{spin}_7}(\mathfrak{spin}_7, \mathrm{End}\,V_\lambda)\leqslant 3.$$

4. The recursion procedure for $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$

The definitions of the conformal weight operator B and the classifying endomorphism K given in the previous section are very similar. With this similarity it should not come as a surprise that the actions of B and K on the space $\mathfrak{W}(V)$ of Weitzenböck formulas obey a simple relation, which is the corner stone of the treatment of Weitzenböck formulas proposed in this article. In the present section we first prove this relation and then use it to construct recursively a basis of K-eigenvectors of $\mathfrak{W}(V_\lambda)$ for the holonomy groups $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$.

4.1 The basic recursion procedure

Recall that the twist τ is defined in the interpretation $\mathfrak{W}(V)=\mathrm{Hom}_{\mathfrak{g}}(T\otimes T\otimes V, V)$ of the space of Weitzenböck formulas as linear maps $T\otimes T\otimes V\longrightarrow V$ by precomposition with the endomorphism $\tau: a\otimes b\otimes v\longmapsto b\otimes a\otimes v$. Generalizing this precomposition we observe that $\mathfrak{W}(V)$ is a right module over the algebra $\mathrm{End}_{\mathfrak{g}}(T\otimes T\otimes V)$ containing τ. Interestingly, both the classifying endomorphism K and the (right) multiplication by the conformal weight operator B

are induced by precomposition with elements in $\mathrm{End}_{\mathfrak{g}}(T \otimes T \otimes V)$, too: K is the precomposition with the $\mathfrak{g}$-invariant endomorphism

$$K: \ T \otimes T \otimes V \longrightarrow T \otimes T \otimes V, \quad a \otimes b \otimes v \longmapsto -\sum_{\nu} X_\nu a \otimes X_\nu b \otimes v$$

while (right) multiplication by B is precomposition with the $\mathfrak{g}$-invariant endomorphism

$$B: \ T \otimes T \otimes V \longrightarrow T \otimes T \otimes V, \quad a \otimes b \otimes v \longmapsto -\sum_{\nu} a \otimes X_\nu b \otimes X_\nu v$$

by Fegan's Lemma 3.3. From this description of the action of the classifying endomorphism K and right multiplication of B on $\mathfrak{W}(V)$ we immediately conclude on $T \otimes T \otimes V_\lambda$ that

$$K + B + \tau B \tau = -\tfrac{1}{2}(\mathrm{Cas}^{\Lambda^2} - 2\mathrm{c}_T^{\Lambda^2} - \mathrm{c}_{V_\lambda}^{\Lambda^2}).$$

However, since Cas^{Λ^2} acts as multiplication with $\mathrm{c}_T^{\Lambda^2}$ on the T-factors and as multiplication with $\mathrm{c}_{V_\lambda}^{\Lambda^2}$ on the V_λ-factor, we obtain the following basic recursion formula.

Theorem 4.1 (Recursion formula). *Let V_λ be an irreducible representation of the holonomy algebra $\mathfrak{g}$. Then the action of K, B and τ on $\mathfrak{W}(V_\lambda) = \mathrm{Hom}_{\mathfrak{g}}(T \otimes T \otimes V_\lambda, V_\lambda)$ by precomposition satisfies*

$$K + B + \tau B \tau = \mathrm{c}_T^{\Lambda^2} = -2\frac{\dim \mathfrak{h}}{\dim T}.$$

We will now explain how this theorem yields a recursion formula for K-eigenvectors. In fact, given an eigenvector $F \in \mathfrak{W}(V)$ for the twist τ and the classifying endomorphism K with eigenvalues t and κ, i.e. $\tau F = tF$ and $KF = \kappa F$, the recursion formula allows us to produce a new τ-eigenvector F_{new} with eigenvalue $-t$. This simple prescription suffices to obtain a complete eigenbasis for $\mathfrak{W}(V)$ of τ- and actually K-eigenvectors in the generic case $\mathfrak{g} = \mathfrak{so}_n$ and, with some modifications, also for the exceptional holonomies $\mathfrak{g} = \mathfrak{g}_2$ and $\mathfrak{g} = \mathfrak{spin}_7$. The quaternionic Kähler case can be dealt with similarly, whereas for Kähler manifolds an easier and more direct approach is possible.

Corollary 4.2 (Basic recursion procedure). *Let $F \in \mathfrak{W}(V)$ be an eigenvector for the involution τ and the classifying endomorphism K of an ideal $\mathfrak{h} \subset \mathfrak{g}$, i.e. $K(F) = \kappa F$ and $\tau(F) = \pm F$. Then the new Weitzenböck formula*

$$F_{\mathrm{new}} := \left(B - \frac{\mathrm{c}_T^{\Lambda^2} - \kappa}{2} \right) \circ F$$

is again a τ-eigenvector in $\mathfrak{W}(V)$ with $\tau(F_{\mathrm{new}}) = \mp F_{\mathrm{new}}$. In particular, we find that

$$\mathbf{1}_{\mathrm{new}} = B \quad \textit{and} \quad B_{\mathrm{new}} = B^2 - \tfrac{1}{4}\mathrm{c}_{\mathfrak{g}}^{\Lambda^2} B.$$

Proof. We observe that the recursion formula in Theorem 4.1 in the form $\tau B \tau = \mathrm{c}_T^{\Lambda^2} - K - B$ implies under the assumptions $K(F) = \kappa F$ and $\tau(F) = \pm F$ that

$$\pm\tau(BF) = (\mathrm{c}_T^{\Lambda^2} - \kappa)F - BF$$

and consequently

$$\pm\tau\left(BF - \frac{\mathrm{c}_T^{\Lambda^2} - \kappa}{2} F \right) = -\left(BF - \frac{\mathrm{c}_T^{\Lambda^2} - \kappa}{2} F \right).$$

The formulas for $\mathbf{1}_{\mathrm{new}}$ and B_{new} are immediate consequences of Lemma 3.10. □

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Recall that a K-eigenvector is automatically a τ-eigenvector. In general, however, the Weitzenböck formula $F_{\text{new}} \in \mathfrak{W}(V)$ does not need to be an eigenvector for K again and it is then not possible to iterate the recursion. Nevertheless, we may avoid the termination of the recursion procedure for most of the irreducible non-symmetric holonomy algebras by using appropriate projections.

We note that any $+1$-eigenvector of τ orthogonal to $\mathbf{1}$ is already a K-eigenvector in the space $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2_0 T, \mathrm{End}\, V_\lambda)$. This is due to the fact that $\mathbf{1}$ spans the second summand of the $+1$-eigenspace of τ. In particular, the orthogonal projection of B_{new} onto the orthogonal complement of $\mathbf{1}$, i.e. the polynomial $B^2 - \frac{1}{4}\mathrm{c}^{\Lambda^2}_{\mathfrak{g}} B + (2/n)\mathrm{c}^{\Lambda^2}_{V_\lambda}$, is a K-eigenvector in $\mathrm{Hom}_{\mathfrak{g}}(\mathrm{Sym}^2_0 T, \mathrm{End}\, V_\lambda)$. More generally, we have the following corollary.

Corollary 4.3 (Orthogonal recursion procedure). *Let $p_0(B), \ldots, p_k(B)$ be a sequence of polynomials obtained by applying the Gram–Schmidt orthonormalization procedure to the powers $1, B, B^2, \ldots, B^k$ of the conformal weight operator B. If all these polynomials are τ-eigenvectors and $p_k(B)$ is moreover a K-eigenvector, then the orthogonal projection $p_{k+1}(B)$ of B^{k+1} onto the orthogonal complement of the span of $1, B, \ldots, B^k$ is a again a τ-eigenvector.*

Proof. By assumption $\mathrm{span}(1, \ldots, B^k) = \mathrm{span}(p_0(B), \ldots, p_k(B))$ is τ-invariant. Moreover, since $p_k(B)$ is a K-eigenvector the basic recursion procedure shows the existence of a polynomial in B of degree $k+1$, which is a τ-eigenvector, so that $\mathrm{span}(1, B, \ldots, B^{k+1})$ is τ-invariant as well. Clearly the orthogonal projection $p_{k+1}(B)$ of B^{k+1} onto the orthogonal complement of $\mathrm{span}\{1, B, \ldots, B^k\}$ is a polynomial in B of degree $k+1$ with

$$\mathrm{span}\{1, B, B^2, \ldots, B^k\} \oplus \mathbb{C}p_{k+1}(B) = \mathrm{span}\{1, B, B^2, \ldots, B^k, B^{k+1}\}.$$

Now the involution τ is symmetric with respect to the scalar product on $\mathrm{End}_{\mathfrak{g}}(T \otimes V)$ and so the orthogonal complement of a τ-invariant space is again τ-invariant. □

4.2 Computation of B-eigenvalues for $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$

In this subsection we will compute the B-eigenvalues for the holonomies $\mathbf{SO}(n)$, $\mathbf{G}_2$ and $\mathbf{Spin}(7)$ by applying the explicit formula of Corollary 3.4. In particular, we will see that with only two exceptions all B-eigenvalues of a given irreducible representation are pairwise different. This information was relevant in the proof of Proposition 3.7.

The $\mathbf{SO}(n)$ *case.* Recall that in § 2 we fixed the notation for the fundamental weights $\omega_1, \ldots, \omega_r$ and the weights $\pm\varepsilon_1, \ldots, \pm\varepsilon_r$ of the defining representation $\mathbb{R}^n$ of $\mathbf{SO}(n)$ with $r := \lfloor n/2 \rfloor$. Moreover, the scalar product $\langle \cdot, \cdot \rangle$ on the dual of a maximal torus was chosen so that the weights $\varepsilon_1, \ldots, \varepsilon_r$ are an orthonormal basis. A highest weight can be written $\lambda = \lambda_1\omega_1 + \cdots + \lambda_r\omega_r = \mu_1\varepsilon_1 + \cdots + \mu_r\varepsilon_r$ with integral coefficients $\lambda_1, \ldots, \lambda_r \geqslant 0$ and coefficients $\mu_1, \ldots, \mu_r$, which are either all integral or all half-integral and decreasing. Independent of the parity of n the conformal weights are

$$b_{+\varepsilon_k} = \mu_k - k + 1, \quad b_{-\varepsilon_k} = -\mu_k - n + k + 1, \quad b_0 = -r$$

according to Corollary 3.4, where the zero weight only appears for n odd. With only a few exceptions the conformal weights are totally ordered and thus pairwise different. In the case n odd the coefficients $\mu_1, \ldots, \mu_r$ are decreasing in the sense $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_r \geqslant 0$ so that we find strict inequalities

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < \cdots < b_{-\varepsilon_r} \leqslant b_0 < b_{+\varepsilon_r} < \cdots < b_{+\varepsilon_1}$$

unless $\mu_r = 0$ or equivalently $\lambda_r = 0$. In the latter case $b_{-\varepsilon_r} \leqslant b_0$ happens to be an equality. However, Lemma 2.2 tells us that the zero weight is irrelevant for highest weights λ with $\lambda_r = 0$.

Without loss of generality, we may thus assume all conformal weights to be different for n odd. Similar considerations in the case of even n based on the inequalities $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{r-1} \geqslant |\mu_r|$ satisfied by the coefficients $\mu_1, \ldots, \mu_r$ of λ lead to

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < \cdots < b_{-\varepsilon_{r-1}} < \{b_{-\varepsilon_r}, b_{+\varepsilon_r}\} < b_{+\varepsilon_{r-1}} < \cdots < b_{+\varepsilon_2} < b_{+\varepsilon_1}$$

where nothing specific can be said about the relation between $b_{-\varepsilon_r}$ and $b_{+\varepsilon_r}$ due to $b_{+\varepsilon_r} - b_{-\varepsilon_r} = 2\mu_r$. This should not be too surprising as the outer automorphism of $\mathbf{SO}(n)$ with n even acts on $\mathfrak{t}^*$ as a reflection along the hyperplane $\mu_r = 0$.

The $\mathbf{G}_2$ *case.* We write the highest weight as $\lambda = a\omega_1 + b\omega_2$ with integers $a, b \geqslant 0$ and use the scalar product defined in § 2 by setting $\langle \varepsilon_1, \varepsilon_1 \rangle = 1 = \langle \varepsilon_2, \varepsilon_2 \rangle$ and $\langle \varepsilon_1, \varepsilon_2 \rangle = \frac{1}{2}$, equivalently $\langle \omega_1, \omega_1 \rangle = 1$, $\langle \omega_2, \omega_2 \rangle = 3$ and $\langle \omega_1, \omega_2 \rangle = \frac{3}{2}$. According to Corollary 3.4 the B eigenvalue for the zero weight is given by $b_0 = -2$, and similarly

$$b_{\pm\varepsilon_1} = -(\tfrac{5}{3} \mp \tfrac{5}{3}) \pm (\tfrac{2}{3}a + b), \quad b_{\pm\varepsilon_2} = -(\tfrac{5}{3} \mp \tfrac{4}{3}) \pm (\tfrac{1}{3}a + b), \quad b_{\pm\varepsilon_3} = -(\tfrac{5}{3} \mp \tfrac{1}{3}) \pm \tfrac{1}{3}a.$$

Again all conformal weights or B-eigenvalues are pairwise different and totally ordered:

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < b_{-\varepsilon_3} < b_0 < b_{+\varepsilon_3} < b_{+\varepsilon_2} < b_{+\varepsilon_1}.$$

The $\mathbf{Spin}(7)$ *case.* Using the fundamental weights $\omega_1, \omega_2, \omega_3$ and the scalar product $\langle \cdot, \cdot \rangle$ introduced in § 2 in terms of the weights $\pm\eta_1, \pm\eta_2, \pm\eta_3$ of the representation $\mathbb{R}^7$ we write the highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ with integers $a, b, c \geqslant 0$ and compute

$$b_{\pm\varepsilon_1} = -(\tfrac{9}{4} \mp \tfrac{9}{4}) \pm (\tfrac{1}{2}a + b + \tfrac{3}{4}c), \quad b_{\pm\varepsilon_2} = -(\tfrac{9}{4} \mp \tfrac{7}{4}) \pm (\tfrac{1}{2}a + b + \tfrac{1}{4}c),$$
$$b_{\pm\varepsilon_3} = -(\tfrac{9}{4} \mp \tfrac{3}{4}) \pm (\tfrac{1}{2}a + \tfrac{1}{4}c), \quad b_{\pm\varepsilon_4} = -(\tfrac{9}{4} \mp \tfrac{1}{4}) \pm (\tfrac{1}{2}a - \tfrac{1}{4}c).$$

In this case we obtain the inequalities

$$b_{-\varepsilon_1} < b_{-\varepsilon_2} < b_{-\varepsilon_3} < \{b_{-\varepsilon_4}, b_{+\varepsilon_4}\} < b_{+\varepsilon_3} < b_{+\varepsilon_2} < b_{+\varepsilon_3}.$$

However, the difference $b_{+\varepsilon_4} - b_{-\varepsilon_4} = a - \frac{1}{2}c + \frac{1}{2}$ does not allow us to draw conclusions about the relation between $b_{-\varepsilon_4}$ and $b_{+\varepsilon_4}$. In particular, for a highest weight λ with $c = 2a + 1$ the two conformal weights $b_{-\varepsilon_4} = b_{+\varepsilon_4}$ agree.

4.3 Basic Weitzenböck formulas for $\mathbf{SO}(n)$, $\mathbf{G_2}$ and $\mathbf{Spin(7)}$

In this section we make the recursion procedure of Corollary 4.2 explicit for the holonomy groups $\mathbf{SO}(n), \mathbf{G}_2$ and $\mathbf{Spin}(7)$. Let us start with the generic Riemannian holonomy algebra $\mathfrak{g} = \mathfrak{so}_n$ with only a single non-trivial ideal $\mathfrak{h} = \mathfrak{g}$. According to (3.15) its classifying endomorphism K has eigenvalues $(1-n), 1$ and -1 with eigenspaces $\mathbb{C}\mathbf{1}$, the orthogonal complement of $\mathbf{1}$ in the τ-eigenspace for the eigenvalue 1 and the τ-eigenspace for the eigenvalue -1 respectively. The orthogonal projection of every τ-eigenvector to the orthogonal complement of $\mathbf{1}$ is thus a K-eigenvector. Consequently, we can modify the recursion procedure such that it associates to an eigenvector F for τ of eigenvalue -1 the K-eigenvector

$$F_{\text{new}} := \left(B + \frac{n-2}{2}\right)F - \frac{1}{n}\langle BF, \mathbf{1}\rangle \mathbf{1}$$

for the eigenvalue 1, while a τ-eigenvector F for the eigenvalue $+1$ orthogonal to $\mathbf{1}$ is mapped to the K-eigenvector

$$F_{\text{new}} := \left(B + \frac{n}{2}\right)F$$

for the eigenvalue -1. For an irreducible $\mathfrak{so}_n$-representation V_λ we thus get a sequence of polynomials $p_0(B), p_1(B), \ldots$ of K-eigenvectors in $\mathfrak{W}(V_\lambda)$. They are defined recursively by

$$p_0(B) := \mathbf{1}, p_1(B) := B \quad \text{and} \quad p_{k+1}(B) := (p_k(B))_{\text{new}} \quad \text{for } k \geqslant 1.$$

Evidently, the different $p_k(B)$ are polynomials of degree k in B and so the eigenvectors $p_0(B), \ldots, p_{d-1}(B) \in \mathfrak{W}(V_\lambda)$ with $d := \deg \min B$ are necessarily linearly independent. According to Proposition 3.7 we always get a complete basis of τ-eigenvectors with the exception of the case $\mathfrak{g} = \mathfrak{so}_{2r}$ and a representation V_λ with $\lambda_{r-1} = \lambda_r$. Here we still have to add a K-eigenvector F_{spin} spanning the orthogonal complement of the image of $\mathbb{C}[B]$ in $\mathfrak{W}(V_\lambda)$.

Note that the polynomials $p_{2k+1}(B), k = 0, 1, \ldots,$ are in the -1-eigenspace of τ. Hence the corresponding Weitzenböck formulas give a pure curvature term. Let N be the number of irreducible components of $T \otimes V_\lambda$; then there are $\lfloor N/2 \rfloor$ linearly independent equations of this type. This result, which is clear from our construction, was proved for the first time in [BH02]. The first eigenvectors in this sequence are $p_0(B) = \mathbf{1}$ and $p_1(B) = B$ as well as

$$p_2(B) = B^2 + \frac{n-2}{2} B + \frac{2}{n} \mathrm{c}^{\Lambda^2}_{V_\lambda}, \tag{4.17}$$

$$p_3(B) = B^3 + (n-1)B^2 + \left(\frac{2}{n} \mathrm{c}^{\Lambda^2}_{V_\lambda} + \frac{n(n-2)}{4} \right) B + \mathrm{c}^{\Lambda^2}_{V_\lambda}. \tag{4.18}$$

Essentially the same procedure can be used in the case $\mathfrak{g} = \mathfrak{g}_2$ to compute a complete K-eigenbasis for the space $\mathfrak{W}(V_\lambda)$ for an irreducible $\mathfrak{g}_2$-representation V_λ. Again there is only one non-trivial ideal $\mathfrak{h} = \mathfrak{g}_2$ and hence only a single classifying endomorphism K. However, the τ-eigenspace in $\mathfrak{W}(V_\lambda)$ for the eigenvalue -1 decomposes into two K-eigenspaces. The recursion procedure gives the K-eigenvectors

$$p_0(B) = \mathbf{1}, \quad p_1(B) = B, \quad p_2(B) = B^2 + 2B + \tfrac{2}{7} \mathrm{c}^{\Lambda^2}_{V_\lambda}. \tag{4.19}$$

Using the recursion procedure again gives a polynomial of degree three in B. Projecting it onto the orthogonal complement of B we obtain

$$p_3(B) = B^3 + \tfrac{13}{3} B^2 + (\tfrac{1}{2} \mathrm{c}^{\Lambda^2}_{V_\lambda} + 4) B + \tfrac{2}{3} \mathrm{c}^{\Lambda^2}_{V_\lambda}. \tag{4.20}$$

We will see in Theorem 6.6 that $p_3(B)$ is in fact a K-eigenvector for the eigenvalue -2, in other words $p_3(B) \in \mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2^\perp, V_\lambda)$ is a Bochner identity. Due to the estimate (3.16) any τ-eigenvector orthogonal to $\mathbf{1}$ and $p_3(B)$ is a K-eigenvector and so we may obtain a complete eigenbasis $p_0(B), \ldots, p_6(B)$ in the $\mathbf{G}_2$ case by applying the Gram–Schmidt orthogonalization to the powers of B and using Corollary 4.3.

In order to make the generalized Bochner identity corresponding to the polynomial $p_3(B)$ explicit we recall that its coefficients as a linear combination of the basis projections pr_ε are the values of the polynomial p_3 at the corresponding B-eigenvalues b_ε. Substituting the explicit formulas for b_ε and for $\mathrm{c}^{\Lambda^2}_{V_\lambda}$ (cf. Remark A.5) we obtain

$$\begin{aligned} F_{\text{Bochner}} := 27 p_3(B) = &+ a(a+3b+3)(2a+3b+4)\mathrm{pr}_{+\varepsilon_1} \\ &- (a+2)(a+3b+5)(2a+3b+6)\mathrm{pr}_{-\varepsilon_1} \\ &- (a+2)(a+3b+3)(2a+3b+4)\mathrm{pr}_{+\varepsilon_2} \\ &+ a(a+3b+5)(2a+3b+6)\mathrm{pr}_{-\varepsilon_2} \\ &- a(a+3b+5)(2a+3b+4)\mathrm{pr}_{+\varepsilon_3} \\ &+ (a+2)(a+3b+3)(2a+3b+6)\mathrm{pr}_{-\varepsilon_3} \\ &+ 6(a^2+3b^2+3ab+5a+9b+6)\mathrm{pr}_0. \end{aligned} \tag{4.21}$$

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Eventually let us discuss the example of $\mathfrak{g} = \mathfrak{spin}_7$. Here the modified recursion procedure gives the three K-eigenvectors

$$p_0(B) = \mathbf{1}, \quad p_1(B) = B, \quad p_2(B) = B^2 + \tfrac{5}{2}B + \tfrac{1}{4}\mathrm{c}^{\Lambda^2}_{V_\lambda} \tag{4.22}$$

and a τ-eigenvector for the eigenvalue -1, which is of third order as a polynomial in B. After projecting it onto the orthogonal complement of B we obtain

$$p_3(B) = B^3 + \frac{11}{2}B^2 + \frac{1}{2\mathrm{c}^{\Lambda^2}_{V_\lambda}}\left(\mathrm{c}^{[4]}_{V_\lambda} + \frac{55}{2}\mathrm{c}^{\Lambda^2}_{V_\lambda}\right)B + \frac{3}{4}\mathrm{c}^{\Lambda^2}_{V_\lambda}, \tag{4.23}$$

where $\mathrm{c}^{[4]}_{V_\lambda}$ is the eigenvalue of the higher Casimir $\mathrm{Cas}^{[4]}$ on the irreducible representation V_λ. Its explicit value is given in Appendix A in Remark A.6.

However, different to the $\mathfrak{g}_2$ case this is no K-eigenvector. Indeed, in § 6 we will see that the space of polynomials in B of degree at most three is not invariant under K. Hence there cannot be a further K-eigenvector expressible as a polynomial of order three in B. In general, the other K-eigenvectors are polynomials of degree seven in B. They are too complicated to be written down, but surprisingly the K-eigenvector for the eigenvalue $-\frac{9}{4}$, i.e. the Bochner identity, for a representation of highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ has the following simple explicit expression:

$$\begin{aligned} F_{\mathrm{Bochner}} = &+c(2b+c+2)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_1} \\ &- (c+2)(2b+c+4)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_1} \\ &- (c+2)(2b+c+2)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_2} \\ &+ c(2b+c+4)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_2} \\ &- c(2b+c+4)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_3} \\ &+ (c+2)(2b+c+2)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_3} \\ &+ (c+2)(2b+c+4)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_4} \\ &- c(2b+c+2)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_4}. \end{aligned} \tag{4.24}$$

This formula is proved in Theorem 6.7. Note that the coefficients of $\mathrm{pr}_{+\varepsilon_4}$ and $\mathrm{pr}_{-\varepsilon_4}$ are different. Hence in the critical case with $c = 2a + 1$, i.e. where $b_{+\varepsilon_4} = b_{-\varepsilon_4}$, this K-eigenvector F_{Bochner} spans the space orthogonal to $\mathbb{C}[B]$ in $\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda)$.

5. Examples of Weitzenböck formulas

In this section we will present a few examples of how to obtain for a given representation V_λ all possible Weitzenböck formulas on sections of the associated bundle $V_\lambda M$. The general procedure is as follows: we first determine the relevant weights ε using the diagrams of § 2. This gives the decomposition of $T \otimes V_\lambda$ into irreducible summands and defines the generalized gradients T_ε. Next we compute the B-eigenvalues b_ε, for example using the general formula of Corollary 3.4, and obtain the universal Weitzenböck formulas of Proposition 3.6. Other Weitzenböck formulas correspond to the B-polynomials constructed in the preceding section. If $F = F(B)$ is such a polynomial then the coefficient of $T^*_\varepsilon T_\varepsilon$ is given as $-F(b_\varepsilon)$.

As a first example we consider the bundle of p-forms on a Riemannian manifold (M^n, g). For simplicity, we assume $n = 2r + 1$ and $p \leqslant r - 1$, i.e. $\mathfrak{g} = \mathfrak{so}_{2r+1}$ and $\lambda = \omega_p$. The relevant weights according to the tables of § 4.2 are ε_1, $-\varepsilon_p$ and ε_{p+1} with the decomposition

$$T \otimes V_\lambda = V_{\lambda+\varepsilon_1} \oplus V_{\lambda-\varepsilon_p} \oplus V_{\lambda+\varepsilon_{p+1}} \cong V_{\lambda+\varepsilon_1} \oplus \Lambda^{p-1} \oplus \Lambda^{p+1}$$

and generalized gradients $T_{\varepsilon_1}, T_{-\varepsilon_p}$ and $T_{\varepsilon_{p+1}}$. To compare these operators with differential and codifferential d, d^* we have to embed Λ^{p-1} respectively Λ^{p+1} into the tensor product $T \otimes \Lambda^p$.

This leads to the following formula:

$$T^*_{-\varepsilon_p}T_{-\varepsilon_p} = \frac{1}{n-p+1}dd^*, \quad T^*_{+\varepsilon_{p+1}}T_{+\varepsilon_{p+1}} = \frac{1}{p+1}d^*d.$$

Next we take the relevant B-eigenvalues from § 4.2; they are

$$b_{+\varepsilon_1} = 1, \quad b_{-\varepsilon_p} = -n+p, \quad b_{\varepsilon_{p+1}} = -p.$$

Since we have only three summands in the decomposition of $T \otimes V_\lambda$ we obtain only one Weitzenböck formula with a pure curvature term, which is the formula given in Proposition 3.6:

$$\begin{aligned} q(R) &= -T^*_{+\varepsilon_1}T_{+\varepsilon_1} + (n-p)T^*_{-\varepsilon_p}T_{-\varepsilon_p} + pT^*_{+\varepsilon_{p+1}}T_{+\varepsilon_{p+1}} \\ &= -T^*_{+\varepsilon_1}T_{+\varepsilon_1} + \frac{n-p}{n-p+1}dd^* + \frac{p}{p+1}d^*d. \end{aligned}$$

If we add the Weitzenböck formula (3.4) for $\nabla^*\nabla$ to this expression for $q(R)$ we obtain the classical Weitzenböck formula for the Laplacian on p-forms:

$$\Delta = \nabla^*\nabla + q(R) = (n-p+1)T^*_{-\varepsilon_p}T_{-\varepsilon_p} + (p+1)T^*_{+\varepsilon_{p+1}}T_{+\varepsilon_{p+1}} = dd^* + d^*d.$$

Let (M^{2r}, g) be a Riemannian spin manifold with spinor bundle $S = S_+ \oplus S_-$. We consider the two bundles $V_{\lambda_\pm}$ defined by the Cartan summand in $S_\pm \otimes T$ with highest weights $\lambda_+ = \omega_1 + \omega_{r-1}$ and $\lambda_- = \omega_1 + \omega_r$. Using the tables of § 2 we find the relevant weights $+\varepsilon_1$, $-\varepsilon_1$ and $+\varepsilon_2$ for both $\lambda_\pm$ and in addition $-\varepsilon_r$ or $+\varepsilon_r$ for λ_+ or λ_- respectively. The corresponding tensor product decomposition is

$$T \otimes V_{\lambda_\pm} = V_{\lambda_\pm + \varepsilon_1} \oplus V_{\lambda_\pm - \varepsilon_1} \oplus V_{\lambda_\pm + \varepsilon_2} \oplus V_{\lambda_\pm \mp \varepsilon_r}.$$

Note that $\lambda_\pm - \varepsilon_1$ is the defining representation for the bundles $S_\pm$ and that $\lambda_\pm \mp \varepsilon_r = \lambda_\mp$. Projecting the covariant derivative of a section of $T \otimes V_{\lambda_\pm}$ onto one of these summands defines four generalized gradients. The fourth operator $T_{\mp\varepsilon_r} : \Gamma(V_{\lambda_\pm}) \longrightarrow \Gamma(V_{\lambda_\mp})$ is usually called the *Rarita–Schwinger operator*. A solution of the Rarita–Schwinger equation is by definition a section of $\psi \in \Gamma(V_{\lambda_\pm})$ with both $T_{\mp\varepsilon_r}\psi = 0$ and $T_{-\varepsilon_1}\psi = 0$.

The B-eigenvalues for $\mathfrak{so}_n$-representations were computed in § 4.2 and in particular

$$b_{+\varepsilon_1} = \tfrac{3}{2}, \quad b_{-\varepsilon_1} = -2r + \tfrac{1}{2}, \quad b_{+\varepsilon_2} = -\tfrac{1}{2}, \quad b_{\pm\varepsilon_r} = -r + \tfrac{1}{2}.$$

Since the decomposition of $T \otimes V_{\lambda_\pm}$ has four summands we will obtain two Weitzenböck formulas with a pure curvature term. The first one is again the universal Weitzenböck formula of Proposition 3.6 corresponding to B, whereas the second corresponds to $p_3(B)$, the degree three polynomial of the recursion procedure defined in (4.18). Its coefficients are the values $p_3(b_\varepsilon)$ for the relevant weights ε. The Casimir operator of an irreducible $\mathfrak{so}_n$-representation V_λ with highest weight λ is computed as $\mathrm{c}_{V_\lambda} = -\langle \lambda + 2\rho, \lambda \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product on $\mathbb{R}^r$. In particular, we have $\mathrm{c}_{V_{\lambda_\pm}} = -\frac{1}{4}r(2r+7)$. Eventually we obtain the following two Weitzenböck formulas on sections of $V_{\lambda_\pm}$:

$$\begin{aligned} q(R) &= -\tfrac{3}{2}T^*_{+\varepsilon_1}T_{+\varepsilon_1} + (2r - \tfrac{1}{2})T^*_{-\varepsilon_1}T_{-\varepsilon_1} + \tfrac{1}{2}T^*_{+\varepsilon_2}T_{+\varepsilon_2} + (r - \tfrac{1}{2})T^*_{\pm\varepsilon_r}T_{\pm\varepsilon_r}, \\ p_3(B)(\nabla^2) &= -(\tfrac{3}{2} + r)(r-1)T^*_{+\varepsilon_1}T_{+\varepsilon_1} + (2r-1)(r^2-1)T^*_{-\varepsilon_1}T_{-\varepsilon_1} \\ &\quad + (r - \tfrac{1}{2})(r+1)T^*_{+\varepsilon_2}T_{+\varepsilon_2} + T_{\pm\varepsilon_r}. \end{aligned}$$

Note that similar Weitzenböck formulas were obtained in [BH02]. More precisely their curvature terms Z_1 and Z_2 are related to B and $p_3(B)$ by the following equations:

$$Z_1 = \frac{(2r+3)(r-1)}{r(2r+1)}B - \frac{3}{r(2r+1)}p_3(B), \quad Z_2 = -\frac{(2r-1)(r+1)}{r(2r+1)}B + \frac{1}{r(2r+1)}p_3(B),$$

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whereas the operators are related by

$$T_{+\varepsilon_1} = G_Z, \quad T_{-\varepsilon_1} = G_\Sigma, \quad T_{+\varepsilon_2} = G_Y, \quad T_{\mp\varepsilon_r} = G_T.$$

In the last part of this section we want to describe for $\mathbf{G}_2$- and $\mathbf{Spin}(7)$-holonomies all pure curvature Weitzenböck formulas on parallel subbundles of the form bundle. In particular, we will present the form Laplacian $\Delta = d^*d + dd^* = \nabla^*\nabla + q(R)$ as a linear combination of the operators $T_\varepsilon^* T_\varepsilon$ and discuss the existence of harmonic forms.

We start with the case of $\mathbf{G}_2$-holonomy. Let $\Gamma_{a,b}$ be the irreducible $\mathbf{G}_2$-representation with highest weight $a\omega_1 + b\omega_2$, $a, b \geqslant 0$, for example $\Gamma_{0,0} = \mathbb{C}$ is the trivial representation, $\Gamma_{1,0} = T$ and $\Gamma_{0,1} = \Lambda^2_{14} \cong \mathfrak{g}_2$. Recall that up to dimension 77 irreducible $\mathbf{G}_2$-representations are uniquely determined by their dimension. However, there are two different irreducible representations in dimension 77: one of them is $[77]^- := \Gamma_{3,0}$, the other is $\Gamma_{0,2}$, the space of $\mathbf{G}_2$-curvature tensors. Moreover,

$$\dim \Gamma_{2,0} = 27, \quad \dim \Gamma_{1,1} = 64.$$

The spaces of 2- and 3-forms have the following decompositions:

$$\Lambda^2 T \cong \Lambda^5 T \cong T \oplus \Lambda^2_{14}, \quad \Lambda^3 T \cong \Lambda^4 T \cong \mathbb{C} \oplus T \oplus \Lambda^3_{27}, \tag{5.25}$$

where the subscripts denote the dimension of the representation. Next we give the relevant weights for the representations $\Gamma_{1,0}, \Gamma_{0,1}$ and $\Gamma_{2,0}$. We start with $\lambda = \omega_2$, i.e. the representation $V_\lambda = \Gamma_{0,1} = \Lambda^2_{14}$; here the relevant weights are $\varepsilon = -\varepsilon_2, \varepsilon_3, \varepsilon_1$ with $\lambda + \varepsilon = \omega_1, 2\omega_1, \omega_1 + \omega_2$ and the corresponding decomposition reads

$$T \otimes \Gamma_{0,1} = \Gamma_{1,0} \oplus \Gamma_{2,0} \oplus \Gamma_{1,1} = T^* \oplus \Lambda^3_{27} T^* \oplus [64].$$

In this case the universal Weitzenböck formula of Proposition 3.6 is the only pure curvature Weitzenböck formula. With the explicit B-eigenvalues given in § 4.2 we find on sections of $\Lambda^2_{14} T^* M$,

$$q(R) = 4T^*_{-\varepsilon_2} T_{-\varepsilon_2} + \tfrac{4}{3} T^*_{+\varepsilon_3} T_{+\varepsilon_3} - T^*_{+\varepsilon_1} T_{+\varepsilon_1},$$

$$\Delta = 5T^*_{-\varepsilon_2} T_{-\varepsilon_2} + \tfrac{7}{3} T^*_{+\varepsilon_3} T_{+\varepsilon_3}.$$

Hence a form ψ in $\Lambda^2_{14} \subset \Lambda^2 T$ is harmonic if and only if $T_{-\varepsilon_2}\psi = 0 = T_{+\varepsilon_3}\psi$ (the manifold is assumed to be compact), i.e. if and only if $\nabla\psi = T_{+\varepsilon_1}\psi$ or equivalently if and only if $\nabla\psi$ is a section of $\Gamma_{11} = [64]$. This statement corresponds to the fact that on a compact manifold a form is harmonic if and only if it is closed and coclosed.

For $\lambda = \omega_1$, i.e. the representation $V_\lambda = \Gamma_{1,0} = T$, the relevant weights are determined as $\varepsilon = -\varepsilon_1, 0, +\varepsilon_2, +\varepsilon_1$ with $\lambda + \varepsilon = 0, \omega_1, \omega_2, 2\omega_1$ leading to the decomposition

$$T \otimes \Gamma_{1,0} = \Gamma_{0,0} \oplus \Gamma_{1,0} \oplus \Gamma_{0,1} \oplus \Gamma_{2,0} = \mathbb{C} \oplus T^* \oplus \Lambda^2_{14} T^* \oplus \Lambda^3_{27} T^*.$$

Here we have two pure curvature Weitzenböck formulas. In fact, both curvature terms are zero, since $q(R) = \mathrm{Ric} = 0$ on $\Gamma_{1,0} = T$. In addition to the universal Weitzenböck formula of Proposition 3.6 we have the equation corresponding to the polynomial $27p_3(B)$ given in (4.21) with $a = 1, b = 0$. After substituting the B-eigenvalues we obtain the following Weitzenböck formulas on 1-forms:

$$0 = 4T^*_{-\varepsilon_1} T_{-\varepsilon_1} + 2T^*_0 T_0 - \tfrac{2}{3} T^*_{+\varepsilon_1} T_{+\varepsilon_1},$$

$$0 = -\tfrac{16}{3} T^*_{-\varepsilon_1} T_{-\varepsilon_1} + \tfrac{8}{3} T^*_0 T_0 - \tfrac{8}{3} T^*_{+\varepsilon_2} T_{+\varepsilon_2} + \tfrac{8}{9} T^*_{+\varepsilon_1} T_{+\varepsilon_1},$$

$$\Delta = 5T^*_{-\varepsilon_1} T_{-\varepsilon_1} + 3T^*_0 T_0 + T^*_{+\varepsilon_2} T_{+\varepsilon_2} + \tfrac{1}{3} T^*_{+\varepsilon_1} T_{+\varepsilon_1}.$$

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It follows that $\Delta \geqslant \frac{1}{3}\nabla^*\nabla$, i.e. there are no non-parallel harmonic 1-forms, which is of course Bochner's theorem in the case of $\mathbf{G}_2$-manifolds.

Next we consider the case $\lambda = 2\omega_1$, i.e. the representation $V_\lambda = \Gamma_{2,0} = \Lambda^3_{27}$. Here the relevant weights are $\varepsilon = -\varepsilon_1, 0, -\varepsilon_3, \varepsilon_2, \varepsilon_1$ with $\lambda + \varepsilon = \omega_1, 2\omega_1, \omega_2, \omega_1 + \omega_2, 3\omega_1$ and with the decomposition

$$\begin{aligned} T \otimes \Gamma_{2,0} &= \Gamma_{1,0} \oplus \Gamma_{2,0} \oplus \Gamma_{0,1} \oplus \Gamma_{1,1} \oplus \Gamma_{3,0} \\ &= T^* \oplus \Lambda^3_{27}T^* \oplus \Lambda^2_{14} \oplus [64] \oplus [77]^-. \end{aligned}$$

Hence we have two pure curvature Weitzenböck formulas on sections of Λ^3_{27}. The first one is the formula for $q(R)$ corresponding to B, while the second corresponds to $-\frac{27}{240}\, p_3(B)$:

$$\begin{aligned} q(R) &= \tfrac{14}{3}T^*_{-\varepsilon_1}T_{-\varepsilon_1} + 2T^*_0 T_0 + \tfrac{8}{3}T^*_{-\varepsilon_3}T_{-\varepsilon_3} - \tfrac{1}{3}T^*_{+\varepsilon_2}T_{+\varepsilon_2} - \tfrac{4}{3}T^*_{+\varepsilon_1}T_{+\varepsilon_1}, \\ 0 &= -\tfrac{7}{6}T^*_{-\varepsilon_1}T_{-\varepsilon_1} + \tfrac{1}{2}T^*_0 T_0 + \tfrac{5}{6}T^*_{-\varepsilon_3}T_{-\varepsilon_3} - \tfrac{2}{3}T^*_{+\varepsilon_2}T_{+\varepsilon_2} + \tfrac{1}{3}T^*_{+\varepsilon_1}T_{+\varepsilon_1}, \\ \Delta &= \tfrac{9}{2}T^*_{-\varepsilon_1}T_{-\varepsilon_1} + \tfrac{7}{2}T^*_0 T_0 + \tfrac{9}{2}T^*_{-\varepsilon_3}T_{-\varepsilon_3}. \end{aligned}$$

It follows that a form ψ in $\Lambda^3_{27} \subset \Lambda^3 T$ is harmonic if and only if $\nabla\psi$ is a section of $\Gamma_{1,1} \oplus \Gamma_{3,0}$. Note that the expression for Δ was obtained by adding the Bochner identity, i.e. the second Weitzenböck formula, to the equation for $\nabla^*\nabla + q(R)$.

Finally, we turn to the case of $\mathbf{Spin}(7)$-holonomy. Irreducible $\mathbf{Spin}(7)$-representations are parametrized as $\Gamma_{a,b,c} = a\omega_1 + b\omega_2 + c\omega_2$. Again $\Gamma_{0,0,0} = \mathbb{C}$ is the trivial representation and $\Gamma_{0,0,1} = T$ denotes the eight-dimensional holonomy representation. We want to describe the generalized gradients for the parallel subbundles of the form bundle. For this we need the following representations, which are also uniquely determined by their dimension:

$$\begin{aligned} &\dim \Gamma_{1,0,0} = 7, \quad \dim \Gamma_{0,1,0} = 21, \quad \dim \Gamma_{1,0,1} = 48, \quad \dim \Gamma_{1,1,0} = 105, \\ &\dim \Gamma_{2,0,0} = 27, \quad \dim \Gamma_{0,0,2} = 35, \quad \dim \Gamma_{2,0,1} = 168, \quad \dim \Gamma_{1,0,2} = 189. \end{aligned}$$

In dimension 112 there are two different irreducible representations denoted by $[112]^a := \Gamma_{0,1,1}$ and $[112]^b := \Gamma_{0,0,3}$. As in the case of $\mathbf{G}_2$-holonomy, we decompose the spaces of differential forms as

$$\begin{aligned} \Lambda^2 T^* &\cong \Lambda^2_7 T^* \oplus \Lambda^2_{21} T^* \cong \Lambda^6 T^*, \\ \Lambda^3 T^* &\cong T^* \oplus \Lambda^3_{48} T^* \cong \Lambda^5 T^*, \\ \Lambda^4 T^* &\cong \mathbb{C} \oplus \Lambda^4_7 T^* \oplus \Lambda^4_{27} T^* \oplus \Lambda^4_{35} T^* \end{aligned} \tag{5.26}$$

into irreducible subspaces, where again the subscripts refer to the dimension.

We start with the representation $V_\lambda = \Gamma_{1,0,0} = \Lambda^2_7$ of highest weight $\lambda = \omega_1$. The relevant weights are $+\varepsilon_1$ and $-\varepsilon_4$ with B-eigenvalues $b_{+\varepsilon_1} = \frac{1}{2}$ and $b_{-\varepsilon_4} = -3$ leading to the decomposition

$$T \otimes \Gamma_{1,0,0} = \Gamma_{1,0,1} \oplus \Gamma_{0,0,1} = \Lambda^3_{48} \oplus T.$$

Because the bundle defined by $\Gamma_{1,0,0}$ can be considered as the subbundle of the spinor bundle orthogonal to the parallel spinor, the curvature endomorphism $q(R)$ is a multiple of the scalar curvature and hence vanishes. Thus we obtain on sections of Λ^2_7 the only Weitzenböck formula:

$$0 = -\tfrac{1}{2}T^*_{+\varepsilon_1}T_{+\varepsilon_1} + 3T^*_{-\varepsilon_4}T_{-\varepsilon_4}.$$

It follows that $\Delta \geqslant \frac{1}{2}\,\nabla^*\nabla$, i.e. there are no non-parallel harmonic forms in Λ^2_7.

For the second component of the space of 2-forms $\Lambda^2_{21} = \Gamma_{0,1,0}$ we have the following relevant weights: $+\varepsilon_1, -\varepsilon_2, e_3$ with B-eigenvalues $1, -5, -\frac{3}{2}$ in the decomposition

$$T \otimes \Gamma_{0,1,0} = \Gamma_{0,1,1} \oplus \Gamma_{0,0,1} \oplus \Gamma_{1,0,1} = [112]^a \oplus T \oplus \Lambda^3_{48}.$$

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On sections of Λ^2_{21} we have the Weitzenböck formula

$$q(R) = -T^*_{+\varepsilon_1}T_{+\varepsilon_1} + 5T^*_{-\varepsilon_2}T_{-\varepsilon_2} + \tfrac{3}{2}T^*_{+\varepsilon_3}T_{+\varepsilon_3}.$$

Hence a form ψ in Λ^2_{21} is harmonic if and only if $\nabla\psi$ is a section of $\Gamma_{0,1,1}$.

The last parallel subbundle of the form bundle with only one pure curvature Weitzenböck formula is $V_\lambda = \Gamma_{2,0,0} = \Lambda^4_{27}$. The relevant weights are $+\varepsilon_1, -\varepsilon_4$ with B-eigenvalues $1, -\frac{7}{2}$ and the decomposition

$$T \otimes \Gamma_{2,0,0} = \Gamma_{2,0,1} \oplus \Gamma_{1,0,1} = [168] \oplus \Lambda^3_{48},$$

with $[168] := \Gamma_{2,0,1}$. On sections of Λ^4_{27} we have the Weitzenböck formula

$$q(R) = -T^*_{+\varepsilon_1}T_{+\varepsilon_1} + \tfrac{7}{2}T^*_{-\varepsilon_2}T_{-\varepsilon_2}.$$

Hence a form ψ in Λ^4_{27} is harmonic if and only if $\nabla\psi$ is a section of $\Gamma_{2,0,1}$.

For the remaining subbundles we have at least two pure curvature Weitzenböck formulas, one of which is a Bochner identity, i.e. with a zero curvature term. We first consider the representation $T = \Gamma_{0,0,1}$ describing 1- and 6-forms on M. The relevant weights are $+\varepsilon_1, -\varepsilon_1, +\varepsilon_2, +\varepsilon_4$ conformal weights or B-eigenvalues $\frac{3}{4}, -\frac{21}{4}, -\frac{1}{4}$ and $-\frac{9}{4}$ respectively. The corresponding decomposition of the representation $T \otimes T$ reads

$$T \otimes \Gamma_{0,0,1} = \Gamma_{0,0,2} \oplus \Gamma_{0,0,0} \oplus \Gamma_{0,1,0} \oplus \Gamma_{1,0,0} = \Lambda^4_{35}T^* \oplus \mathbb{C} \oplus \Lambda^2_{21}T^* \oplus \Lambda^2_7 T^*.$$

In this case we have the universal Weitzenböck formula and the Bochner identity (4.24) for $(a, b, c) = (0, 0, 1)$. Since $q(R) = \mathrm{Ric}$ on the tangent bundle we obtain two zero curvature Weitzenböck formulas on sections of T:

$$\begin{aligned} 0 &= -\tfrac{3}{4}T^*_{+\varepsilon_1}T_{+\varepsilon_1} + \tfrac{21}{4}T^*_{-\varepsilon_1}T_{-\varepsilon_1} + \tfrac{1}{4}T^*_{+\varepsilon_2}T_{+\varepsilon_2} + \tfrac{9}{4}T^*_{+\varepsilon_4}T_{+\varepsilon_4},\\ 0 &= +15T^*_{+\varepsilon_1}T_{+\varepsilon_1} - 105T^*_{-\varepsilon_1}T_{-\varepsilon_1} - 45T^*_{+\varepsilon_2}T_{+\varepsilon_2} + 75T^*_{+\varepsilon_4}T_{+\varepsilon_4}.\end{aligned}$$

Evidently the first equation tells us that $\Delta \geqslant \frac{1}{4}\nabla^*\nabla$ so that every harmonic 1-form is necessarily parallel. Of course, this is Bochner's theorem reproved in the case of $\mathbf{Spin}(7)$-manifolds. Another direct consequence is the well-known fact that any Killing vector field on a compact $\mathbf{Spin}(7)$-manifold has to be parallel. Indeed, Killing vector fields are vector fields $X \in \Gamma(TM)$, for which $\nabla X^\sharp \in \Gamma(T^*M \otimes T^*M)$ is skew and thus a 2-form. On $\mathbf{Spin}(7)$-manifolds this implies that $T_{+\varepsilon_1}X = 0 = T_{-\varepsilon_1}X$ and so all generalized gradients vanish on X.

Next we consider the representation $\Lambda^4_{35}T^* = \Gamma_{0,0,2}$ with relevant weights $\varepsilon_1, -\varepsilon_1, \varepsilon_2, \varepsilon_4$, conformal weights or B-eigenvalues $\frac{3}{2}, -6, 0, -\frac{5}{2}$ and decomposition

$$T \otimes \Gamma_{0,0,2} = \Gamma_{0,0,3} \oplus \Gamma_{0,0,1} \oplus \Gamma_{0,1,1} \oplus \Gamma_{1,0,1} = [112]^b \oplus T^* \oplus [112]^a \oplus \Lambda^3_{48}T^*.$$

Thus there are two pure curvature Weitzenböck formulas on sections of Λ^4_{35}. For the second we take $\frac{1}{96}F_{\mathrm{Bochner}}$ and obtain

$$\begin{aligned} q(R) &= -\tfrac{3}{2}T^*_{+\varepsilon_1}T_{+\varepsilon_1} + 6T^*_{-\varepsilon_1}T_{-\varepsilon_1} + \tfrac{5}{2}T^*_{+\varepsilon_4}T_{+\varepsilon_4},\\ 0 &= \tfrac{1}{2}T^*_{+\varepsilon_1}T_{+\varepsilon_1} - 2T^*_{-\varepsilon_1}T_{-\varepsilon_1} - T^*_{+\varepsilon_2}T_{+\varepsilon_2} + \tfrac{3}{2}T^*_{+\varepsilon_4}T_{+\varepsilon_4},\\ \Delta &= 5T^*_{-\varepsilon_1}T_{-\varepsilon_1} + 5T^*_{+\varepsilon_4}T_{+\varepsilon_4}.\end{aligned}$$

Note that in order to obtain the optimal expression for the operator Δ it was not sufficient to take its definition $\Delta = \nabla^*\nabla + q(R)$, we still had to add a multiple of the Bochner identity.

Finally, we consider the representation $\Lambda^3_{48}T^* = \Gamma_{1,0,1}$. According to the table at the end of § 2 the relevant weights are $+\varepsilon_1, -\varepsilon_1, +\varepsilon_2, -\varepsilon_3, \varepsilon_4, -\varepsilon_4$ with conformal weights or B-eigenvalues

$\frac{5}{4}, -\frac{23}{4}, \frac{1}{4}, -\frac{15}{4}, -\frac{7}{4}$ and $-\frac{11}{4}$ respectively leading to

$$\begin{aligned} T \otimes \Gamma_{1,0,1} &= \Gamma_{1,0,2} \oplus \Gamma_{1,0,0} \oplus \Gamma_{1,1,0} \oplus \Gamma_{0,1,0} \oplus \Gamma_{2,0,0} \oplus \Gamma_{0,0,2} \\ &= [189] \oplus \Lambda^2_7 T^* \oplus [105] \oplus \Lambda^2_{21} T^* \oplus \Lambda^4_{27} T^* \oplus \Lambda^4_{35} T^*. \end{aligned}$$

On sections of the associated bundle $\Lambda^3_{48} T^*M \subset \Lambda^3 T^*M$ one has three curvature Weitzenböck formulas, the formula corresponding to B and the Bochner identity $\frac{1}{84} F_{\text{Bochner}}$:

$$\begin{aligned} q(R) &= -\tfrac{5}{4} T^*_{\varepsilon_1} T_{\varepsilon_1} + \tfrac{23}{4} T^*_{-\varepsilon_1} T_{-\varepsilon_1} - \tfrac{1}{4} T^*_{\varepsilon_2} T_{\varepsilon_2} + \tfrac{15}{4} T^*_{-\varepsilon_3} T_{-\varepsilon_3} + \tfrac{7}{4} T^*_{\varepsilon_4} T_{\varepsilon_4} + \tfrac{11}{4} T^*_{-\varepsilon_4} T_{-\varepsilon_4}, \\ 0 &= \tfrac{1}{4} T^*_{\varepsilon_1} T_{\varepsilon_1} - \tfrac{45}{28} T^*_{-\varepsilon_1} T_{-\varepsilon_1} - \tfrac{3}{4} T^*_{\varepsilon_2} T_{\varepsilon_2} + \tfrac{27}{28} T^*_{-\varepsilon_3} T_{-\varepsilon_3} + \tfrac{5}{4} T^*_{\varepsilon_4} T_{\varepsilon_4} - \tfrac{9}{28} T^*_{-\varepsilon_4} T_{-\varepsilon_4}, \\ \Delta &= \tfrac{36}{7} T^*_{-\varepsilon_1} T_{-\varepsilon_1} + \tfrac{40}{7} T^*_{-\varepsilon_3} T_{-\varepsilon_3} + 4 T^*_{\varepsilon_4} T_{\varepsilon_4} + \tfrac{24}{7} T^*_{-\varepsilon_4} T_{-\varepsilon_4}. \end{aligned}$$

Consequently, a 3-form $\psi \in \Gamma(\Lambda^3_{48} T^*M)$ is harmonic if and only if its covariant derivative $\nabla\psi$ takes values in $([189] \oplus [105])M \subset T^*M \otimes \Lambda^3_{48} T^*M$ everywhere. Recall that we denote by VM the vector bundle associated to the representation V of the holonomy group G.

6. Bochner identities in $\mathbf{G_2}$- and Spin(7)-holonomies

The aim of this section is to provide a proof of the Bochner identities for the holonomies $\mathfrak{g}_2$ and $\mathfrak{spin}_7$ and thus to complete the description of the space of Weitzenböck formulas in these cases. Interestingly, it seems necessary to introduce a fairly more abstract point of view of Weitzenböck formulas in order to get to this point.

6.1 Universal Weitzenböck classes and the Kostant theorem

The essential additional twist we will employ in this section is that we will base the study of Weitzenböck formulas on the study of the action of central elements of the universal enveloping algebra. As a byproduct we get a explicit formula for a central element of order four in the universal enveloping algebra of $\mathfrak{spin}_7$ and a central element of order four in the universal enveloping algebra $\mathcal{U}\mathfrak{g}_2$.

The universal enveloping algebra $\mathcal{U}\mathfrak{g}$ of a Lie algebra $\mathfrak{g}$ is the associative algebra with 1 generated freely by the vector space $\mathfrak{g}$ subject only to the commutator relation $XY - YX = [X, Y]$. Thus $\mathcal{U}\mathfrak{g}$ is spanned by monomials of the form $X_1 \ldots X_r$ in elements $X_1, \ldots, X_r$ of $\mathfrak{g}$ and the filtration $\mathcal{U}^{\leqslant \bullet}\mathfrak{g}$ by the degree r of these monomials makes $\mathcal{U}\mathfrak{g}$ a filtered algebra. Even more important for our purposes is the Hopf algebra structure of $\mathcal{U}\mathfrak{g}$ with the cocommutative comultiplication

$$\Delta : \mathcal{U}\mathfrak{g} \longrightarrow \mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}, \quad \mathcal{Q} \longmapsto \sum \Delta_L \mathcal{Q} \otimes \Delta_R \mathcal{Q} \tag{6.27}$$

defined as the unique algebra homomorphism sending $X \in \mathfrak{g}$ to $\Delta X := X \otimes 1 + 1 \otimes X$ in $\mathcal{U}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g}$. Defining Δ in this way clearly implies that, for all $d, r \geqslant 0$,

$$\Delta(\mathcal{U}^{\leqslant d+r}\mathfrak{g}) \subset \mathcal{U}^{<d}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g} + \mathcal{U}\mathfrak{g} \otimes \mathcal{U}^{\leqslant r}\mathfrak{g}. \tag{6.28}$$

An integral part of the structure of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ is the algebra homomorphism $\mathcal{U}\mathfrak{g} \longrightarrow \operatorname{End} V$ associated to a representation V of $\mathfrak{g}$. For finite-dimensional representations V the images of these algebra homomorphisms are easily characterized.

LEMMA 6.1 (Bicommutant theorem). *Consider a finite-dimensional representation V of a semisimple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and the induced representation $\mathcal{U}\mathfrak{g} \longrightarrow \operatorname{End} V$ of $\mathcal{U}\mathfrak{g}$. The image of this algebra homomorphism is precisely the commutant of the algebra $\operatorname{End}_{\mathfrak{g}} V$ of $\mathfrak{g}$-invariant* *endomorphisms*

$$\operatorname{im}(\mathcal{U}\mathfrak{g} \longrightarrow \operatorname{End} V) = \{A \in \operatorname{End} V \mid [A, F] = 0 \text{ for all } F \in \operatorname{End}_{\mathfrak{g}} V\}.$$

In particular, the map $\operatorname{Zent} \mathcal{U}\mathfrak{g} \longrightarrow \operatorname{Zent} \operatorname{End}_{\mathfrak{g}} V$ *is surjective for* V *finite dimensional.*

The Bicommutant theorem is actually a special motivating example of von Neumann's bicommutant theorem. Observe that every $*$-subalgebra of $\operatorname{End} V$ is necessarily von Neumann for a finite-dimensional vector space V. The image of $\mathcal{U}\mathfrak{g}$ in $\operatorname{End} V$ is the subalgebra generated by the $*$-closed subspace $\mathfrak{g}$ of $\operatorname{End} V$ and thus von Neumann with commutant $\operatorname{End}_{\mathfrak{g}} V$.

Coming back to Weitzenböck formulas, we conclude by Schur's lemma that for irreducible representations V_λ the algebra homomorphism $\mathcal{U}\mathfrak{g} \longrightarrow \operatorname{End} V_\lambda$ is surjective and hence the same is true for the algebra homomorphism

$$\Phi: \operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}\mathfrak{g}) \longrightarrow \operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \operatorname{End} V_\lambda) = \mathfrak{W}(V_\lambda)$$

where $\operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \operatorname{End} V_\lambda)$ is one of the interpretation of the space $\mathfrak{W}(V_\lambda)$ of Weitzenböck formulas on $V_\lambda M$. Motivated by this surjection we will call $\operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}\mathfrak{g})$ the space of universal Weitzenböck formulas. With the universal enveloping algebra $\mathcal{U}\mathfrak{g}$ being a module over its center the space $\operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}\mathfrak{g})$ of universal Weitzenböck formulas is naturally a module for $\operatorname{Zent} \mathcal{U}\mathfrak{g}$, too, and the filtration $\operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}^{\leqslant \bullet}\mathfrak{g})$ turns it into a filtered module for the filtration $\operatorname{Zent}^{\leqslant \bullet} \mathcal{U}\mathfrak{g} := \operatorname{Zent} \mathcal{U}\mathfrak{g} \cap \mathcal{U}^{\leqslant \bullet}\mathfrak{g}$ of the center.

Definition 6.2 (Universal Weitzenböck classes). The space of universal Weitzenböck formulas $\mathfrak{W}^{\leqslant \bullet} := \operatorname{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}^{\leqslant \bullet}\mathfrak{g})$ is a filtered module over the center $\operatorname{Zent}^{\leqslant \bullet} \mathcal{U}\mathfrak{g}$ of the universal enveloping algebra $\mathcal{U}\mathfrak{g}$. It splits into the direct sum of filtered $\operatorname{Zent} \mathcal{U}\mathfrak{g}$-submodules called universal Weitzenböck classes:

$$\mathfrak{W}^{\leqslant \bullet} = \bigoplus_{\alpha} \mathfrak{W}^{\leqslant \bullet}_{W_\alpha} := \bigoplus_{\alpha} \operatorname{Hom}_{\mathfrak{g}}(W_\alpha, \mathcal{U}^{\leqslant \bullet}\mathfrak{g}).$$

It is clear from the definition that with $F \in \mathfrak{W}^{\leqslant k}$ also $F|_{W_\alpha} \in \mathfrak{W}^{\leqslant k}_{W_\alpha}$. Moreover, the powers B^k of the conformal weight operator B are in the image of $\mathfrak{W}^{\leqslant k}$ under the surjection Φ. Indeed, B^k is the image of the invariant map $p_k : T \otimes T \longrightarrow \mathcal{U}^{\leqslant k}\mathfrak{g}$ defined by

$$p_k(a \otimes b) = \sum_{\mu_1, \ldots, \mu_{k-1}} \operatorname{pr}_{\mathfrak{g}}(a \wedge t_{\mu_1}) \operatorname{pr}(t_{\mu_1} \wedge t_{\mu_2}) \cdots \operatorname{pr}_{\mathfrak{g}}(t_{\mu_{k-1}} \wedge b),$$

where $\operatorname{pr}_{\mathfrak{g}} : T \otimes T \longrightarrow \mathfrak{g} \subset \Lambda^2 T$ is the same orthogonal projection used before in the definition of B. Under the vector space identification $\mathcal{U}\mathfrak{g} \cong \operatorname{Sym} \mathfrak{g}$ we may consider $p_k(a \otimes b)$ as the polynomial $p_k(a \otimes b)[X] = \langle X^k a, b \rangle$ on $\mathfrak{g}$. It is important for our considerations below that the space of universal Weitzenböck formulas is a free module over $\operatorname{Zent} \mathcal{U}\mathfrak{g}$.

Theorem 6.3 (Kostant's theorem). *For every finite-dimensional representation* V *the space* $\operatorname{Hom}_{\mathfrak{g}}(V, \mathcal{U}\mathfrak{g})$ *is a free* $\operatorname{Zent} \mathcal{U}\mathfrak{g}$*-module, whose rank over* $\operatorname{Zent} \mathcal{U}\mathfrak{g}$ *agrees with the multiplicity of the zero weight in* V*:*

$$\operatorname{Hom}^{\leqslant \bullet}_{\mathfrak{g}}(V, \mathcal{U}\mathfrak{g}) \cong \operatorname{Zent} \mathcal{U}\mathfrak{g} \otimes \operatorname{Hom}^{\bullet}_{\mathfrak{t}}(V, \mathbb{C}).$$

In particular, the module

$$\operatorname{Hom}^{\leqslant \bullet}_{\mathfrak{g}}(\mathfrak{g}, \mathcal{U}\mathfrak{g}) \cong \operatorname{Zent} \mathcal{U}\mathfrak{g} \otimes \operatorname{Prim}^{\bullet + 1} \mathfrak{g}$$

is generated freely as a filtered $\operatorname{Zent} \mathcal{U}\mathfrak{g}$*-module by the primitive elements of* $\operatorname{Zent} \mathcal{U}\mathfrak{g}$ *with degrees shifted by* -1.

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As an example we consider holonomy $\mathfrak{g}_2$ and the spaces which are mapped under Φ onto the K-eigenspaces. We refer to Appendix A for the other holonomies. Then

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}_2}(\mathbb{C},\mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2,\\ \mathrm{Hom}_{\mathfrak{g}_2}(\mathrm{Sym}^2_0 T,\mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2\langle F_2, F_4, F_6\rangle,\\ \mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2,\mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2\langle F_1, F_5\rangle,\\ \mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2^{\perp},\mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2\langle G_3\rangle,\end{aligned}$$

where F_1, F_2, G_3, F_4, F_5 and F_6 are free generators of degree $1, 2, \ldots, 6$. The numbers of generators, i.e. the dimension of the corresponding zero weight space, can be read off from the table in (2.2). The degree of the generators, also called generalized exponents, can be obtained by decomposing $\mathrm{Sym}^k\mathfrak{g}_2$ into irreducible components (e.g. using the program LiE) and by determining the multiplicity of W_α in this decomposition for sufficiently many k. As mentioned in the theorem, the degrees of the generators F_1, F_5 are the degrees of the generators C_2, C_6 of $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2$ shifted by one.

It follows from Kostant's theorem that a basis in the eigenspace $\mathfrak{W}_{W_\alpha}(V_\lambda)$ of the classifying endomorphism K may be obtained as the image under the surjective representation map $\mathfrak{W}_{W_\alpha} \longrightarrow \mathfrak{W}_{W_\alpha}(V_\lambda)$ of certain free generators for the universal Weitzenböck classes $\mathfrak{W}_{W_\alpha}$. Indeed, the module multiplication with $\mathcal{Q}\in\mathrm{Zent}\,\mathcal{U}\mathfrak{g}$ in $\mathfrak{W}$ turns in $\mathfrak{W}(V_\lambda)$ into multiplication with the value of the central character for λ on $\mathcal{Q}$, because

$$\mathcal{Q}|_{V_\lambda} =: \chi_\lambda(\mathcal{Q})\,\mathrm{id}_{V_\lambda}, \quad \chi_\lambda(\mathcal{Q}) = \frac{1}{\dim V_\lambda}\,\mathrm{tr}_{V_\lambda}\,\mathcal{Q}.$$

In general, the value of the central character on $\mathcal{Q}\in\mathrm{Zent}\,\mathcal{U}\mathfrak{g}$ is a polynomial in the highest weight λ invariant under the Weyl group of $\mathfrak{g}$. At least in principle we know the central characters of the higher Casimirs $\mathrm{Cas}^{[k]}\in\mathrm{Zent}^{\leqslant k}\mathcal{U}\mathfrak{g}$ defined in (3.12) as traces of the powers of the conformal weight operator, since (3.13) implies that

$$\chi_\lambda(\mathrm{Cas}^{[k]}) = \sum_\varepsilon b_\varepsilon^k \frac{\dim V_{\lambda+\varepsilon}}{\dim V_\lambda}. \tag{6.29}$$

In order to proceed we use the diagonal Δ of the Hopf algebra $\mathcal{U}\mathfrak{g}$ together with the representation of $\mathcal{U}\mathfrak{g}$ on the euclidean vector space T to define an algebra homomorphism

$$\Delta:\ \mathrm{Zent}\,\mathcal{U}\mathfrak{g} \xrightarrow{\Delta} (\mathcal{U}\mathfrak{g}\otimes\mathcal{U}\mathfrak{g})^{\mathfrak{g}} \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(T\otimes T,\mathcal{U}\mathfrak{g}) = \mathfrak{W}$$

by

$$(\Delta\mathcal{Q})_{a\otimes b} = \sum \langle a, (\Delta_L\mathcal{Q})b\rangle \Delta_R\mathcal{Q} \text{ for all } \mathcal{Q}\in\mathrm{Zent}\,\mathcal{U}\mathfrak{g}.$$

A particularly nice property of Δ is that the image of $\Delta\mathcal{Q}\in\mathfrak{W}$ under the representation map $\Phi:\mathfrak{W}\longrightarrow\mathfrak{W}(V_\lambda)$ can be written in the following way:

$$\Delta\mathcal{Q} = \sum_{\varepsilon\subset\lambda} \chi_{\lambda+\varepsilon}(\mathcal{Q})\mathrm{pr}_\varepsilon \in \mathfrak{W}(V_\lambda). \tag{6.30}$$

In fact, working our way through the identification $\mathrm{Hom}_{\mathfrak{g}}(T\otimes T, \mathrm{End}\,V_\lambda) = \mathrm{End}_{\mathfrak{g}}(T\otimes V_\lambda)$ we find the usual tensor product action of $\mathcal{Q}\in\mathrm{Zent}\,\mathcal{U}\mathfrak{g}$ on $T\otimes V_\lambda$:

$$\begin{aligned}\Phi(\Delta\mathcal{Q})(b\otimes v) &= \sum_\mu t_\mu\otimes(\Delta\mathcal{Q})_{t_\mu\otimes b}v = \sum_\mu t_\mu\otimes\langle t_\mu, (\Delta_L\mathcal{Q})b\rangle\Delta_R\mathcal{Q}v\\ &= \sum (\Delta_L\mathcal{Q})b\otimes(\Delta_R\mathcal{Q})v = \Delta\mathcal{Q}(b\otimes v) = \mathcal{Q}(b\otimes v).\end{aligned}$$

The last Δ is the restriction of the comultiplication to $\mathrm{Zent}\,\mathcal{U}\mathfrak{g} \subset \mathcal{U}\mathfrak{g}$, which is precisely the action of $\mathcal{Q} \in \mathrm{Zent}\,\mathcal{U}\mathfrak{g}$ on $T \otimes V$. Hence we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Zent}\,\mathcal{U}\mathfrak{g} \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathcal{U}\mathfrak{g}) \arrow[d, "\Phi"] \arrow[r, phantom, "\cong"] & \bigoplus_\alpha \mathrm{Hom}_{\mathfrak{g}}(W_\alpha, \mathcal{U}\mathfrak{g}) \arrow[d] \\
\mathrm{End}_{\mathfrak{g}}(T \otimes V_\lambda) \arrow[r, "\cong"] & \mathrm{Hom}_{\mathfrak{g}}(T \otimes T, \mathrm{End}\,V_\lambda) \arrow[r, phantom, "\cong"] & \bigoplus_\alpha \mathrm{Hom}_{\mathfrak{g}}(W_\alpha, \mathrm{End}\,V_\lambda)
\end{tikzcd}$$

where the right vertical arrow is the restriction of the representation map Φ onto $\mathfrak{W}_{W_\alpha}$. Recall that the left square consists of algebra and the right square of $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}$-module homomorphisms, and that moreover all vertical arrows are surjective maps.

Example 6.4 (Conformal weight operator). A special case of this construction is the relation between the conformal weight operator B and the Casimir. The image of the Casimir $\mathrm{Cas}^{\Lambda^2} \in \mathrm{Zent}\,\mathcal{U}\mathfrak{g}$ becomes

$$\Delta(\mathrm{Cas}^{\Lambda^2}) = -2B + (\mathrm{Cas}_T^{\Lambda^2} + \mathrm{Cas}_{V_\lambda}^{\Lambda^2})$$

because $\Delta(X^2) = X^2 \otimes 1 + 2X \otimes X + 1 \otimes X^2$ for every $X \in \mathfrak{g}$ and Fegan's Lemma 3.3. Moreover, since Δ is an algebra homomorphism we conclude that

$$p(B) = \Delta p(-\tfrac{1}{2}\mathrm{Cas}^{\Lambda^2} + \tfrac{1}{2}(\mathrm{Cas}_T^{\Lambda^2} + \mathrm{Cas}_{V_\lambda}^{\Lambda^2}))$$

for every polynomial $p(B)$ in the conformal weight operator B. In particular, the space of polynomials in B is in the image under Δ of the subalgebra generated by Cas^{Λ^2}.

The crucial additional information we get from introducing the universal Weitzenböck classes is the filtration degree of the generators of the $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}$-modules $\mathfrak{W}_{W_\alpha}$. In order to prove the Bochner identities for holonomy $\mathfrak{spin}_7$ we still need the following lemma.

Lemma 6.5 (Filtration property of Δ). *Consider the Weitzenböck class $\mathfrak{W}_{W_\alpha}$ associated to an irreducible subspace $W_\alpha \subset T \otimes T$. If there is no non-trivial, $\mathfrak{g}$-equivariant map from W_α to $\mathcal{U}^{<d}\mathfrak{g}$ for some $d \geqslant 1$, i.e. if $\mathfrak{W}_{W_\alpha}^{<d} = \{0\}$, then the composition of Δ with the restriction res_{W_α} to $\mathfrak{W}_\alpha$ is filtered*

$$\mathrm{res}_{W_\alpha} \circ \Delta : \ \mathrm{Zent}^{\leqslant d+\bullet}\mathcal{U}\mathfrak{g} \longrightarrow \mathfrak{W}_{W_\alpha}^{\leqslant \bullet}, \qquad \mathcal{Q} \longmapsto \Delta\mathcal{Q}|_{W_\alpha}$$

of degree $-d$. In particular, the restriction $\Delta\mathcal{Q}|_{W_\alpha} = 0$ vanishes for all $\mathcal{Q} \in \mathrm{Zent}^{<2d}\mathcal{U}\mathfrak{g}$.

Proof. By the filtration property (6.28) of the comultiplication we can write the diagonal $\Delta\mathcal{Q}$ of an element $\mathcal{Q} \in \mathrm{Zent}^{\leqslant d+r}\mathcal{U}\mathfrak{g}$ in a not necessarily unique way as a sum of two terms $\Delta\mathcal{Q} = \Delta\mathcal{Q}^{<d} + \Delta\mathcal{Q}^{\leqslant r}$ satisfying $\Delta\mathcal{Q}^{<d} \in (\mathcal{U}^{<d}\mathfrak{g} \otimes \mathcal{U}\mathfrak{g})^{\mathfrak{g}}$ and $\Delta\mathcal{Q}^{\leqslant r} \in (\mathcal{U}\mathfrak{g} \otimes \mathcal{U}^{\leqslant r}\mathfrak{g})^{\mathfrak{g}}$ respectively. Both these summands give rise to $\mathfrak{g}$-equivariant, linear maps $T \otimes T \longrightarrow \mathcal{U}\mathfrak{g}$ through the pairing of $T \otimes T$ with the left $\mathcal{U}\mathfrak{g}$-factor, explicitly

$$(\Delta\mathcal{Q}^{<d})_{a \otimes b} := \sum \langle a, \Delta\mathcal{Q}_L^{<d} b\rangle \Delta\mathcal{Q}_R^{<d}$$

with essentially the same formula for $\Delta\mathcal{Q}^{\leqslant r}$. By construction the map $T \otimes T \longrightarrow \mathcal{U}\mathfrak{g}$ associated to $\Delta\mathcal{Q}^{\leqslant r}$ maps into $\mathcal{U}^{\leqslant r}\mathfrak{g}$, while the map associated to $\Delta\mathcal{Q}^{<d}$ vanishes upon restriction to $W_\alpha \subset T \otimes T$, because by assumption there is no non-trivial, $\mathfrak{g}$-invariant, pairing of W_α with the left image of $\Delta\mathcal{Q}^{<d}$ defined by

$$\mathrm{span}\{\Delta\mathcal{Q}_L^{<d} \mid \Delta\mathcal{Q}^{<d} = \sum \Delta\mathcal{Q}_L^{<d} \otimes \Delta\mathcal{Q}_R^{<d}\} \subset \mathcal{U}^{<d}\mathfrak{g}.$$

For the second statement we note that $\Delta\mathcal{Q}|_{W_\alpha} \in \mathfrak{W}_\alpha^{<d} = \{0\}$ for all $\mathcal{Q} \in \mathrm{Zent}^{\leqslant 2d-1}\mathcal{U}\mathfrak{g}$. □

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6.2 Proof of the Bochner identities in holonomies $\mathfrak{g}_2$ and $\mathfrak{spin}_7$

Let us now discuss the details of the proof of the additional Bochner identity in $\mathbf{G}_2$-holonomy. Applying the Gram–Schmidt orthogonalization process of Corollary 4.3 to the powers $1, B, B^2$ and B^3 of the conformal weight operator, we obtained in (4.19) and (4.20) a sequence $p_0(B), p_1(B), p_2(B), p_3(B)$ of τ-eigenvectors. In order to proceed with the recursion procedure it remains to be shown that $p_3(B)$ is a K-eigenvector.

We know that $p_3(B)$ is a -1 eigenvector of τ, orthogonal to B and expressible as a polynomial in B of degree three. Thus $p_3(B)$ is an element in the image of $\mathfrak{W}^{\leqslant 3}$ in $\mathfrak{W}(V_\lambda)$ and can be written as a sum $p_3(B) = p_3(B)_{\mathfrak{g}_2} + p_3(B)_{\mathfrak{g}_2^\perp}$ of two vectors $p_3(B)_{\mathfrak{g}_2}$ and $p_3(B)_{\mathfrak{g}_2^\perp}$ in the image of $\mathfrak{W}^{\leqslant 3}_{\mathfrak{g}_2}$ and $\mathfrak{W}^{\leqslant 3}_{\mathfrak{g}_2^\perp}$ in $\mathfrak{W}(V_\lambda)$ respectively. However, the image of $\mathfrak{W}^{\leqslant 3}_{\mathfrak{g}_2}$ in $\mathfrak{W}(V_\lambda)$ is spanned by B, because the filtered $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2$-module $\mathfrak{W}_{\mathfrak{g}_2}$ is generated by two elements in degrees one and five and the representation $\mathfrak{W}_{\mathfrak{g}_2} \longrightarrow \mathfrak{W}_{\mathfrak{g}_2}(V_\lambda)$ turns module multiplication into multiplication by the central character χ_λ. Consequently, the vector $p_3(B)$ is orthogonal to the image of $\mathfrak{W}^{\leqslant 3}_{\mathfrak{g}_2}$ in $\mathfrak{W}(V_\lambda)$ and lies in the eigenspace $\mathfrak{W}_{\mathfrak{g}_2^\perp}(V_\lambda)$ of the classifying endomorphism K.

THEOREM 6.6 (Bochner identity in $\mathbf{G}_2$-holonomy). *The following cubic polynomial in the conformal weight operator B defines an eigenvector for the classifying endomorphisms K of eigenvalue -2:*

$$p_3(B) := B^3 + \tfrac{13}{3}B^2 + (\tfrac{1}{2}\mathrm{c}^{\Lambda^2}_{V_\lambda} + 4)B + \tfrac{2}{3}\mathrm{c}^{\Lambda^2}_{V_\lambda}.$$

Inserting the eigenvalues or conformal weights b_ε of B we arrive at (4.21).

In the last part of this section we will prove the Bochner identity for holonomy $\mathfrak{spin}_7$. As in the $\mathfrak{g}_2$ case we apply the Gram–Schmidt orthogonalization process of Corollary 4.3 to the powers $1, B, B^2$ and B^3 and obtain in (4.22) and (4.23) a sequence $p_0(B)$, $p_1(B)$, $p_2(B)$, $p_3(B)$ of τ-eigenvectors. Again $p_3(B)$ is a (-1)-eigenvector of τ, orthogonal to B and expressible as a polynomial in B of degree three so that the summands in the decomposition $p_3(B) = p_3(B)_{\mathfrak{spin}_7} + p_3(B)_{\mathfrak{spin}_7^\perp}$ are in the image of $\mathfrak{W}^{\leqslant 3}_{\mathfrak{spin}_7}$ and $\mathfrak{W}^{\leqslant 3}_{\mathfrak{spin}_7^\perp}$ in $\mathfrak{W}(V_\lambda)$ respectively. Of course, we want to extract the Bochner identity $p_3(B)_{\mathfrak{spin}_7^\perp}$ from $p_3(B)$. At this point the argument in the $\mathbf{Spin}(7)$ case becomes more complicated, because the $\mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7$-module $\mathfrak{W}_{\mathfrak{spin}_7}$ has generators in degree one, three and five so that the image of $\mathfrak{W}^{\leqslant 3}_{\mathfrak{spin}_7}$ in $\mathfrak{W}_{\mathfrak{spin}_7}(V_\lambda)$ has dimension two. Even with $p_3(B)$ orthogonal to B we may thus not conclude that the component $p_3(B)_{\mathfrak{spin}_7} = 0$ vanishes. The idea to cope with this complication is to construct an element $Q_\lambda \in \mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7$ depending polynomially on the highest weight λ such that $\Delta Q_\lambda \in \mathfrak{W}_{\mathfrak{spin}_7}(V_\lambda)$ is orthogonal to B. The Bochner identity is then the projection of $p_3(B)$ onto the orthogonal complement of ΔQ_λ.

In the four-dimensional space $\mathrm{Zent}^{\leqslant 4}\mathcal{U}\mathfrak{spin}_7$ we look for an element $\mathcal{Q}_\lambda$ as a linear combination of the base vectors $\mathbf{1}, \mathrm{Cas}, \mathrm{Cas}^2$ and $\mathrm{Cas}^{[4]}$ with unknown coefficients. We know $\Delta\mathrm{Cas}$ and $\Delta\mathrm{Cas}^2$ from Example 6.4 and $\Delta\mathrm{Cas}^{[4]}$ from (6.29) and (6.30) so that the conditions

$$\langle \Delta Q_\lambda, \mathbf{1}\rangle = 0, \quad \langle \Delta Q_\lambda, B\rangle = 0, \quad \langle \Delta Q_\lambda, B^2\rangle = 0$$

turn into three linear independent equations for the four unknown coefficients. Using a computer algebra system to do the necessary calculations, we find the convenient solution

$$\begin{aligned}\mathcal{Q}_\lambda = {}& 2\mathrm{c}^{\Lambda^2}_{V_\lambda}\mathrm{Cas}^{[4]} - 160\mathrm{c}^{\Lambda^2}_{V_\lambda}(\mathrm{Cas}^{\Lambda^2})^2 + (320(\mathrm{c}^{\Lambda^2}_{V_\lambda})^2 - 1184\mathrm{c}^{\Lambda^2}_{V_\lambda} - 4\mathrm{c}^{[4]}_{V_\lambda})\mathrm{Cas}^{\Lambda^2}\\ &+ (-160(\mathrm{c}^{\Lambda^2}_{V_\lambda})^3 + 2\mathrm{c}^{\Lambda^2}_{V_\lambda}\mathrm{c}^{[4]}_{V_\lambda} + 1712(\mathrm{c}^{\Lambda^2}_{V_\lambda})^2 - 9408\mathrm{c}^{\Lambda^2}_{V_\lambda} - 21\mathrm{c}^{[4]}_{V_\lambda})\end{aligned}$$

in $\mathrm{Zent}^{\leqslant 4}\mathcal{U}\mathfrak{spin}_7$. We denote the eigenvalue of the central element $\mathrm{Cas}^{[4]} \in \mathcal{U}\mathfrak{spin}_7$ on the irreducible representation V_λ by $\mathrm{c}^{[4]}_{V_\lambda}$ in analogy with the eigenvalues of Cas^{Λ^2}.

By construction $\Delta\mathcal{Q}_\lambda$ is orthogonal to $1, B$ and B^2 and we conclude that ΔQ_λ is indeed an eigenvector for the classifying endomorphism K. In fact, the component $\Delta\mathcal{Q}_\lambda|_{\mathfrak{spin}_7^\perp} = 0$ vanishes according to Lemma 6.5, because $\mathcal{Q}_\lambda$ has degree four and there is no non-trivial equivariant map $\mathfrak{spin}_7^\perp \longrightarrow \mathcal{U}^{<3}\mathfrak{spin}_7$. Similarly the components $\Delta\mathcal{Q}_\lambda|_{\mathrm{Sym}^2_\circ T} = 0$ and $\Delta\mathcal{Q}_\lambda|_{\mathbb{C}} = 0$ are trivial, since the image of $\mathfrak{W}^{\leqslant 2}_{\mathrm{Sym}^2_\circ T}$ in $\mathfrak{W}(V_\lambda)$ has dimension one and is spanned by $p_2(B)$, while $\mathfrak{W}_{\mathbb{C}}(V_\lambda)$ is spanned by $p_0(B)$. With $\Delta\mathcal{Q}_\lambda = \Delta\mathcal{Q}_\lambda|_{\mathfrak{spin}_7}$ being an eigenvector of K orthogonal to $p_1(B) = B$ the problematic component $p_3(B)_{\mathfrak{spin}_7}$ of $p_3(B)$ must be a multiple of $\Delta\mathcal{Q}_\lambda$. In consequence, the complementary component $p_3(B)_{\mathfrak{spin}_7^\perp}$ of $p_3(B)$ is the projection of $p_3(B)$ onto the orthogonal complement of ΔQ_λ and may serve as the $\mathfrak{spin}_7$-Bochner identity. Using again a computer algebra system for the necessary calculations we find that this projection of $p_3(B)$ to the orthogonal complement of $\Delta\mathcal{Q}_\lambda$ agrees with the endomorphism $F_{\mathrm{Bochner}} \in \mathfrak{W}(V_\lambda)$ specified in (4.24).

THEOREM 6.7 (Bochner identity in **Spin**(7)-holonomy). *The endomorphism $F_{\mathrm{Bochner}} \in \mathfrak{W}(V_\lambda)$ defined in (4.24) with components*

$$\begin{aligned} F_{\mathrm{Bochner}} = &+c(2b+c+2)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_1} \\ &-(c+2)(2b+c+4)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_1} \\ &-(c+2)(2b+c+2)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_2} \\ &+c(2b+c+4)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_2} \\ &-c(2b+c+4)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_3} \\ &+(c+2)(2b+c+2)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_3} \\ &+(c+2)(2b+c+4)(2a+2b+c+4)\mathrm{pr}_{+\varepsilon_4} \\ &-c(2b+c+2)(2a+2b+c+6)\mathrm{pr}_{-\varepsilon_4} \end{aligned}$$

is an eigenvector of the classifying endomorphism K for the eigenvalue $-\frac{9}{4}$.

Appendix A. Module generators and higher Casimirs

Remark A.1 (Module generators for $\mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r+1}$). The center of the universal enveloping algebra of $\mathfrak{so}_{2r+1}, r \geqslant 1$, is a free polynomial algebra $\mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r+1} = \mathbb{C}[P^{[2]}, P^{[4]}, \ldots, P^{[2r]}]$ in r generators of degree $2, 4, \ldots, 2r$. Moreover,

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{so}_{2r+1}}(\mathbb{C}, \mathcal{U}\mathfrak{so}_{2r+1}) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r+1}, \\ \mathrm{Hom}_{\mathfrak{so}_{2r+1}}(\mathrm{Sym}^2_0 T, \mathcal{U}\mathfrak{so}_{2r+1}) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r+1}\langle F_2, F_4, \ldots, F_{2r}\rangle, \\ \mathrm{Hom}_{\mathfrak{so}_{2r+1}}(\mathfrak{so}_{2r+1}, \mathcal{U}\mathfrak{so}_{2r+1}) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r+1}\langle F_1, F_3, \ldots, F_{2r-1}\rangle. \end{aligned}$$

Remark A.2 (Module generators for $\mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r}$). The center of the universal enveloping algebra $\mathcal{U}\mathfrak{so}_{2r}$ of $\mathfrak{so}_{2r}, r \geqslant 2$, is a free polynomial algebra $\mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r} = \mathbb{C}[P^{[2]}, P^{[4]}, \ldots, P^{[2r-2]}, E^{[r]}]$ in $r-1$ generators of degree $2, 4, \ldots, 2r-2$ respectively and one additional generator in degree r. Moreover,

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{so}_{2r}}(\mathbb{C}, \mathcal{U}\mathfrak{so}_{2r}) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r}, \\ \mathrm{Hom}_{\mathfrak{so}_{2r}}(\mathrm{Sym}^2_0 T, \mathcal{U}\mathfrak{so}_{2r}) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r}\langle F_2, F_4, \ldots, F_{2r-2}\rangle, \\ \mathrm{Hom}_{\mathfrak{so}_{2r}}(\mathfrak{so}_{2r}, \mathcal{U}\mathfrak{so}_{2r}) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{so}_{2r}\langle F_1, F_3, \ldots, F_{2r-3}, G_{r-1}\rangle. \end{aligned}$$

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Remark A.3 (Module generators for $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2$). The center of the universal enveloping algebra $\mathcal{U}\mathfrak{g}_2$ of $\mathfrak{g}_2$ is a free polynomial algebra $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2 = \mathbb{C}[\mathrm{Cas}^{[2]}, \mathrm{Cas}^{[6]}]$ in two generators of degree two and six. Moreover,

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{g}_2}(\mathbb{C}, \mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2,\\
\mathrm{Hom}_{\mathfrak{g}_2}(\mathrm{Sym}^2_0 T, \mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2\langle F_2, F_4, F_6\rangle,\\
\mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2, \mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2\langle F_1, F_5\rangle,\\
\mathrm{Hom}_{\mathfrak{g}_2}(\mathfrak{g}_2^{\perp}, \mathcal{U}\mathfrak{g}_2) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2\langle G_3\rangle.
\end{aligned}$$

Remark A.4 (Module generators for $\mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7$). The center of the universal enveloping algebra $\mathcal{U}\mathfrak{spin}_7$ of $\mathfrak{spin}_7$ is a free polynomial algebra $\mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7 = \mathbb{C}[\mathrm{Cas}^{[2]}, \mathrm{Cas}^{[4]}, \mathrm{Cas}^{[6]}]$ in three generators of degree two, four and six. Moreover,

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{spin}_7}(\mathbb{C}, \mathcal{U}\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7,\\
\mathrm{Hom}_{\mathfrak{spin}_7}(\mathrm{Sym}^2_0 T, \mathcal{U}\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7\langle F_2, F_4, F_6\rangle,\\
\mathrm{Hom}_{\mathfrak{spin}_7}(\mathfrak{spin}_7, \mathcal{U}\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7\langle F_1, F_3, F_5\rangle,\\
\mathrm{Hom}_{\mathfrak{spin}_7}(\mathfrak{spin}_7^{\perp}, \mathcal{U}\mathfrak{spin}_7) &\cong \mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7\langle G_3\rangle.
\end{aligned}$$

Remark A.5 (Higher Casimirs for $\mathbf{G}_2$). The eigenvalues of the generators $\mathrm{Cas}^{[2]}$ and $\mathrm{Cas}^{[6]}$ of $\mathrm{Zent}\,\mathcal{U}\mathfrak{g}_2$ of degrees two and six respectively on the irreducible representation V_λ of highest weight $\lambda = a\omega_1 + b\omega_2$ are given by

$$\begin{aligned}
\tfrac{3}{4}\mathrm{c}^{[2]}_{V_\lambda} &= a^2 + 3ab + 3b^2 + 5a + 9b,\\
\tfrac{243}{11}\mathrm{c}^{[6]}_{V_\lambda} &= 4a^6 + 36a^5b + 117a^4b^2 + 162a^3b^3 + 81a^2b^4 + 60a^5 + 414a^4b + 954a^3b^2\\
&\quad + 810a^2b^3 + 162ab^4 - 408a^4 - 2808a^3b - 8829a^2b^2 - 12\,636ab^3 - 6804b^4\\
&\quad - 6580a^3 - 33\,174a^2b - 61\,362ab^2 - 40\,824b^3 - 6396a^2 - 32\,508ab\\
&\quad - 27\,756b^2 + 56\,520a + 100\,440b.
\end{aligned}$$

Remark A.6 (Higher Casimirs for $\mathbf{Spin}(7)$). The eigenvalues of the generators $\mathrm{Cas}^{[2]}$, $\mathrm{Cas}^{[4]}$ and $\mathrm{Cas}^{[6]}$ of $\mathrm{Zent}\,\mathcal{U}\mathfrak{spin}_7$ of degrees two, four and six respectively on the irreducible representation V_λ of highest weight $\lambda = a\omega_1 + b\omega_2 + c\omega_3$ are

$$\begin{aligned}
2\mathrm{c}^{[2]}_{V_\lambda} &= 4a^2 + 8b^2 + 3c^2 + 8ab + 4ac + 8bc + 20a + 32b + 18c,\\
32\mathrm{c}^{[4]}_{V_\lambda} &= 16a^4 + 128b^4 + 21c^4 + 192a^2b^2 + 72a^2c^2 + 240b^2c^2 + 32a^3c + 64a^3b + 256b^3c\\
&\quad + 256b^3a + 56c^3a + 112c^3b + 192a^2bc + 384b^2ac + 240c^2ab + 160a^3 + 1024b^3\\
&\quad + 252c^3 + 768a^2b + 432a^2c + 1536b^2a + 1632b^2c + 1056c^2b + 552c^2a\\
&\quad + 1632abc + 800a^2 + 1152c^2 + 3040b^2 + 3040ab + 1760ac + 3424bc\\
&\quad + 2000a + 3968b + 2376c,\\
512\mathrm{c}^{[6]}_{V_\lambda} &= 64a^6 + 2048b^6 + 183c^6 + 384a^5b + 192a^5c + 6144b^5c + 6144b^5a + 732c^5a\\
&\quad + 1464c^5b + 1920a^4b^2 + 720a^4c^2 + 7680b^4a^2 + 9600b^4c^2 + 1260c^4a^2 + 4920c^4b^2\\
&\quad + 1920a^4bc + 15\,360b^4ac + 4920c^4ab + 5120a^3b^3 + 1120a^3c^3 + 8960b^3c^3\\
&\quad + 7680a^3b^2c + 4800a^3c^2b + 15\,360b^3a^2c + 19\,200b^3c^2a + 6720c^3a^2b + 13\,440c^3b^2a\\
&\quad + 14\,400a^2b^2c^2 + 960a^5 + 24\,576b^5 + 3294c^5 + 7680a^4b + 4320a^4c + 61\,440b^4a\\
&\quad + 65\,280b^4c + 11\,100c^4a + 22\,080c^4b + 30\,720a^3b^2 + 11\,040a^3c^2 + 61\,440b^3a^2
\end{aligned}$$

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$$
\begin{aligned}
&+ 84\,480b^3c^2 + 15\,120c^3a^2 + 60\,000c^3b^2 + 32\,640a^3bc + 130\,560b^3ac + 60\,000c^3ab \\
&+ 97\,920a^2b^2c + 63\,360a^2c^2b + 126\,720b^2c^2a + 9600a^4 + 167\,424b^4 + 32592c^4 \\
&+ 67\,200a^3b + 38\,400a^3c + 334\,848b^3a + 365\,568b^3c + 88\,032c^3a + 175\,584c^3b \\
&+ 234\,624a^2b^2 + 92\,832a^2c^2 + 364\,128b^2c^2 + 257\,664a^2bc + 548\,352b^2ac \\
&+ 364\,128c^2ab + 56\,000a^3 + 684\,032b^3 + 193\,464c^3 + 413\,952a^2b + 251\,808a^2c \\
&+ 993\,024b^2a + 1158\,912b^2c + 397\,968c^2a + 790\,656c^2b + 1125\,888abc + 160\,000a^2 \\
&+ 1321\,856b^2 + 562\,848c^2 + 1189\,760ab + 759\,040ac + 1607\,552bc + 200\,000a \\
&+ 863\,744b + 606\,240c.
\end{aligned}
$$

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