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DECAYING SOLUTIONS OF $2m^{\text{th}}$ ORDER ELLIPTIC PROBLEMS

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ABSTRACT. We consider a semilinear elliptic problem $(-\Delta)^m u = f(x,u)$ in R^n, $(n > 2m)$. Under suitable conditions on f, we show the existence of a decaying positive solution. We do not employ radial arguments. Our main tools are weighted spaces, various applications of the Mountain Pass Theorem and L^p regularity estimates of Agmon. We answer an open question of Kusano, Naito and Swanson [Canad. J. Math. **40**(1988), 1281–1300] in the superlinear case: $|f(x,u)| \leq g(x)u^s$ $(1 < s < \frac{n+2m}{n-2m})$, and improve the results of Dalmasso [C. R. Acad. Sci. Paris **308**(1989), 411–414] for the case $m = 2, f(x,u) = g(|x|)u^s$ $(1 < s < \frac{n+4}{n-4})$.

1. **Introduction.** We are concerned in this paper with the existence of nontrivial positive decaying solutions to the $2m^{\text{th}}$ order elliptic problem:

$$(1) \qquad \begin{cases} \ell u = f(x,u), & x \in R^n \\ \lim\limits_{|x| \to \infty} u(x) = 0 \end{cases}$$

where $\ell = (-\Delta)^m$, Δ denotes the Laplacean and $n > 2m$. Of special interest to us is the case of $f(x,t)$ purely superlinear with subcritical growth:

$$|f(x,t)| \leq g(x)t^s$$

for $1 < s < (n+2m)/(n-2m)$ and some $g(x)$ to be specified below. At the end of the paper, however, we indicate the simple changes needed to deal with purely sublinear or mixed sublinear-superlinear problems.

Unlike the case $m = 1$ there are few studies of higher order problems such as (1). We mention, in particular, the results of Dalmasso, [6,7], Fukagai, [8], Kusano, Naito and Swanson, [10,11,12], Kusano and Swanson, [13], and Usami, [16] and the references therein. Most of these results were obtained for radially symmetric cases: $f(x,t) = f(|x|,t)$, whence approaches using ordinary differential equations were applicable. The most closely related paper to this work is [6], where the superlinear case is studied and the existence of a positive decaying solution is obtained for

$$\Delta^2 u = g(|x|)u^s, \quad x \in R^n$$

with $1 < s < (n+4)/(n-4)$, under the assumption $\int_0^\infty r^3 g(r)\,dr < \infty$. Motivation for our manuscript also came from [12] where sublinear and mixed sublinear-superlinear

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problems were considered. Finally we mention that higher order elliptic problems in non-radial cases were studied in the paper of Allegretto and Huang, [3] and the paper of Bernis, [5]. In [3], however only the existence of solutions bounded above and below by positive constants was considered, and the approach used did not seem at all suited to the problem of decaying solutions, while in [5], different $2m^{\text{th}}$ order problems were considered.

We wish to establish general non-radial conditions under which (1) has a positive decaying solution. Our main tools are weighted spaces, various applications of the Mountain Pass Theorem, L^p existence and regularity estimates. The general philosophy follows the one given in [4] for second order problems although the details are quite different. After some preliminary discussions, we present our main results. In particular, we show that (1) has solutions in the subcritical case if $|f(x,t)| \leq g(x)|t|^s$ with $g \in L^{p_0} \cap L^\infty$ for some p_0. We conclude the paper with some remarks, extensions and presentation of illustrative examples which explicitly compare our results to earlier work. For example, we show that the integration condition

$$\int_0^\infty r^3 g(r)\, dr < \infty$$

given in [6] is much stronger than our condition even in some radial cases. Finally, we note that our examples answer an open question raised in [12, p. 1298] for the superlinear case.

2. **Notation and preliminary results.** For any Banach space, we denote by $B_r(x)$ the ball of radius r centered at x, with $B_r \stackrel{\Delta}{=} B_r(0)$. Let $\sigma(x) = (1 + |x|^2)^{-1}$.

LEMMA 0. *Let $\varphi \in C_0^2(R^n)$. Then, for $n > 2i \geq 2$:*
(a) $c \int_{R^n} \sigma^i \varphi^2 \leq \int_{R^n} \sigma^{i-1} |\nabla \varphi|^2$,
(b) $c \int_{R^n} \sigma^{i-1} |\nabla \varphi|^2 \leq \int_{R^n} \sigma^{i-2} |\Delta \varphi|^2$,

where $c = c(n, i)$ is a constant independent of φ.

PROOF.
(a) By the Divergence Theorem:

$$\int_{R^n} \sigma^i \varphi^2 = -\frac{1}{n} \int_{R^n} x \cdot \nabla (\sigma^i \varphi^2) = -\frac{2}{n} \int_{R^n} \left[\sigma^i \varphi x \cdot \nabla \varphi - i\sigma^i \frac{|x|^2}{1 + |x|^2} \varphi^2 \right],$$

yielding:

$$\begin{aligned}\left(1 - \frac{2i}{n}\right) \int_{R^n} \sigma^i \varphi^2 &\leq \frac{2}{n} \int_{R^n} \sigma^{i-\frac{1}{2}} |\varphi| |\nabla \varphi| \frac{|x|}{\sqrt{1 + |x|^2}} \\ &\leq \frac{2}{n} \int_{R^n} \left(\sigma^{\frac{i}{2}} |\varphi|\right) \cdot \left(\sigma^{\frac{i-1}{2}} |\nabla \varphi|\right),\end{aligned}$$

and the first inequality is immediate.

(b) Applying again the Divergence Theorem gives

$$\begin{aligned}
\int_{R^n} \sigma^{i-1}|\nabla\varphi|^2 &= -\int_{R^n} \varphi \sum_j D_j(\sigma^{i-1}D_j\varphi) \\
&= \int_{R^n} \sigma^{i-1}\varphi(-\Delta\varphi) + (i-1)\sum_j \int_{R^n} 2\varphi D_j\varphi\sigma^i x_j \\
&= \int_{R^n} \sigma^{i-1}\varphi(-\Delta\varphi) + (i-1)\sum_j \int_{R^n} D_j(\varphi^2)\sigma^i x_j \\
&= \int_{R^n} \sigma^{i-1}\varphi(-\Delta\varphi) - (i-1)\int_{R^n} \varphi^2 \sum_j D_j(\sigma^i x_j) \\
&= \int_{R^n} \sigma^{i-1}\varphi(-\Delta\varphi) - (i-1)\int_{R^n} \sigma^i\left[n - \frac{2i|x|^2}{1+|x|^2}\right]\varphi^2 \\
&\leq \int_{R^n} \sigma^{i-1}\varphi(-\Delta\varphi) \leq \int_{R^n}\big(\sigma^{\frac{1}{2}}|\varphi|\big)\big(\sigma^{\frac{i}{2}-1}|\Delta\varphi|\big).
\end{aligned}$$

Inequality (b) now follows from Hölder's Inequality and (a).

With the aid of Lemma 0 and some elementary limit arguments we can set up weighted spaces suitable for the consideration of (1). Let $W^{k,p}$ denote the Sobolev spaces of functions with k weak derivatives in L^p. The space $W^{k,p}_{\text{loc}}$ is defined in the usual way. In $W^{m,2}_{\text{loc}}$ we introduce

$$\|\varphi\|^2 = \begin{cases} \sum_{i=0}^{N+1} \int_{R^n} \sigma^{m-2i}|\Delta^i\varphi|^2 + \sum_{i=0}^{N} \int_{R^n} \sigma^{m-(2i+1)}|\nabla(\Delta^i\varphi)|^2 & \text{if } m = 2(N+1) \\ \sum_{i=0}^{N} \int_{R^n} \sigma^{m-2i}|\Delta^i\varphi|^2 + \sum_{i=0}^{N} \int_{R^n} \sigma^{m-(2i+1)}|\nabla(\Delta^i\varphi)|^2 & \text{if } m = 2N+1 \end{cases}$$

and let $\tilde{E} = \{\varphi \mid \|\varphi\| < \infty\}$. We note that $\tilde{E}$ is a Hilbert space (for completeness we need only recall the completeness of weighted L^2 spaces and the fact that ∇ and Δ are closed maps) with the obvious inner product. Let E be the closure of $C_0^\infty(R^n)$ in $\tilde{E}$ with respect to $\|\ \|$. To establish properties of E it is useful to recall:

LEMMA 1. *Let $v \in W^{1,2}(B_r)$ be a weak solution of $-\Delta v = f$. If $f \in W^{k,2}(B_r)$ then for $r' < r$ we have $v \in W^{k+2,2}(B_{r'})$ and*

$$\|v\|_{W^{k+2,2}(B_{r'})} \leq C\{\|v\|_{L^2(B_r)} + \|f\|_{W^{k,2}(B_r)}\}$$

where $C = C(n,k,r,r')$.

PROOF. In fact this lemma is [9, Theorem 8.10]. We need only observe that $\|v\|_{W^{1,2}}$ in [9, Theorem 8.10] may be replaced by $\|v\|_{L^2}$. See, e.g., the remark after the proof of [9, Theorem 8.8].

We note the following properties of E:

LEMMA 2. *For any $u \in E$,*
(a) $\Delta^i u \in W^{1,2}_{\text{loc}}(R^n)$, for $i = 0,\dots,N$.
(b) $\|u\| \sim \|u\|_\ell$, where $\|u\|_\ell^2 = \begin{cases} \int_{R^n}|\Delta^{N+1}u|^2 & m = 2(N+1) \\ \int_{R^n}|\nabla\Delta^N u|^2 & m = 2N+1 \end{cases}$

(c) $u \in W^{m,2}(B_r)$ *and*

$$\|u\|_{W^{m,2}(B_r)} \leq C\|u\|, \quad C = C(n,m,r)$$

(d) $u \in L^{2n/(n-2m)}(R^n)$, *and* $\|u\|_{L^{2n/(n-2m)}(R^n)} \leq C\|u\|$.

PROOF. Part (a) is immediate by the definition of E. Applying Lemma 0 and a limit argument N times (for the case $m = 2(N+1)$, the other case is similar) gives:

$$\begin{aligned}
\|u\|_\ell^2 &= \frac{1}{2}\int_{R^n} |\Delta^{N+1}u|^2 + \frac{1}{2}\int_{R^n} |\Delta^{N+1}u|^2 \\
&\geq \frac{1}{2}\int_{R^n} |\Delta^{N+1}u|^2 + \frac{c}{2}\int_{R^n} \sigma|\nabla\Delta^N u|^2 \\
&\vdots \\
&\geq \sum_{i=0}^{N+1} \frac{1}{2}\Big(\frac{c}{2}\Big)^{m-2i} \int_{R^n} \sigma^{m-2i}|\Delta^i u|^2 + \sum_{i=0}^{N} \frac{1}{2}\Big(\frac{c}{2}\Big)^{m-(2i+1)} \int_{R^n} \sigma^{m-(2i+1)}|\nabla(\Delta^i u)|^2,
\end{aligned}$$

whence $\|u\|_\ell \geq C\|u\|$. We conclude that (b) holds. To prove (c), choose $r_0 > r$, and note that $u \in W^{m,2}(B_r)$ by construction. If $m = 2(N+1)$, then let $v = \Delta^N u, f = \Delta^{N+1}u$. Observing that $f \in L^2(R^n)$ and applying Lemma 1, we obtain $\Delta^N u \in W^{2,2}(B_{r_1})$ $(r < r_1 < r_0)$ and

$$\|\Delta^N u\|_{W^{2,2}(B_{r_1})} \leq C\{\|\Delta^N u\|_{L^2(B_{r_0})} + \|\Delta^{N+1}u\|_{L^2(B_{r_0})}\}.$$

Iterating this process yields $u \in W^{2(N+1),2}(B_r)$ and

$$\begin{aligned}
\|u\|_{W^{2(N+1),2}(B_r)} &\leq C\sum_{i=0}^{N+1} \|\Delta^i u\|_{L^2(B_{r_0})} \\
&\leq C\sum_{i=0}^{N+1} \|\Delta^i u\|_{L^2_{\sigma^{2(N+1-i)}}}(R^n) \\
&\leq C\|u\|
\end{aligned}$$

where $C = C(n,m,r)$. If $m = 2N+1$, we start with $v = \Delta^{N-1}u, f = \Delta^N u$ and $f \in W^{1,2}_{\mathrm{loc}}(R^n)$ by Part (a). It follows from Lemma 1 that $\Delta^{N-1}u \in W^{3,2}(B_{r_1})$ $(r < r_1 < r_0)$ and

$$\|\Delta^{N-1}u\|_{W^{3,2}(B_{r_1})} \leq C\{\|\Delta^{N-1}u\|_{L^2(B_{r_0})} + \|\Delta^N u\|_{L^2(B_{r_0})} + \|\nabla(\Delta^N u)\|_{L^2(B_{r_0})}\}.$$

The rest of the proof is the same as above, and (c) holds. Finally to prove (d), note that by Sobolev's Inequality, for $u \in C_0^\infty(R^n)$:

$$\begin{aligned}
\|u\|^2_{L^{2n/(n-2m)}(R^n)} &\leq C\int_{R^n} \sum_{|\alpha|=m} |D^\alpha u|^2 \leq \begin{cases} C\int_{R^n} |\Delta^{N+1}u|^2 \\ C\int_{R^n} |\nabla(\Delta^N u)|^2 \end{cases} \\
&\leq C\|u\|_\ell^2
\end{aligned}$$

for some constant C independent of u.

We note that Lemma 2 implies that:

(i) $\|u\|_\ell$ is an equivalent norm on E;

(ii) E can be imbedded into $W^{m,2}_{\mathrm{loc}}(R^n) \cap L^{2n/(n-2m)}(R^n)$.

3. **Results.** We now state our hypotheses of f:

(1°) $f \in C^{\alpha}_{\text{loc}}(R^n \times R), f(x,0) \equiv 0, 0 < f(x,t)$ in $\Omega \times R^+$ for some open set $\Omega \subseteq R^n$; $0 \leq f(x,t)$ in $R^n \times R^+$.

(2°) $|f(x,t)| \leq f_0(x)+f_1(x)|t|^s, 1 < s < \frac{n+2m}{n-2m}, f_0 \in L^\infty \cap L^2_{\sigma^{-m}}(R^n), f_1 \in L^\infty \cap L^{p_0}(R^n)$, $p_0 = \frac{2n}{2n-(s+1)(n-2m)}$;

(3°) Either $f_0 \equiv 0$ in (2) or $\lim_{t\to 0+} \frac{f(x,t)}{\sigma^m(x)t} = 0$ uniformly w.r.t. x in R^n.

(4°) There exists $\mu > 2$ such that

$$\mu F(x,t) \leq tf(x,t) \text{ for } (x,t) \in R^n \times R^+,$$

where $F(x,t) = \int_0^t f(x,\xi)\,d\xi$.

Let $G(u) = \int_{R^n} F(x,u)$ and $J(u) = \frac{1}{2}\|u\|_\ell^2 - G(u)$ for $u \in E$. Under our assumptions on f and by Lemma 2(d) G and J are well-defined.

LEMMA 3.

(a) G and J are weakly lower semicontinuous on E with $G'(u)(\varphi) = \int_{R^n} f(x,u)\varphi$.

(b) G' is continuous and compact from E to E.

PROOF. Since the proof follows the lines of the one given in [4] for $m = 1$, we only sketch the basic ideas.

(a) Let $u_k \longrightarrow u$ weakly in E. Then $\{u_k\}$ is bounded in E and we observe:

$$\begin{aligned}|G(u_k) - G(u)| \leq \int_{B_r} &|F(x,u_k) - F(x,u)| \\ &+ C\{\|f_0\|_{L^2_{\sigma^{-m}}(R^n\setminus B_r)}(\|u_k\|_\ell + \|u\|_\ell) \\ &+ \|f_1\|_{L^{p_0}(R^n\setminus B_r)}(\|u_k\|_\ell^{s+1} + \|u\|_\ell^{s+1})\}.\end{aligned}$$

The weak lower semicontinuity of G now follows by the boundedness of $\{u_k|_{B_r}\}$ in $W^{m,2}(B_r)$ (see Lemma 2) and the compactness of $W^{m,2}(B_r) \hookrightarrow L^P(B_r)$ for $1 \leq p < \frac{2n}{n-2m}$, (see, e.g. [1, Theorem 6.2]). The semicontinuity of J is then obvious.

For differentiability of G, we show that: given any $\varepsilon > 0$, there exists a $\delta = \delta(\varepsilon,u) > 0$ such that

$$\left|\int_{R^n} F(x,u+\varphi) - \int_{R^n} F(x,u) - \int_{R^n} f(x,u)\varphi\right| < \varepsilon\|\varphi\|_\ell$$

for all $\varphi \in E$ with $\|\varphi\|_\ell \leq \delta$. Observe that $f_0 \in L^\infty \cap L^2_{\sigma^{-m}}(R^n), f_1 \in L^{p_0}(R^n)$ and

$$\begin{aligned}&\left|\int_{R^n\setminus B_r} F(x,u+\varphi) - F(x,u) - f(x,u)\varphi\right| \\ &\qquad \leq \int_{R^n\setminus B_r} 2f_0|\varphi| + f_1(|u| + |\varphi|)^s|\varphi| + f_1|u|^s|\varphi| \\ &\qquad \leq C\{\|f_0\|_{L^2_{\sigma^{-m}}(R^n\setminus B_r)} + \|f_1\|_{L^{p_0}(R^n\setminus B_r)}(\|u\|_\ell^s + \|\varphi\|_\ell^s)\}\|\varphi\|_\ell \\ &\qquad < \frac{\varepsilon}{2}\|\varphi\|_\ell\end{aligned}$$

for sufficiently large r and $\|\varphi\|_\ell \leq 1$. To estimate the integral on the bounded domain:

$$\left|\int_{B^r} F(x,u+\varphi) - F(x,u) - f(x,u)\varphi\right| < \frac{\varepsilon}{2}\|\varphi\|_\ell,$$

we need only follow the arguments in [15, Prop. B10], since they are the same in nature.

(b) Let $\tilde{u} = G'(u)$, that is: $\langle \tilde{u}, \varphi \rangle = \int_{R^n} f(x,u)\varphi$, where $\langle \cdot, \cdot \rangle$ denotes the inner product induced by $\| \cdot \|_\ell$. For continuity, it suffices to show that for any sequence $u_k \longrightarrow u$ in E there exists a subsequence $\{u_{k_j}\}$ such that $\tilde{u}_{k_j} \longrightarrow \tilde{u}$ in E. Note that $\{u_k\}$ has a subsequence $u_{k_j} \longrightarrow u$ pointwise in R^n and

$$\|\tilde{u}_k - \tilde{u}\|_\ell \le C\{ \|f(\cdot,u_k) - f(\cdot,u)\|_{L^{2n/(n+2m)}(B_r)} \\ + \|f_0\|_{L^2_{\sigma^{-m}}(R^n\setminus B_r)} + \|f_1\|_{L^{p_0}(R^n\setminus B_r)}(\|u_k\|_\ell^s + \|u\|_\ell^s)\}.$$

The continuity of G' follows from $|f(x,t)|^{2n/(n+2m)} \le C_1 + C_2|t|^{2ns/(n+2m)}$ in B_r with $1 < \frac{2ns}{n+2m} < \frac{2n}{n-2m}$ and the continuity properties of the Nemytskii operator. To show compactness, note:

$$\langle \tilde{u}, \varphi \rangle = \int_{B_r} f(x,u)\varphi + \int_{R^n\setminus B_r} f(x,u)\varphi.$$

The first term defines a map from E to E : $G_r'(u)(\varphi) = \int_{B_r} f(x,u)\varphi$. $G_r'(\cdot)$ is obviously compact. Indeed, we again note that any bounded sequence $\{u_k\}$ in E has a Cauchy subsequence in $L^p(B_r)$ with $1 \le p < \frac{2n}{n-2m}$, say $\{u_k\}$ itself. The compactness immediately follows from the estimate below:

$$\|G_r'(u_k) - G_r'(u_j)\|_\ell \le C\|f(\cdot,u_k) - f(\cdot,u_j)\|_{L^{2n/(n+2m)}(B_r)}.$$

We estimate the second term by

$$\left| \int_{R^n\setminus B_r} f(x,u)\varphi \right| \le C\Big(\|f_0\|_{L^2_{\sigma^{-m}}(R^n\setminus B_r)} + \|f_1\|_{L^{p_0}(R^n\setminus B_r)} \cdot \|u\|_\ell^s\Big)\|\varphi\|_\ell.$$

Therefore, $G'(u)$ is a limit map of a sequence of compact maps under the norm $\| \cdot \|_\ell$, whence $G'(u)$ is also compact.

The critical points of J, i.e., $u \in E$ such that

$$J'(u)(\varphi) = \langle u, \varphi \rangle - \int_{R^n} f(x,u)\varphi = 0$$

are the weak solutions of $\ell u = f(x,u)$. We state our main result.

THEOREM 4. *Under conditions* (1°)–(4°) *on* f, (1) *has a positive classical solution* u *with* $D^\alpha u \longrightarrow 0$ *as* $|x| \longrightarrow \infty$ *for* $|\alpha| \le 2m-1$.

PROOF. In view of Lemma 3 we can apply Mountain Pass Theorem arguments by suitably modifying the procedure given in [15]. Without loss of generality, we set $f(x,t) = 0$ if $t \le 0$. If $f_0 \equiv 0$, then:

$$J(u) \ge \frac{1}{2}\|u\|_\ell^2 - \frac{1}{s+1}\int_{R^n} f_1|u|^{s+1} \ge \frac{1}{2}\|u\|_\ell^2 - C\|f_1\|_{L^{p_0}}\|u\|_\ell^{s+1}.$$

If $\lim_{t\to 0^+} \frac{f(x,t)}{\sigma^m(x)t} = 0$ uniformly w.r.t. $x \in R^n$, then $|f(x,t)| \le \varepsilon\sigma^m(x)|t| + C(\varepsilon)|t|^{\frac{n+2m}{n-2m}}$, $C(\varepsilon)$ independent of x. It follows that:

$$J(u) \ge \Big(\frac{1}{2} - \varepsilon C(n)\Big)\|u\|_\ell^2 - C(\varepsilon,n)\|u\|_\ell^{2n/(n-2m)}.$$

Hence the conditions (2°) and (3°) yield $J(u) \geq \alpha$ for all $u \in \partial B_\rho(0)$, some $\rho, \alpha > 0$. From assumptions (1°) and (4°) we have: $0 < \mu F(x,t) \leq tf(x,t)$ for all $x \in \Omega$ and $t \geq 0$. We may assume Ω is bounded. Integrating shows that there exist $a_1, a_2 > 0$ such that $F(x,t) \geq a_1 t^\mu - a_2$ for $x \in \Omega$, $t > 0$. Let $w \in C_0^\infty(\Omega)$ with $w(x) \geq 0, \not\equiv 0$, and let β be a positive number. We observe that

$$J(\beta w) \leq \frac{1}{2}\beta^2 \|w\|_\ell^2 - \beta^\mu \int_\Omega a_1 w^\mu + a_2|\Omega|$$

yields $J(\beta w) < 0$ for β large. Finally, to verify the *PS* condition, let $\{u_k\} \subseteq E$ with $J(u_k) \leq C$ and $J'(u_k)(\cdot) \longrightarrow 0$. We observe that:

$$\begin{aligned} C &\geq \frac{1}{2}\|u_k\|_\ell^2 - \int_{u_k(x)\geq 0} F(x,u_k) \\ &\geq \frac{1}{2}\|u_k\|_\ell^2 - \frac{1}{\mu}\int_{u_k(x)\geq 0} f(x,u_k)u_k \\ &\geq \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_\ell^2 - \frac{1}{\mu}\left(\|u_k\|_\ell^2 - \int_{u_k(x)\geq 0} f(x,u_k)u_k\right) \\ &= \left(\frac{1}{2} - \frac{1}{\mu}\right)\|u_k\|_\ell^2 - \frac{1}{\mu}J'(u_k)(u_k) \end{aligned}$$

It follows that $\{u_k\}$ is bounded. Note that $u_k = G'(u_k) + J'(u_k)$. The *PS* condition is now immediate from the compactness of G'. We conclude that $J(\cdot)$ has a nontrivial critical point, say u. Lemma 5 below shows that $u \in C^{2m}(R^n)$ and $D^\alpha u \longrightarrow 0$ as $|x| \longrightarrow \infty$ for $|\alpha| \leq 2m - 1$. To see that u is positive, we observe that $f(x,u) \geq 0$ and $(-\Delta)^{m-1}u$ is a solution of the following problem:

$$\begin{cases} (-\Delta)v = f(x,u) \\ \lim\limits_{|x|\to\infty} v = 0. \end{cases}$$

If we choose x_0 and apply the Maximum Principle on B_r, r large, we conclude: $v(x_0) \geq \inf_{|x|=r} v(x)$. Letting $r \longrightarrow \infty$, we obtain: $(-\Delta)^{m-1}u \geq 0$. Similarly, we can show that $(-\Delta)^{m-2}u \geq 0, \ldots, -\Delta u \geq 0$ and $u \geq 0$. Since u is nontrivial, by the Mountain Pass Theorem, then $u > 0$ by [9, Th. 8.18].

LEMMA 5. *Let u be any critical point of $J(\cdot)$.*
(a) u is a classical solution of $(-\Delta)^m u = f(x,u)$ and furthermore, for some $p = p(q)$,

$$\|u\|_{W^{2m,q}(B_1(x))} \leq H\left(\|u\|_{L^{2n/(n-2m)}(B_2(x))}, \|f_0\|_{L^p(B_2(x))}\right)$$

where $q \geq \frac{2n}{n-2m}$, H is a continuous function, dependent on n, m, q and $\|f_1\|_{L^\infty}$, with $H(0,0) = 0$.
(b) $D^\alpha u \longrightarrow 0$ as $|x| \longrightarrow \infty$ for $|\alpha| \leq 2m - 1$.

PROOF. Let u be a critical point of J:

$$\langle u, \varphi\rangle - \int_{R^n} f(x,u)\varphi = 0, \quad \varphi \in E.$$

Suppose $\delta = \frac{n+2m}{n-2m} - s < 1$ (the case $\delta \geq 1$ will be discussed below) and $u \in L^{q_0}\big(B_{r_i}(x)\big)$ with $q_0 = 2n/[(n-2m)(1-\delta)^i]$ for some integer i and $1 < r_i \leq 2$. Note that this is true for $i = 0$ by construction. Observe that for any $\varphi \in C_0^\infty\big(B_{r_i}(x)\big)$,

$$\begin{aligned}\big|\big(u, (-\Delta)^m \varphi\big)\big| &= |\langle u, \varphi\rangle| = \Big|\int_{B_{r_i}(x)} f(\cdot, u)\varphi\Big| \\ &\leq \int_{B_{r_i}(x)} f_0|\varphi| + f_1|u|^s|\varphi| \\ &\leq C\{\|f_0\|_{L^p(B_{r_i}(x))} + \|u\|^s_{L^{q_0}(B_{r_i}(x))}\}\|\varphi\|_{L^{q'}(B_{r_i}(x))}\end{aligned}$$

where $p \geq q, q' : \frac{1}{q} + \frac{1}{q'} = 1$, $q = \frac{q_0}{s}$, and $C = C(n, m, \|f_1\|_{L^\infty})$. We obtain by the regularity theorem [2, Th. 6.1] that:

$$u \in W^{2m,q}\big(B_{r_{i+1}}(x)\big), \quad 1 < r_{i+1} < r_i,$$

$$\begin{aligned}\|u\|_{W^{2m,q}(B_{r_{i+1}}(x))} &\leq C\Big[\|f_0\|_{L^p(B_{r_i}(x))} + \|u\|^s_{L^{q_0}(B_{r_i}(x))} + \|u\|_{L^q(B_{r_i}(x))}\Big] \\ &\leq C\Big[\|f_0\|_{L^p(B_{r_i}(x))} + \|u\|^s_{L^{q_0}(B_{r_i}(x))} + \|u\|_{L^{q_0}(B_{r_i}(x))}\Big].\end{aligned}$$

Consequently, it follows from Sobolev's Embedding Theorem [1, Th. 5.4] that $u \in L^{q_1}\big(B_{r_{i+1}}(x)\big)$ with:

$$q_1 = \frac{2n}{(n-2m)(1-\delta)^i\big[\frac{n+2m}{n-2m} - \delta\big] - 4m}$$

unless the denominator is nonpositive, in which case clearly $u \in L^q$ for q large. Note that this must occur for $i \geq i_0$, some $i_0 = i_0(n, m, \delta)$. If $\delta \geq 1$, then $i_0 = 0$. However, we find in any case:

$$\begin{aligned}q_1 &\geq \frac{2n}{(n-2m)(1-\delta)^i\Big[\frac{n+2m}{n-2m} - \delta - \frac{4m}{(n-2m)(1-\delta)^i}\Big]} \\ &\geq \frac{2n}{(n-2m)(1-\delta)^{i+1}}.\end{aligned}$$

We conclude that for any large chosen q we can show by iterating a finite number of times (depending on q) that $u \in W^{2m,q}$. It follows that $u \in C^{2m-1}$ and, consequently, $f\big(x, u(x)\big) \in C^\alpha_{\mathrm{loc}}$. Setting $v = (-\Delta)^{m-1}u$ and employing [9, Th. 9.19] we conclude $v \in C^2$ and, consequently, $u \in C^{2m}$. The function H consists of sums and products and is constructed merely by keeping track of the bound on $\|u\|_{W^{2m,q_i}}$ in each of the iteration steps.

It follows from (a) that

$$D^\alpha u \in W^{2m-|\alpha|,q}\big(B_1(x)\big), \quad |\alpha| \leq 2m-1,$$

$$\|D^\alpha u\|_{W^{2m-|\alpha|,q}(B_1(x))} \leq H\Big(\|u\|_{L^{2n/(n-2m)}(B_2(x))}, \|f_0\|_{L^p(B_2(x))}\Big).$$

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(b) is an immediate consequence of Sobolev's Embedding Theorems on $B_1(x)$ and the observation that

$$\lim_{|x|\to\infty} \|u\|_{L^{2n/(n-2m)}\left(B_2(x)\right)} = \lim_{|x|\to\infty} \|f_0\|_{L^p\left(B_2(x)\right)} = 0,$$

for any large p.

4. **Examples and comparison.** We conclude with illustrative examples showing the connection with earlier work. For simplicity we assume all coefficients are in C^∞ unless otherwise mentioned.

EXAMPLE 1. Consider the superlinear biharmonic problem

$$(1') \qquad \begin{cases} \Delta^2 u = g(|x|)u^s, & x \in R^n \\ \lim_{|x|\to\infty} u = 0 \end{cases}$$

where $n \geq 5$, $1 < s < \frac{n+4}{n-4}$, $g \in L^\infty$, $0 < g(|x|) = O(|x|^{-a})$ at ∞. The conditions (1°), (3°) and (4°) are satisfied. We conclude that (1′) has a positive solution by our theorem if: $\frac{2na}{2n-(s+1)(n-4)} > n$ i.e. $a > \frac{n-4}{2}\left[\frac{n+4}{n-4} - s\right] := a_*(s)$. The graph of $a_*(s)$ is a straight line with $4 = a_*(1) > a_*(s) > a_*\left(\frac{n+4}{n-4}\right) = 0$ for $1 < s < \frac{n+4}{n-4}$.

As mentioned before, to the best of our knowledge only Dalmasso studied in [6] the superlinear biharmonic problem. The existence of a positive solution to (1′) is obtained there by the assumption:

$$\int_0^\infty r^3 g(r)\, dr < \infty.$$

If $g(|x|) = O(|x|^{-a})$, the integration condition implies $a > 4$, which is much stronger than our condition in this case: $a > \frac{n-4}{2}\left[\frac{n+4}{n-4} - s\right]$.

EXAMPLE 2. Consider the general superlinear pluriharmonic problem

$$(2) \qquad \begin{cases} (-\Delta)^m u = g(x)u^s, & x \in R^n \\ \lim_{|x|\to\infty} D^\alpha u = 0, & 0 \leq |\alpha| \leq 2m-1 \end{cases}$$

where $n > 2m$, $1 < s < \frac{n+2m}{n-2m}$, $0 < g(x) \in L^\infty \cap L^{p_0} = \frac{2n}{2n-(s+1)(n-2m)}$. In this case, the conditions of our theorem are satisfied, thus (2) has a classical positive solution. We note in passing that if we replace u^s by $u|u|^{s-1}$ in (2), since J in this case is an even functional, then the corresponding problem has infinitely many solutions, of which at least one is positive, as a consequence of results in [15, Theorem 9.12]. Observe that this example answers the open question posed in [12, p. 1298].

EXAMPLE 3. In [12] various other problems were considered. One such was the sublinear problem:

$$(3) \qquad (-\Delta)^m u = g(|x|)u^\gamma$$

for $-1 < \gamma < 1$, $n \geq 2m+1$. With some modifications, we can consider problems similar to (3) in case $0 \leq \gamma < 1$. Specifically, consider

$$\left\{\begin{array}{l}(-\Delta)^m u = f(x,u)\\ \lim\limits_{|x|\to\infty} u = 0.\end{array}\right. \tag{4}$$

Assume f is nonnegative, smooth and

(i) $f(x,t) \geq g_1(x)t^{\gamma_1}$ as $t \to 0^+$, $f(x,t) \leq g_2(x)t^{\gamma_2}$ for $t \geq 0$ with $0 \leq \gamma_1, \gamma_2 < 1$.

(ii) $g_1(x) \geq 0, \not\equiv 0$, and $g_2(x) \in L^\infty \cap L^{p_0}$, $p_0 = \frac{2n}{2n-(\gamma_2+1)(n-2m)}$.

While Mountain Pass arguments do not apply (since we cannot guarantee that $J(\varphi) \geq a$ for $\varphi \in \partial B_\rho$, some $\rho > 0$, $a > 0$), we note that J is bounded below. Indeed,

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|u\|_\ell^2 - \int_{R^n} \frac{1}{\gamma_2+1} g_2(x)|u|^{\gamma_2+1}\\ &\geq \frac{1}{2}\|u\|_\ell^2 - \frac{C}{\gamma_2+1}\|g_2\|_{p_0}\|u\|_\ell^{\gamma_2+1}\\ &= \|u\|_\ell^{\gamma_2+1}\left(\frac{1}{2}\|u\|_\ell^{1-\gamma_2} - \frac{C}{\gamma_2+1}\|g_2\|_{p_0}\right), \end{aligned}$$

where $f(x,t) = 0$ for $t < 0$ is assumed. Also J and G are weakly lower semicontinuous, and G' is compact by (ii) and the proof of Lemma 3. (Note that in this case, the estimates in the proof of Lemma 3 are even simpler. For instance, $|G(u_k) - G(u)| \leq \int_{B_r} |F(x,u_k) - F(x,u)| + C\|g_2\|_{L^{p_0}(R^n\setminus B_r)}(\|u_k\|_\ell^{\gamma_2+1} + \|u\|_\ell^{\gamma_2+1})$.) We observe that:

$$J(\beta\varphi) \leq \frac{\beta^2}{2}\|\varphi\|_\ell^2 - \frac{\beta^{\gamma_1+1}}{\gamma_1+1}\int_{R^n} g_1(x)|\varphi|^{\gamma_1+1} < 0$$

for some $\varphi \in C_0^\infty(R^n)$ if β is small enough. Therefore (4) has a positive solution u such that $J(u) = \inf\{J(v) \mid v \in E\} < 0$. The case $\gamma < 0$ does not seem accessible to our methods.

EXAMPLE 4. Another situation considered in [12] was a mixed sublinear-superlinear problem. We can also consider these problems as follows. Let:

$$(-\Delta)^m u = f(x,u) = g(x)u^\alpha + h(x)u^\beta$$

with $0 < \alpha < 1 < \beta < \frac{n+2m}{n-2m}$, $0 \leq g \in L^\infty \cap L^q$, $q = \frac{2n}{2n-(\alpha+1)(n-2m)}$, $0 \leq h \in L^\infty \cap L^p$, $p = \frac{2n}{2n-(\beta+1)(n-2m)}$, and $h \not\equiv 0$. We are still able to employ the Mountain Pass Theorem to obtain the existence of a decaying positive solution. In this case J is weakly lower semicontinuous and differentiable, while G' is compact in the same way as mentioned in Example 3. Observe that for some $(0 \leq)\ w \in C_0^\infty(R^n)$,

$$J(tw) \leq \frac{t^2}{2}\|w\|_\ell^2 - \frac{t^{\beta+1}}{\beta+1}\int_{R^n} hw^{\beta+1} < 0$$

for large $t > 0$, and that the (PS) condition follows from the estimate

$$\begin{aligned} J(u) &= \left(\frac{1}{2} - \frac{1}{\beta+1}\right)\|u\|_\ell^2 + \frac{1}{\beta+1}J'(u)(u) - \frac{(\beta-\alpha)}{(\alpha+1)(\beta+1)}\int_{R^n} gu^{\alpha+1}\\ &\geq \left(\frac{1}{2} - \frac{1}{\beta+1}\right)\|u\|_\ell^2 + \frac{1}{\beta+1}J'(u)(u) - \frac{C(\beta-\alpha)}{(\alpha+1)(\beta+1)}\|g\|_q\|u\|_\ell^{\alpha+1}. \end{aligned}$$

But the step $J(\varphi) \geq a$ for $\varphi \in \partial B_\rho$, some $\rho > 0$, $a > 0$ no longer follows as before. Observe, however, that for some C_1, C_2,

$$\begin{aligned} J(\varphi) &\geq \frac{1}{2}\|\varphi\|_\ell^2 - C_1\|g\|_q\|\varphi\|_\ell^{\alpha+1} - C_2\|h\|_p\|\varphi\|_\ell^{\beta+1} \\ &= \frac{1}{2}\|\varphi\|_\ell^2\{1 - 2C_1\|g\|_q\|\varphi\|_\ell^{\alpha-1} - 2C_2\|h\|_p\|\varphi\|_\ell^{\beta-1}\} \\ &\geq a, \end{aligned}$$

for all $\varphi \in \partial B_\rho(0)$, if for some $\rho > 0$:

$$1 - 2C_1\|g\|_q\rho^{\alpha-1} - 2C_2\|h\|_p\rho^{\beta-1} \equiv H(\rho) > 0.$$

Elementary differentiation shows that the maximum for $H(\rho)$ is attained at $\rho_0 = \left(\frac{(1-\alpha)C_1\|g\|_q}{(\beta-1)C_2\|h\|_p}\right)^{\frac{1}{\beta-\alpha}}$. We obtain the existence of u by Mountain Pass arguments if $H(\rho_0) > 0$, i.e.,

$$2(C_1\|g\|_q)^{\frac{\beta-1}{\beta-\alpha}}(C_2\|h\|_p)^{\frac{1-\alpha}{\beta-\alpha}}\left[\left(\frac{\beta-1}{1-\alpha}\right)^{\frac{1-\alpha}{\beta-\alpha}} + \left(\frac{1-\alpha}{\beta-1}\right)^{\frac{\beta-1}{\beta-\alpha}}\right] < 1.$$

In order to apply this result, we need estimates on C_1, C_2. Such estimates are known and essentially given explicitly in [14, p. 56]:

$$C_1 = \frac{1}{\alpha+1}\left[n^{-(m+1)/2}\left(\frac{n-1}{v_n^{1/n}}\right)^m \frac{\Gamma(n/2-m)}{\Gamma(n/2)}\right]^{\alpha+1},$$

$$C_2 = \frac{1}{\beta+1}\left[n^{-(m+1)/2}\left(\frac{n-1}{v_n^{1/n}}\right)^m \frac{\Gamma(n/2-m)}{\Gamma(n/2)}\right]^{\beta+1},$$

where v_n is the volume of $B_1(0)$. This result does not seem easily comparable to [12, Th. 5] since they are different. We observe that for this example some embedding estimates are important in general. Note, however, that if $g = 0$ then we recover the superlinear result of Theorem 4.

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