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STRONGLY CLEAN POWER SERIES RINGS

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Abstract An element a in a ring R with identity is called strongly clean if it is the sum of an idempotent and a unit that commute. And $a \in R$ is called strongly π-regular if both chains $aR \supseteq a^2R \supseteq \cdots$ and $Ra \supseteq Ra^2 \supseteq \cdots$ terminate. A ring R is called strongly clean (respectively, strongly π-regular) if every element of R is strongly clean (respectively, strongly π-regular). Strongly π-regular elements of a ring are all strongly clean. Let σ be an endomorphism of R. It is proved that for $\Sigma r_i x^i \in R[\![x, \sigma]\!]$, if r_0 or $1 - r_0$ is strongly π-regular in R, then $\Sigma r_i x^i$ is strongly clean in $R[\![x, \sigma]\!]$. In particular, if R is strongly π-regular, then $R[\![x, \sigma]\!]$ is strongly clean. It is also proved that if R is a strongly π-regular ring, then $R[x, \sigma]/(x^n)$ is strongly clean for all $n \geqslant 1$ and that the group ring of a locally finite group over a strongly regular or commutative strongly π-regular ring is strongly clean.

Keywords: strongly clean ring; power series ring; strongly π-regular ring; group ring

2000 *Mathematics subject classification:* Primary 16U99; 16S99

1. Introduction

Rings are associative with identity. An element in a ring is called clean if it is the sum of an idempotent and a unit [**8**]. An element a in a ring R is called strongly clean if $a = e + u$, where $e^2 = e \in R$ and u is a unit of R such that $eu = ue$. In this case we also say that $a = e + u$ is a strongly clean expression of a in R. A ring R is called clean (respectively, strongly clean) if every element of R is clean (respectively, strongly clean). Note that clean and strongly clean rings are the 'additive analogues' of unit-regular and strongly regular rings, respectively, because a ring R is unit-regular if and only if every element of R is the product of an idempotent and a unit (in either order) and R is strongly regular if and only if every element of R is the product of an idempotent and a unit that commute. Local rings are obviously strongly clean. An element $a \in R$ is called right π-regular if the chain $aR \supseteq a^2R \supseteq \cdots$ terminates. The left π-regular elements are defined analogously. An element $a \in R$ is called strongly π-regular if it is both left and right π-regular, and R is called a strongly π-regular ring if every element is strongly π-regular. According to Dischinger [**6**], R is strongly π-regular if and only if every element of R is right π-regular if and only if every element of R is left π-regular. According to

Burgess and Menal [**2**], strongly π-regular rings are strongly clean; in particular one-sided perfect rings are strongly clean. Strongly clean rings were introduced by Nicholson [**9**], where their connection with strongly π-regular rings and hence to Fitting's lemma were discussed. In particular, it was proved in [**9**] that every strongly π-regular element is strongly clean.

According to Han and Nicholson [**7**], a ring R is clean if and only if the power series ring $R[\![x]\!]$ is clean. But it is unknown when a power series ring is strongly clean. For a local ring R, $R[\![x]\!]$ is certainly strongly clean (being local). But for any ring R, $R[\![x]\!]$ is never strongly π-regular because x is not strongly π-regular. Our main result says that, for a ring R, $\Sigma_{i\geqslant 0} r_i x^i \in R[\![x,\sigma]\!]$ is a strongly clean element if either r_0 or $1-r_0$ is strongly π-regular in R. In particular, $R[\![x,\sigma]\!]$ is strongly clean for any strongly π-regular ring R. This gives a new family of strongly clean rings, neither local nor strongly π-regular.

It was proved in [**7**] that if R is a Boolean ring and G is a locally finite group, then the group ring RG is a clean ring. Here it is proved that the group ring RG of a locally finite group G over a strongly regular or commutative strongly π-regular ring R is strongly π-regular (and hence strongly clean). This gives an affirmative answer to the question in [**7**] of whether the group ring RG of a locally finite group G over a commutative (von Neumann) regular ring R is clean. It is also proved here that, for a strongly π-regular ring R, $R[x,\sigma]/(x^n)$ is strongly clean for all $n \geqslant 1$. We write $\mathbb{Z}$ for the ring of integers, $\mathbb{Z}_n$ for the ring of integers modulo n, and $\mathbb{Z}_{(p)}$ for the localization of the ring $\mathbb{Z}$ at the ideal generated by the prime number p. As usual, $\mathbb{Q}$ is the field of rationals, $U(R)$ stands for the group of units of R, and $J(R)$ denotes the Jacobson radical of R. The $n \times n$ upper triangular matrix ring over R is denoted $\mathbb{T}_n(R)$.

2. Strongly clean power series

When is $R[\![x]\!]$ strongly clean? Here we present a new family of strongly clean rings through power series rings. If R is a ring and $\sigma : R \to R$ is a ring homomorphism (with $\sigma(1) = 1$), let $R[\![x,\sigma]\!]$ denote the ring of skew formal power series over R; that is, all formal power series in x with coefficients from R with multiplication defined by $xr = \sigma(r)x$ for all $r \in R$. Note that, by [**1**] or [**9**], an element $a \in R$ is strongly π-regular if and only if there exists $n \geqslant 1$ such that $a^n = eu = ue$, where $e^2 = e \in R$ and $u \in U(R)$ and a, e and u all commute. The main result of this section is the following theorem.

Theorem 2.1. *Let R be a ring and $r = \Sigma r_i x^i \in R[\![x,\sigma]\!]$. If either r_0 or $1 - r_0$ is a strongly π-regular element of R, then r is a strongly clean element of $R[\![x,\sigma]\!]$.*

Proof. Because r is strongly clean in $R[\![x,\sigma]\!]$ if and only if $1-r$ is also strongly clean in $R[\![x,\sigma]\!]$, we need only to prove the claim for the case where r_0 is a strongly π-regular element of R. So write $r_0^m = f_0 w_0 = w_0 f_0$, where $f_0^2 = f_0 \in R$ and $w_0 \in U(R)$ and r_0, f_0 and w_0 all commute. Let $n = 2m$. Then $r_0^n = f_0 w_0^2 = w_0^2 f_0$. Next we show that there exist $e = \Sigma e_i x^i, u = \Sigma u_i x^i \in R[\![x,\sigma]\!]$ such that

$$r = e + u, \quad eu = ue, \quad e^2 = e \text{ and } u \in U(R[\![x,\sigma]\!]).$$

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Choose $e_0 = 1 - f_0$ and $u_0 = r_0 - (1 - f_0)$. Then $u_0 \in U(R)$ by the proof of [**9**, Theorem 1] and hence $r_0 = e_0 + u_0$ is a strongly clean expression of r_0 in R. Now let $w = w_0^2$. Then

$$r_0^m = (1 - e_0)w_0 = w_0(1 - e_0), \tag{2.1}$$

$$r_0^n = (1 - e_0)w = w(1 - e_0). \tag{2.2}$$

Thus, r_0, e_0, w and u_0 all commute and

$$r_0^m e_0 = e_0 r_0^m = 0, \quad e_0 r_0^{n-1} = r_0^{n-1} e_0 = r_0^{m-1} r_0^m e_0 = 0. \tag{2.3}$$

Note that $e^2 = e$ is equivalent to

$$e_m = e_m \sigma^m(e_0) + e_{m-1}\sigma^{m-1}(e_1) + \cdots + e_1\sigma(e_{m-1}) + e_0 e_m \tag{E_m}$$

for $m = 0, 1, 2, \dots$, and $eu = ue$ is equivalent to

$$\begin{aligned} &e_m\sigma^m(u_0) + e_{m-1}\sigma^{m-1}(u_1) + \cdots + e_1\sigma(u_{m-1}) + e_0 u_m \\ &\qquad = u_m\sigma^m(e_0) + u_{m-1}\sigma^{m-1}(e_1) + \cdots + u_1\sigma(e_{m-1}) + u_0 e_m \end{aligned} \tag{F_m}$$

for $m = 0, 1, 2, \dots$, and $r = e + u$ is the same as

$$r_m = e_m + u_m \tag{G_m}$$

for $m = 0, 1, 2, \dots$. Clearly, e_0, u_0 satisfy (E_0), (F_0) and (G_0). Since $u_0 \in U(R)$, u is a unit of $R[\![x, \sigma]\!]$ no matter how we choose u_i for $i \geqslant 1$. Thus, it suffices to show that there exist e_i, u_i $(i = 1, 2, \dots)$ such that (E_m), (F_m) and (G_m) are satisfied for all $m \geqslant 1$. Assume that $e_0, \dots, e_k$, $u_0, \dots, u_k$ have been obtained so that (E_m), (F_m) and (G_m) are satisfied for all $m = 0, 1, \dots, k$. We next find e_{k+1} and u_{k+1} that satisfy (E_{k+1}), (F_{k+1}) and (G_{k+1}). Let

$$\begin{aligned} s_0 &= l_0 = m_0 = 0, \\ s_k &= e_1\sigma(u_k) + e_2\sigma^2(u_{k-1}) + \cdots + e_k\sigma^k(u_1), \\ l_k &= u_1\sigma(e_k) + u_2\sigma^2(e_{k-1}) + \cdots + u_k\sigma^k(e_1), \\ m_k &= e_1\sigma(e_k) + e_2\sigma^2(e_{k-1}) + \cdots + e_k\sigma^k(e_1), \\ t_k &= e_0 r_{k+1} - r_{k+1}\sigma^{k+1}(e_0) + s_k - l_k. \end{aligned}$$

Then s_k, l_k, m_k and t_k are well-defined elements of R.

Claim 1. $m_k\sigma^{k+1}(e_0) = e_0 m_k$.

Proof of Claim 1. Noting that (E_m) holds for $m = 1, 2, \dots, k$, we have

$$\begin{aligned} &m_k\sigma^{k+1}(e_0) - e_0 m_k \\ &\quad = [e_1\sigma(e_k) + e_2\sigma^2(e_{k-1}) + \cdots + e_k\sigma^k(e_1)]\sigma^{k+1}(e_0) \\ &\qquad - e_0[e_1\sigma(e_k) + e_2\sigma^2(e_{k-1}) + \cdots + e_k\sigma^k(e_1)] \\ &\quad = e_1\sigma[e_k\sigma^k(e_0)] + e_2\sigma^2[e_{k-1}\sigma^{k-1}(e_0)] + \cdots + e_k\sigma^k[e_1\sigma(e_0)] \\ &\qquad - e_0 e_1\sigma(e_k) - e_0 e_2\sigma^2(e_{k-1}) - \cdots - e_0 e_k\sigma^k(e_1) \end{aligned}$$

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$$
\begin{aligned}
&= e_1\sigma[e_k - e_0e_k - e_1\sigma(e_{k-1}) - \cdots - e_{k-1}\sigma^{k-1}(e_1)] \\
&\quad + e_2\sigma^2[e_{k-1} - e_0e_{k-1} - e_1\sigma(e_{k-2}) - \cdots - e_{k-2}\sigma^{k-2}(e_1)] \\
&\quad + \cdots \\
&\quad + e_k\sigma^k(e_1 - e_0e_1) \\
&\quad - e_0e_1\sigma(e_k) - e_0e_2\sigma^2(e_{k-1}) - \cdots - e_0e_k\sigma^k(e_1) \\
&= [e_1 - e_1\sigma(e_0) - e_0e_1]\sigma(e_k) + [-e_1\sigma(e_1) + e_2 - e_2\sigma^2(e_0) - e_0e_2]\sigma^2(e_{k-1}) \\
&\quad + \cdots + [-e_1\sigma(e_{k-1}) - \cdots - e_{k-1}\sigma^{k-1}(e_0) + e_k - e_k\sigma^k(e_0) - e_0e_k]\sigma^k(e_1) \\
&= 0\sigma(e_k) + 0\sigma^2(e_{k-1}) + \cdots + 0\sigma^k(e_1) = 0.
\end{aligned}
$$

Claim 2. $e_0t_k + t_k\sigma^{k+1}(e_0) = t_k + m_k\sigma^{k+1}(r_0) - r_0m_k$.

Proof of Claim 2. Because of (E_m) and (F_m) for $m = 1, 2, \dots, k$, we have

$$
\begin{aligned}
&s_k\sigma^{k+1}(e_0) + e_0s_k \\
&\quad = [e_1\sigma(u_k) + e_2\sigma^2(u_{k-1}) + \cdots + e_k\sigma^k(u_1)]\sigma^{k+1}(e_0) \\
&\qquad + e_0[e_1\sigma(u_k) + e_2\sigma^2(u_{k-1}) + \cdots + e_k\sigma^k(u_1)] \\
&\quad = e_1\sigma(u_k)\sigma^{k+1}(e_0) + e_2\sigma^2(u_{k-1})\sigma^{k+1}(e_0) + \cdots + e_k\sigma^k(u_1)\sigma^{k+1}(e_0) \\
&\qquad + [e_1 - e_1\sigma(e_0)]\sigma(u_k) \\
&\qquad + [e_2 - e_2\sigma^2(e_0) - e_1\sigma(e_1)]\sigma^2(u_{k-1}) \\
&\qquad + \cdots \\
&\qquad + [e_k - e_k\sigma^k(e_0) - e_{k-1}\sigma^{k-1}(e_1) - \cdots - e_1\sigma(e_{k-1})]\sigma^k(u_1) \\
&\quad = e_1\sigma[u_k + u_k\sigma^k(e_0) - e_0u_k - e_1\sigma(u_{k-1}) - \cdots - e_{k-1}\sigma^{k-1}(u_1)] \\
&\qquad + e_2\sigma^2[u_{k-1} + u_{k-1}\sigma^{k-1}(e_0) - e_0u_{k-1} - \cdots - e_{k-2}\sigma^{k-2}(u_1)] \\
&\qquad + \cdots \\
&\qquad + e_{k-1}\sigma^{k-1}[u_2 + u_2\sigma^2(e_0) - e_0u_2 - e_1\sigma(u_1)] \\
&\qquad + e_k\sigma^k[u_1 + u_1\sigma(e_0) - e_0u_1] \\
&\quad = e_1\sigma(u_k) + e_2\sigma^2(u_{k-1}) + \cdots + e_k\sigma^k(u_1) \\
&\qquad + e_1\sigma[u_k\sigma^k(e_0) - e_0u_k - e_1\sigma(u_{k-1}) - \cdots - e_{k-1}\sigma^{k-1}(u_1)] \\
&\qquad + \cdots \\
&\qquad + e_{k-1}\sigma^{k-1}[u_2\sigma^2(e_0) - e_0u_2 - e_1\sigma(u_1)] \\
&\qquad + e_k\sigma^k[u_1\sigma(e_0) - e_0u_1] \\
&\quad = s_k - e_1\sigma[u_{k-1}\sigma^{k-1}(e_1) + \cdots + u_1\sigma(e_{k-1}) + u_0e_k - e_k\sigma^k(u_0)] \\
&\qquad - e_2\sigma^2[u_{k-2}\sigma^{k-2}(e_1) + \cdots + u_1\sigma(e_{k-2}) + u_0e_{k-1} - e_{k-1}\sigma^{k-1}(u_0)] \\
&\qquad - \cdots \\
&\qquad - e_{k-1}\sigma^{k-1}[u_1\sigma(e_1) + u_0e_2 - e_2\sigma^2(u_0)] \\
&\qquad - e_k\sigma^k[u_0e_1 - e_1\sigma(u_0)]
\end{aligned}
$$

$$
\begin{aligned}
&= s_k - e_1\sigma[u_0e_k - e_k\sigma^k(u_0)] - e_2\sigma^2[u_0e_{k-1} - e_{k-1}\sigma^{k-1}(u_0)] - \cdots \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - e_k\sigma^k[u_0e_1 - e_1\sigma(u_0)] \\
&\quad - [e_1\sigma(u_{k-1}) + e_2\sigma^2(u_{k-2}) + \cdots + e_{k-1}\sigma^{k-1}(u_1)]\sigma^k(e_1) \\
&\quad - [e_1\sigma(u_{k-2}) + e_2\sigma^2(u_{k-3}) + \cdots + e_{k-2}\sigma^{k-2}(u_1)]\sigma^{k-1}(e_2) \\
&\quad - \cdots \\
&\quad - [e_1\sigma(u_1)]\sigma^2(e_{k-1}) \\
&= s_k - I_1 - I_2,
\end{aligned}
$$

where

$$I_1 = s_{k-1}\sigma^k(e_1) + s_{k-2}\sigma^{k-1}(e_2) + \cdots + s_1\sigma^2(e_{k-1}),$$
$$I_2 = e_1\sigma[u_0e_k - e_k\sigma^k(u_0)] + e_2\sigma^2[u_0e_{k-1} - e_{k-1}\sigma^{k-1}(u_0)] + \cdots + e_k\sigma^k[u_0e_1 - e_1\sigma(u_0)].$$

Similarly, it can be verified that

$$e_0l_k + l_k\sigma^{k+1}(e_0) = l_k - J_1 - J_2,$$

where

$$J_1 = e_1\sigma(l_{k-1}) + e_2\sigma^2(l_{k-2}) + \cdots + e_{k-1}\sigma^{k-1}(l_1)$$

and

$$
\begin{aligned}
J_2 = [e_k\sigma^k(u_0) - u_0e_k]\sigma^k(e_1) + [e_{k-1}\sigma^{k-1}(u_0) - u_0e_{k-1}]\sigma^{k-1}(e_2) + \cdots \\
+ [e_1\sigma(u_0) - u_0e_1]\sigma(e_k).
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
J_1 &= e_1\sigma(l_{k-1}) + e_2\sigma^2(l_{k-2}) + \cdots + e_{k-1}\sigma^{k-1}(l_1) \\
&= e_1\sigma[u_{k-1}\sigma^{k-1}(e_1) + u_{k-2}\sigma^{k-2}(e_2) + \cdots + u_1\sigma(e_{k-1})] \\
&\quad + e_2\sigma^2[u_{k-2}\sigma^{k-2}(e_1) + u_{k-3}\sigma^{k-3}(e_2) + \cdots + u_1\sigma(e_{k-2})] \\
&\quad + \cdots \\
&\quad + e_{k-1}\sigma^{k-1}(u_1\sigma(e_1)) \\
&= [e_1\sigma(u_{k-1}) + e_2\sigma^2(u_{k-2}) + \cdots + e_{k-1}\sigma^{k-1}(u_1)]\sigma^k(e_1) \\
&\quad + [e_1\sigma(u_{k-2}) + e_2\sigma^2(u_{k-3}) + \cdots + e_{k-2}\sigma^{k-2}(u_1)]\sigma^{k-1}(e_2) \\
&\quad + \cdots \\
&\quad + [e_1\sigma(u_1)]\sigma^2(e_{k-1}) \\
&= s_{k-1}\sigma^k(e_1) + s_{k-2}\sigma^{k-1}(e_2) + \cdots + s_1\sigma^2(e_{k-1}) = I_1,
\end{aligned}
$$

and

$$
\begin{aligned}
-I_2 + J_2 = &-e_1\sigma[u_0e_k - e_k\sigma^k(u_0)] - e_2\sigma^2[u_0e_{k-1} - e_{k-1}\sigma^{k-1}(u_0)] - \cdots \\
&- e_k\sigma^k[u_0e_1 - e_1\sigma(u_0)] \\
&+ [e_k\sigma^k(u_0) - u_0e_k]\sigma^k(e_1) + [e_{k-1}\sigma^{k-1}(u_0) - u_0e_{k-1}]\sigma^{k-1}(e_2) + \cdots \\
&+ [e_1\sigma(u_0) - u_0e_1]\sigma(e_k)
\end{aligned}
$$

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$$
\begin{aligned}
&= [e_1\sigma(e_k) + e_2\sigma^2(e_{k-1}) + \cdots + e_k\sigma^k(e_1)]\sigma^{k+1}(u_0) \\
&\quad - e_1\sigma(u_0)\sigma(e_k) - e_2\sigma^2(u_0)\sigma^2(e_{k-1}) - \cdots - e_k\sigma^k(u_0)\sigma^k(e_1) \\
&\quad - u_0[e_k\sigma^k(e_1) + e_{k-1}\sigma^{k-1}(e_2) + \cdots + e_1\sigma(e_k)] \\
&\quad + e_k\sigma^k(u_0)\sigma^k(e_1) + e_{k-1}\sigma^{k-1}(u_0)\sigma^{k-1}(e_2) + \cdots + e_1\sigma(u_0)\sigma(e_k) \\
&= m_k\sigma^{k+1}(u_0) - u_0 m_k.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
e_0(s_k - l_k) + (s_k - l_k)\sigma^{k+1}(e_0) &= [s_k\sigma^{k+1}(e_0) + e_0 s_k] - [l_k\sigma^{k+1}(e_0) + e_0 l_k] \\
&= (s_k - I_1 - I_2) - (l_k - J_1 - J_2) \\
&= s_k - l_k - I_2 + J_2 \\
&= s_k - l_k + m_k\sigma^{k+1}(u_0) - u_0 m_k \\
&= s_k - l_k + m_k\sigma^{k+1}(r_0 - e_0) - (r_0 - e_0)m_k \\
&= s_k - l_k + m_k\sigma^{k+1}(r_0) - r_0 m_k \quad \text{(by Claim 1).} \qquad (2.4)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
e_0 t_k + t_k\sigma^{k+1}(e_0) &= e_0[e_0 r_{k+1} - r_{k+1}\sigma^{k+1}(e_0) + s_k - l_k] \\
&\qquad + [e_0 r_{k+1} - r_{k+1}\sigma^{k+1}(e_0) + s_k - l_k]\sigma^{k+1}(e_0) \\
&= e_0 r_{k+1} - r_{k+1}\sigma^{k+1}(e_0) + e_0(s_k - l_k) + (s_k - l_k)\sigma^{k+1}(e_0) \\
&= e_0 r_{k+1} - r_{k+1}\sigma^{k+1}(e_0) + s_k - l_k + m_k\sigma^{k+1}(r_0) - r_0 m_k \quad \text{(by (2.4))} \\
&= t_k + m_k\sigma^{k+1}(r_0) - r_0 m_k,
\end{aligned}
$$

proving Claim 2.

Claim 3. $e_0 t_k\sigma^{k+1}(e_0) = e_0 m_k\sigma^{k+1}(r_0) - r_0 e_0 m_k$.

Proof of Claim 3. Multiplying the equality in Claim 2 by e_0 from the left, we obtain

$$
e_0 t_k + e_0 t_k\sigma^{k+1}(e_0) = e_0 t_k + e_0 m_k\sigma^{k+1}(r_0) - r_0 e_0 m_k.
$$

Thus, Claim 3 follows.

For each integer $i \geqslant 0$, let

$$
c_i = e_0 r_0^i, \quad b_i = \sigma^{k+1}(c_i) = \sigma^{k+1}(e_0 r_0^i). \qquad (2.5)
$$

Claim 4. *Choose*

$$
e_{k+1} = -\sum_{i=0}^{m-1} c_i(t_k b + m_k)b^i + \sum_{i=0}^{m-1} a^i(a t_k - m_k)b_i + m_k,
$$

where

$$
a = w^{-1}r_0^{n-1}, \qquad b = \sigma^{k+1}(a) = \sigma^{k+1}(w^{-1}r_0^{n-1}). \qquad (2.6)
$$

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Then

$$e_{k+1} = e_{k+1}\sigma^{k+1}(e_0) + e_k\sigma^k(e_1) + \cdots + e_1\sigma(e_k) + e_0e_{k+1}.$$

That is

$$e_{k+1} = e_{k+1}\sigma^{k+1}(e_0) + e_0e_{k+1} + m_k.$$

Proof of Claim 4. Notice that the following hold:

$$c_0 = e_0, \quad b_0 = \sigma^{k+1}(e_0) \quad (\text{by } (2.5)), \tag{2.7}$$

$$c_m = e_0r_0^m = 0 \quad (\text{by } (2.3)), \tag{2.8}$$

$$b_m = \sigma^{k+1}(e_0r_0^m) = 0, \tag{2.9}$$

$$c_0a = e_0w^{-1}r_0^{n-1} = e_0r_0^{n-1}w^{-1} = 0 \quad (\text{by } (2.3)), \tag{2.10}$$

$$bb_0 = \sigma^{k+1}(a)\sigma^{k+1}(c_0) = \sigma^{k+1}(ac_0) = \sigma^{k+1}(c_0a) = 0, \tag{2.11}$$

$$b_0b_i = b_ib_0 = b_i, \tag{2.12}$$

$$c_0c_i = c_ic_0 = c_i. \tag{2.13}$$

Therefore, we have

$$\begin{aligned} e_{k+1}\sigma^{k+1}(e_0) &= e_{k+1}b_0 \\ &= -\sum_{i=0}^{m-1} c_i(t_kb + m_k)b^ib_0 + \sum_{i=0}^{m-1} a^i(at_k - m_k)b_ib_0 + m_kb_0 \\ &= -c_0(t_kb + m_k)b_0 + \sum_{i=0}^{m-1} a^i(at_k - m_k)b_i + m_kb_0 \quad (\text{by } (2.11)) \\ &= \sum_{i=0}^{m-1} a^i(at_k - m_k)b_i - c_0t_kbb_0 - c_0m_kb_0 + m_kb_0 \\ &= \sum_{i=0}^{m-1} a^i(at_k - m_k)b_i - e_0m_k\sigma^{k+1}(e_0) + m_k\sigma^{k+1}(e_0) \quad (\text{by } (2.11)) \\ &= \sum_{i=0}^{m-1} a^i(at_k - m_k)b_i \quad (\text{by Claim 1}) \end{aligned}$$

and

$$\begin{aligned} e_0e_{k+1} &= -\sum_{i=0}^{m-1} c_0c_i(t_kb + m_k)b^i + \sum_{i=0}^{m-1} c_0a^i(at_k - m_k)b_i + c_0m_k \\ &= -\sum_{i=0}^{m-1} c_i(t_kb + m_k)b^i + c_0(at_k - m_k)b_0 + c_0m_k \\ &= -\sum_{i=0}^{m-1} c_i(t_kb + m_k)b^i - c_0m_kb_0 + e_0m_k \quad (\text{by } (2.10)) \\ &= -\sum_{i=0}^{m-1} c_i(t_kb + m_k)b^i \quad (\text{by Claim 1}). \end{aligned}$$

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Hence,

$$e_0e_{k+1} + e_{k+1}\sigma^{k+1}(e_0) + m_k = -\sum_{i=0}^{m-1} c_i(t_kb + m_k)b^i + \sum_{i=0}^{m-1} a^i(at_k - m_k)b_i + m_k$$
$$= e_{k+1}.$$

Claim 5. *Choose* $u_{k+1} = r_{k+1} - e_{k+1}$*. Then we have*

$$e_{k+1}\sigma^{k+1}(u_0) + e_k\sigma^k(u_1) + \cdots + e_1\sigma(u_k) + e_0u_{k+1}$$
$$= u_{k+1}\sigma^{k+1}(e_0) + u_k\sigma^k(e_1) + \cdots + u_1\sigma(e_k) + u_0e_{k+1}.$$

That is

$$e_{k+1}\sigma^{k+1}(u_0) + e_0u_{k+1} + s_k = u_{k+1}\sigma^{k+1}(e_0) + u_0e_{k+1} + l_k. \tag{2.14}$$

Proof of Claim 5. Equation (2.14) is equivalent to

$$e_{k+1}\sigma^{k+1}(r_0 - e_0) + e_0(r_{k+1} - e_{k+1}) + s_k = (r_{k+1} - e_{k+1})\sigma^{k+1}(e_0) + (r_0 - e_0)e_{k+1} + l_k.$$

That is

$$r_0e_{k+1} - e_{k+1}\sigma^{k+1}(r_0) = e_0r_{k+1} - r_{k+1}\sigma^{k+1}(e_0) + s_k - l_k = t_k.$$

So it suffices to show that

$$r_0e_{k+1} - e_{k+1}\sigma^{k+1}(r_0) = t_k. \tag{2.15}$$

Because

$$r_0c_i = r_0e_0r_0^i = e_0r_0^{i+1} = c_{i+1}, \tag{2.16}$$

(2.17)–(2.21) hold:

$$b_i\sigma^{k+1}(r_0) = \sigma^{k+1}(e_0r_0^i)\sigma^{k+1}(r_0) = \sigma^{k+1}(e_0r_0^{i+1}) = b_{i+1}, \tag{2.17}$$
$$r_0a = r_0w^{-1}r_0^{n-1} = w^{-1}r_0^n = 1 - e_0 \quad \text{(by (2.2))}, \tag{2.18}$$
$$b\sigma^{k+1}(r_0) = \sigma^{k+1}(ar_0) = \sigma^{k+1}(r_0a) = 1 - \sigma^{k+1}(e_0), \tag{2.19}$$
$$r_0a^2 = (1 - e_0)a = a - e_0a = a - c_0a = a \quad \text{(by (2.10))}, \tag{2.20}$$
$$b^2\sigma^{k+1}(r_0) = \sigma^{k+1}(a^2r_0) = \sigma^{k+1}(a) = b. \tag{2.21}$$

Thus,

$$\begin{aligned}
r_0e_{k+1} - e_{k+1}\sigma^{k+1}(r_0)& \\
&= -\sum_{i=0}^{m-1} r_0c_i(t_kb+m_k)b^i + \sum_{i=0}^{m-1} r_0a^i(at_k-m_k)b_i + r_0m_k \\
&\quad + \sum_{i=0}^{m-1} c_i(t_kb+m_k)b^i\sigma^{k+1}(r_0) - \sum_{i=0}^{m-1} a^i(at_k-m_k)b_i\sigma^{k+1}(r_0) - m_k\sigma^{k+1}(r_0) \\
&= -\sum_{i=0}^{m-1} c_{i+1}(t_kb+m_k)b^i + r_0(at_k-m_k)b_0 + (1-e_0)(at_k-m_k)b_1 \\
&\quad + \sum_{i=2}^{m-1} a^{i-1}(at_k-m_k)b_i + r_0m_k \\
&\quad + c_0(t_kb+m_k)\sigma^{k+1}(r_0) + c_1(t_kb+m_k)(1-\sigma^{k+1}(e_0)) \\
&\quad + \sum_{i=2}^{m-1} c_i(t_kb+m_k)b^{i-1} \\
&\quad - \sum_{i=0}^{m-1} a^i(at_k-m_k)b_{i+1} - m_k\sigma^{k+1}(r_0) \quad \text{(by (2.16)–(2.21))} \\
&= -c_1(t_kb+m_k) - c_m(t_kb+m_k)b^{m-1} - (at_k-m_k)b_1 - a^{m-1}(at_k-m_k)b_m \\
&\quad + r_0(at_k-m_k)b_0 + (1-e_0)(at_k-m_k)b_1 + c_0(t_kb+m_k)\sigma^{k+1}(r_0) \\
&\quad + c_1(t_kb+m_k)(1-\sigma^{k+1}(e_0)) + r_0m_k - m_k\sigma^{k+1}(r_0) \\
&= -c_1(t_kb+m_k)\sigma^{k+1}(e_0) - e_0(at_k-m_k)b_1 + r_0(at_k-m_k)b_0 \\
&\quad + c_0(t_kb+m_k)\sigma^{k+1}(r_0) + r_0m_k - m_k\sigma^{k+1}(r_0) \quad \text{(by (2.8), (2.9))} \\
&= -c_1t_kb\sigma^{k+1}(e_0) - e_0at_kb_1 + r_0at_kb_0 + c_0t_kb\sigma^{k+1}(r_0) - c_1m_k\sigma^{k+1}(e_0) \\
&\quad + e_0m_kb_1 - r_0m_kb_0 + c_0m_k\sigma^{k+1}(r_0) + r_0m_k - m_k\sigma^{k+1}(r_0) \\
&= (1-e_0)t_kb_0 + c_0t_k[1-\sigma^{k+1}(e_0)] - r_0e_0m_k\sigma^{k+1}(e_0) + e_0m_k\sigma^{k+1}(e_0)\sigma^{k+1}(r_0) \\
&\quad - r_0m_k\sigma^{k+1}(e_0) + e_0m_k\sigma^{k+1}(r_0) \\
&\quad + r_0m_k - m_k\sigma^{k+1}(r_0) \quad \text{(by (2.10), (2.11), (2.18), (2.19))} \\
&= t_k\sigma^{k+1}(e_0) + e_0t_k - 2e_0t_k\sigma^{k+1}(e_0) - r_0e_0m_k + e_0m_k\sigma^{k+1}(r_0) \\
&\quad - r_0e_0m_k + e_0m_k\sigma^{k+1}(r_0) + r_0m_k - m_k\sigma^{k+1}(r_0) \quad \text{(by Claim 1)} \\
&= t_k - 2e_0t_k\sigma^{k+1}(e_0) - 2r_0e_0m_k + 2e_0m_k\sigma^{k+1}(r_0) \qquad \text{(by Claim 2)} \\
&= t_k \quad \text{(by Claim 3)},
\end{aligned}$$

verifying Claim 5.

Thus, by Claims 4 and 5, e_{k+1} and u_{k+1} satisfy (E_{k+1}), (F_{k+1}) and (G_{k+1}). The proof is complete by the induction principle. □

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Corollary 2.2. *If R is a strongly π-regular ring and σ is an endomorphism of R, then $R[\![x;\sigma]\!]$ is a strongly clean ring.*

The proof of Theorem 2.1 works for the next theorem.

Theorem 2.3. *Let σ be an endomorphism of R, $n \geqslant 1$ and $r = \sum_{i=0}^{n-1} r_i x^i \in R[x,\sigma]/(x^n)$. If r_0 or $1 - r_0$ is strongly π-regular in R, then r is strongly clean in $R[x,\sigma]/(x^n)$.*

Corollary 2.4. *If R is a strongly π-regular ring and σ is an endomorphism of R, then $R[x,\sigma]/(x^n)$ is strongly clean for all $n \geqslant 1$.*

Remark 2.5.

(i) A ring R is said to satisfy the condition $(*)$ if for each $a \in R$, either a or $1 - a$ is strongly π-regular. Both local rings and strongly π-regular rings satisfy $(*)$. If R_1 is a local ring that is not strongly π-regular and R_2 is a strongly π-regular ring that is not local, then $R = R_1 \times R_2$ is neither local nor strongly π-regular, but R satisfies $(*)$. Thus, the assumption in Theorem 2.1 unifies local and strongly π-regular rings.

(ii) The condition $(*)$ is sufficient for $R[\![x,\sigma]\!]$ to be strongly clean, but it is not necessary. Let $R = \mathbb{T}_2(\mathbb{Z}_{(2)})$ and let

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \in R.$$

It can be verified easily that neither A nor $I - A$ is strongly π-regular. But

$$R[\![x]\!] = \mathbb{T}_2(\mathbb{Z}_{(2)})[\![x]\!] \cong \mathbb{T}_2(\mathbb{Z}_{(2)}[\![x]\!])$$

is strongly clean by [**9**, Example 2], because $\mathbb{Z}_{(2)}[\![x]\!]$ is a commutative local ring.

3. Other extensions

If G is a group, we denote the group ring of G over R by RG. In this section, C_n stands for the cyclic group of order n and C_∞ denotes the infinite cyclic group. It was proved in [**7**, Proposition 4] that if R is a Boolean ring and G is a locally finite group, then RG is clean. This is a consequence of the next result, the proof of which uses an idea in [**4**]. Recall that a ring R is strongly regular if, for any $a \in R$, $a \in a^2R$.

Theorem 3.1. *If R is a strongly regular or commutative strongly π-regular ring and G is a locally finite group, then RG is strongly π-regular.*

Proof. By [**4**, Corollary 3.2], it suffices to show that $(R/P)G$ is strongly π-regular for any prime ideal P of R. If R is commutative strongly π-regular, then R/P is a commutative strongly π-regular domain; so R/P is a field. If R is strongly regular, then R/P is a prime, regular ring whose idempotents are central; so R/P is a division ring. Either way, for any finite subgroup G_1 of G, $(R/P)G_1$ is Artinian (by [**5**, Theorem 1]) and hence strongly π-regular. Hence, $(R/P)G$ is strongly π-regular. □

Theorem 3.1 gives an affirmative answer to the question in [**7**] of whether the group ring RG of a locally finite group G over a commutative regular ring R is clean. Because $\mathbb{Z}_{(7)}C_3$ is not clean [**7**], 'strongly π-regular' in Theorem 3.1 cannot be replaced by 'strongly clean'. But if RG is commutative clean, then G must be locally finite.

Proposition 3.2. *Let R be a commutative ring and G be an abelian group. If RG is clean, then G is locally finite.*

Proof. Suppose G is not locally finite. Then G is not torsion, so $G/T(G)$ is non-trivial and torsion free, where $T(G)$ is the torsion subgroup of G. Then $R(G/T(G))$ is clean, being an image of RG. So we can assume that G is torsion free. If G has rank greater than 1, then G has a torsion-free quotient G' of rank 1. Since RG' is clean again, we can assume that G is of rank 1. So G is isomorphic to a subgroup of $(\mathbb{Q},+)$. Since R is commutative, it has a quotient R' which is a field. Because $R'G$ is clean (being an image of RG), we can assume that R is a field. Take $g \in G$ such that $g^{-1} \neq g$. Since $g+g^{-1}$ is clean in RG, there exists a finitely generated subgroup G_1 of G such that $g \in G_1$ and $g+g^{-1}$ is clean in RG_1. Because every finitely generated subgroup of $(\mathbb{Q},+)$ is cyclic, G_1 is cyclic. Write $G_1=\langle h\rangle$. Then $g=h^k$, $g^{-1}=h^{-k}$ for some positive integer k. There is a natural isomorphism $R\langle h\rangle \cong R[x,x^{-1}]$ with $h^k+h^{-k} \longleftrightarrow x^k+x^{-k}$. Thus, x^k+x^{-k} is clean in $R[x,x^{-1}]$. But this is impossible because all the idempotents of $R[x,x^{-1}]$ are in R and all the units of $R[x,x^{-1}]$ are in $\{ax^i : 0 \neq a \in R,\ i \in \mathbb{Z}\}$. The contradiction shows that G is locally finite. □

It is proved in [**3**] that a ring R is semiperfect if and only if R is a clean ring containing no infinite set of orthogonal idempotents. This result can be used to give many examples of clean and non-clean rings. For example, for a finite group G and a prime p, $\mathbb{Z}_{(p)}G$ is Noetherian; so $\mathbb{Z}_{(p)}G$ is clean if and only if it is semiperfect. It is known [**7**] that if R is semiperfect, then RC_2 is clean. Below we will see that C_2 is the only non-trivial cyclic group having this property. The proof of the next example follows by Proposition 3.2.

Example 3.3. If R is a commutative ring, then RC_∞ is not clean.

Example 3.4. If $k \geqslant 2$, then $\mathbb{Z}_{(5)}C_{2^k}$ is not clean.

Proof. In $\mathbb{Z}_5$[X], $X^4-1=(X-\bar{1})(X-\bar{4})(X-\bar{2})(X-\bar{3})$. But in $\mathbb{Z}_{(5)}[X]$, $X^4-1=(X-1)(X+1)(X^2+1)$ with X^2+1 irreducible. So $\mathbb{Z}_{(5)}C_4$ is not semiperfect by [**10**, Theorem 5.8]. Hence, $\mathbb{Z}_{(5)}C_4$ is not clean. For $k \geqslant 2$, $\mathbb{Z}_{(5)}C_4$ is an image of $\mathbb{Z}_{(5)}C_{2^k}$, so $\mathbb{Z}_{(5)}C_{2^k}$ is not clean. □

Example 3.5. If $p \neq 2$ is a prime, then there exists a prime q such that $\mathbb{Z}_{(q)}C_p$ is not clean.

Proof. Because $\mathbb{Z}_{(7)}C_3$ is not clean [**7**], we can assume that $p \geqslant 5$. By Euler's theorem, p divides 2^p-1 and p divides 4^p-1.

Claim. *Either p is not the only prime divisor of 2^p-1, or 4^p-1 has a prime divisor which is neither p nor 3.*

If the claim does not hold, then

$$2^p - 1 = p^n \quad \text{and} \quad 4^p - 1 = 3^s p^t,$$

where $n \geqslant 1$, $s \geqslant 1$ and $t \geqslant 1$. Thus, $3^s p^t = (2^p)^2 - 1 = (2^p+1)(2^p-1) = (2^p+1)p^n$. It must be that $n = t$. This gives $3^s = 2^p + 1$ because $p \neq 3$, and so $s \geqslant 4$.

If $s = 2k$ is even, then $k \geqslant 2$ and $2^p = (3^k)^2 - 1 = (3^k+1)(3^k-1)$. So $3^k + 1 = 2^l$ and $3^k - 1 = 2^{p-l}$, where $l \geqslant 3$ and $p - l \geqslant 3$. Thus, $2 \cdot 3^k = 2^l + 2^{p-l}$, and hence 2 divides 3^k: a contradiction.

If s is odd, then

$$\begin{aligned} 2^p &= (2+1)^s - 1 \\ &= \binom{s}{0} + \binom{s}{1}2 + \binom{s}{2}2^2 + \cdots + \binom{s}{s}2^s - 1 \\ &= \binom{s}{1}2 + \binom{s}{2}2^2 + \cdots + \binom{s}{s}2^s. \end{aligned}$$

This shows that 2 divides s, a contradiction. Therefore, the claim is proved. Let $\Phi_p(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$. It is well-known that $\Phi_p(X)$ is irreducible in $\mathbb{Q}[X]$ (applying Eisenstein's criterion to $\Phi_p(X+1)$). By the claim, there exist two cases.

Case 1. $2^p - 1$ has a prime divisor q with $q \neq p$. Thus, $q > 2$ and q divides $2^{p-1} + 2^{p-2} + \cdots + 2 + 1$. So $\bar{2}$ is a root of $\Phi_p(X)$ in $\mathbb{Z}_q$. Because $\Phi_p(X)$ is irreducible in $\mathbb{Z}_{(q)}$, $\mathbb{Z}_{(q)}C_p$ is not semiperfect by [**10**, Theorem 5.8]. Hence, $\mathbb{Z}_{(q)}C_p$ is not clean.

Case 2. $4^p - 1$ has a prime divisor q with $q \neq p$ and $q \neq 3$. Thus, $q > 4$ and q divides $4^{p-1} + 4^{p-2} + \cdots + 4 + 1$. So $\bar{4}$ is a root of $\Phi_p(X)$ in $\mathbb{Z}_q$. As above, $\mathbb{Z}_{(q)}C_p$ is not clean.

□

Example 3.6. If $n > 2$, then there exists a prime q such that $\mathbb{Z}_{(q)}C_n$ is not clean.

Proof. If n has an odd prime divisor p, then C_p is a quotient of C_n. By Example 3.5, there exists a prime q such that $\mathbb{Z}_{(q)}C_p$ is not clean. Because $\mathbb{Z}_{(q)}C_p$ is an image of $\mathbb{Z}_{(q)}C_n$, $\mathbb{Z}_{(q)}C_n$ is not clean. If $n = 2^k$, then $k \geqslant 2$. By Example 3.4, $\mathbb{Z}_{(5)}C_n$ is not clean. □

Proposition 3.7. *Let $n \geqslant 2$. The following are equivalent:*

(i) *RC_n is clean for every semiperfect ring R;*

(ii) *RC_n is clean for every local ring R;*

(iii) *$n = 2$.*

Proof. By Example 3.6 and [**7**, Proposition 3]. □

If RH is clean for every finitely generated subgroup H of a group G, then RG is clean. The converse does not hold: let $R = \mathbb{Z}_{(7)}$ and let $G = S_3$ be the symmetric group of order 6. Then RG is semiperfect (and so is clean) by [**10**, Lemma 6.1], but RC_3 is not clean.

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