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Nagoya Math. J.
Vol. 175 (2004), 165–170

BERGMAN COMPLETENESS OF HYPERCONVEX MANIFOLDS

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Abstract. We proved that any hyperconvex manifold has a complete Bergman metric.

§1. Introduction

Let M be an $n-$dimensional complex manifold. Let $\mathcal{H}$ denote the Hilbert space of holomorphic $n-$forms on M such that $|\int_M f \wedge \bar{f}| < \infty$. Let $h_0, h_1, \cdots$ be a complete orthonormal basis for $\mathcal{H}$. Then the $2n-$form defined on $M \times M$ given by $K_M = \sum_{j=0}^{\infty} h_j \wedge \bar{h}_j$ is called the Bergman kernel form of M. Let $z = (z_1, \cdots, z_n)$ be a local coordinate system in M and let $K_M(z) = K_M^*(z) dz_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_1 \wedge \cdots \wedge d\bar{z}_n$ where K_M^* is a locally defined function. Then $\beta := \partial\bar{\partial} \log K_M^*$ is a well-defined Hermitian form of bidegree (1,1), whenever K_M^* is nonzero. We call β the Bergman metric if it is everywhere positive definite. Let us recall

Definition. A complex manifold M is called hyperconvex if there exists a strictly plurisubharmonic (psh) function $\rho : M \to [-1, 0)$ such that $\{x \in M : \rho(x) < c\}$ is relatively compact in M for every $c < 0$.

The purpose of this note is to show the following

Theorem 1. *Every hyperconvex manifold has a complete Bergman metric.*

Theorem 1 was conjectured by S. Kobayashi [11]. In the special case of bounded hyperconvex domains $\Omega \subset \mathbf{C}^n$, it suffices to show that the volume of $\{g_\Omega(\cdot, y) < -1\}$ tends to zero as $y \to \partial\Omega$, where $g_\Omega(\cdot, y)$ denotes the

Received July 16, 2003.
2000 Mathematics Subject Classification: 32F45.
*Supported by JSPS.

pluricomplex Green function of Ω (cf. Chen [3] and Herbort [8] independently), and this property was verified by Blocki-Pflug [2] (independently Herbort [8]). The case of hyperconvex Riemann surfaces was shown in [4].

Combining with a theorem of Ohsawa-Sibony [13], we obtain

COROLLARY 2. *Every bounded pseudoconvex domain with C^2 boundary in a complex manifold with positive holomorphic bisectional curvature (eg.* $\mathbf{P}^n$*) is Bergman complete.*

Greene-Wu [7] proved the the existence of a bounded smooth strictly psh exhaustion function under the following curvature condition. Hence

COROLLARY 3. *Let M be a complete Kähler manifold with a pole o such that its sectional curvature* K *is non-positive and in addition satisfies*

$$\mathrm{K} \leq -\frac{1+\epsilon}{r^2 \log r}$$

for some constant $\epsilon > 0$ outside a compact subset of M, where r denotes the distance function based at o. Then M has a complete Bergman metric.

In [7], Bergman completeness has been shown in the case when M is a simply-connected complete Kähler manifold such that the sectional curvature is suitably negatively pinched, for instance, pinched between negative constants. Their result was extended in [4] by only assuming that the curvature is bounded from above by $-A/r^2$.

This work was done during the author's stay at Nagoya University. He would like to thank the Department of Mathematics for their hospitality, especially to Professor Takeo Ohsawa for his constant encouragement and discussion. The author also thanks the referee for valuable suggestions.

§2. Proof of Theorem 1

Let $g_M(\cdot, y)$ be the pluricomplex Green function on M, i.e.,

$$g_M(x, y) = \sup\{u(x)\}$$

where the superum is taken over all negative functions $u \in PSH(M)$ satisfying the property that the function $u - \log|z|$ is bounded from above in a deleted neighborhood of y for some holomorphic local coordinates z centered at y, that is, $z(y) = 0$. Since M is hyperconvex, $g_M(\cdot, y)$ is non-trivial (cf. [4]). Set $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$. It is easy to see that $dd^c = 2i\partial\bar{\partial}$. As in [2], the following inequality of Blocki is again crucial.

PROPOSITION 4. (cf.[1]) *Let Ω be a smooth bounded domain in a complex manifold M. Assume that $u, v \in C^\infty(\overline{\Omega})$ are non-positive psh functions such that $u = 0$ on $\partial\Omega$. Then*

$$\int_\Omega |u|^n (dd^c v)^n \le n! \|v\|_\infty^{n-1} \int_\Omega |v| (dd^c u)^n.$$

Proof. Note that

$$\begin{aligned}\int_\Omega (-u)^n (dd^c v)^n &= n \int_\Omega (-u)^{n-1} du \wedge d^c v \wedge (dd^c v)^{n-1} \\ &= n \int_\Omega (-u)^{n-1} dv \wedge d^c u \wedge (dd^c v)^{n-1} \\ &= n \int_\Omega (-u)^{n-1} (-v) dd^c u \wedge (dd^c v)^{n-1} \\ &\quad + n(n-1) \int_\Omega (-u)^{n-2} v du \wedge d^c u \wedge (dd^c v)^{n-1} \\ &\le n \|v\|_\infty \int_\Omega (-u)^{n-1} dd^c u \wedge (dd^c v)^{n-1}\end{aligned}$$

where the first and third equalities follow from Stokes' theorem and the second one from the fact that the $(1,1)$ parts of $du \wedge d^c v$ and $dv \wedge d^c u$ coincide, the inequality follows from $du \wedge d^c u = 2i\partial u \wedge \bar\partial u \ge 0$. The desired inequality is obtained by repeating the argument $n-1$ times.

LEMMA 5. *Let M be a hyperconvex manifold. For any $y \in M$, the following inequality holds:*

$$\int_M |g_M(\cdot, y)|^n (dd^c \rho)^n \le n! (2\pi)^n |\rho(y)|.$$

Proof. For any positive integer j, we set $M_j = \{x \in M : \rho(x) < 1/j\}$. Let $y \in M$ and let $g_{M_j}(\cdot, y)$ denote the pluricomplex Green function on M_j for all sufficiently large j. Since M_j is hyperconvex, for any fixed j, the function $\max\{g_{M_j}(\cdot, y), -k\} + \frac{1}{k}(\rho - 1/j)$ is a continuous strictly psh function and approaches to zero at ∂M_j for each integer $k > 0$ (cf. [6]). According to a well-known theorem of Richberg [14], there is a smooth psh function $\{g_{j,k}\}$ on M_j such that

$$\left| g_{j,k} - \max\{g_{M_j}(\cdot, y), -k\} - \frac{1}{k}(\rho - 1/j) \right| < \frac{1}{2k} |\rho - 1/j|,$$

which implies $g_{j,k} < 0$, $g_{j,k}(x) \to 0$ as $x \to \partial M_j$ and $g_{j,k} \to g_{M_j}(\cdot, y)$ as $k \to \infty$. Hence we can take a sequence of positive numbers $\{\lambda_k\}$ with $\lambda_k \to 0$ as $k \to \infty$ such that

$$M_{j-1} \subset\subset M_{j,k} := \{x \in M_j : g_{j,k}(x) < -\lambda_k\} \subset\subset M_j$$

and $M_{j,k}$ has a smooth boundary by Sard's theorem. By Proposition 4, we have

$$\begin{aligned}\int_{M_{j-1}} |g_{j,k} + \lambda_k|^n (dd^c\rho)^n &\leq \int_{M_{j,k}} |g_{j,k} + \lambda_k|^n (dd^c\rho)^n \\ &\leq n! \int_{M_{j,k}} |\rho| (dd^c g_{j,k})^n \\ &\leq n! \int_{M_j} |\rho| (dd^c g_{j,k})^n.\end{aligned}$$

According to [6], letting $k \to \infty$, we obtain

$$\begin{aligned}\int_{M_{j-1}} |g_{M_j}(\cdot, y)|^n (dd^c\rho)^n &\leq n! \int_{M_j} |\rho| (dd^c g_{M_j}(\cdot, y))^n \\ &= n!(2\pi)^n |\rho(y)|\end{aligned}$$

where the equality follows from $(dd^c g_{M_j}(\cdot, y))^n = (2\pi)^n \delta_y$ on M_j. The desired inequality is then obtained by letting $j \to \infty$ since $g_{M_j}(\cdot, y) \searrow g_M(\cdot, y)$.

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Proof of Theorem 1. The existence of the Bergman metric of a hyperconvex manifold was shown in [4]. Take a smooth function χ on $\mathbf{R}$ such that $\chi = 1$ on $(-\infty, -1]$ and $\chi = 0$ on $[0, \infty)$. Let $f \in \mathcal{H}$ and $\{y_k\}_{k=1}^{\infty}$ be a sequence of points which has no adherent point in M. Set

$$\begin{aligned}\eta_k &= \chi\left(-\log(-g_M(\cdot, y_k) + 1) + \log 2\right) f. \\ \varphi_k &= 2n g_M(\cdot, y_k) - \log(-g_M(\cdot, y_k) + 1)\end{aligned}$$

Let us first proceed the proof under the assumption that η_k, φ_k are smooth and φ_k is strictly psh. By the well-known L^2 estimates (cf. [5], [12]), we can solve the equation $\overline{\partial} u_k = \overline{\partial} \eta_k$ in such a way that

$$\begin{aligned}\left|\int_M u_k \wedge \bar{u}_k e^{-\varphi_k}\right| &\le \int_M |\overline{\partial}\eta_k|^2_{\partial\bar{\partial}\varphi_k} e^{-\varphi_k} dV_{\varphi_k} \\ &\le C_1 \left|\int_{A_k} f \wedge \overline{f}\right|\end{aligned}$$

since

$$\partial\bar{\partial}\varphi_k \ge \frac{\partial g_M(\cdot, y_k)\bar{\partial} g_M(\cdot, y_k)}{(-g_M(\cdot, y_k) + 1)^2}.$$

Here C_1 is a constant depending only on $\sup|\chi'|$ and $A_k = \{x \in M : g_M(\cdot, y_k) < -1\}$. The general case follows from a standard limiting procedure as follows: By a similar argument as in the proof of Lemma 5, one can approximate $g_M(\cdot, y_k)$ by a sequence of negative smooth strictly psh functions on M and solve the $\bar{\partial}-$equation with $g_M(\cdot, y_k)$ replaced by such functions, then take a limit.

Hence the function $F_k = \eta_k - u_k$ is holomorphic on M which satisfies $F_k(y_k) = f(y_k)$ and $\left|\int_M F_k \wedge \overline{F}_k\right| \le C_2 \left|\int_{A_k} f \wedge \overline{f}\right|$. It follows that

$$\frac{f(y_k) \wedge \bar{f}(y_k)}{K_M(y_k)} \le C_2 \left|\int_{A_k} f \wedge \overline{f}\right|. \tag{1}$$

For any $\epsilon > 0$, there is a relative compact subset M_ϵ so that

$$\left|\int_{M \setminus M_\epsilon} f \wedge \bar{f}\right| < \epsilon. \tag{2}$$

By Lemma 5, we have

$$\begin{aligned}\int_{M_\epsilon \cap A_k} (dd^c \rho)^n &\le \int_M |g_M(\cdot, y_k)|^n (dd^c\rho)^n \\ &\le n!(2\pi)^n |\rho(y_k)|.\end{aligned}$$

This shows that one can choose a k_ϵ such that for all $k > k_\epsilon$,

$$\left|\int_{M_\epsilon \cap A_k} f \wedge \bar{f}\right| \le \sup_{M_\epsilon} \left|\frac{f \wedge \bar{f}}{(dd^c\rho)^n}\right| \cdot \int_{M_\epsilon \cap A_k} (dd^c\rho)^n < \epsilon \tag{3}$$

since $\rho(y_k) \to 0$ as $k \to \infty$. By (1)–(3) and the well-known Kobayashi's criterion [10], the Bergman metric on M is complete.

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