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Cyclicity of period annulus for a class of quadratic reversible systems with a nonrational first integral

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In this paper, we study the quadratic perturbations of a one-parameter family of reversible quadratic systems whose first integral contains the logarithmic function. By the criterion function for determining the lowest upper bound of the number of zeros of Abelian integrals, we obtain that the cyclicity of either period annulus is two. To the best of our knowledge, this is the first result for the cyclicity of period annulus of the one-parameter family of reversible quadratic systems whose first integral contains the logarithmic function. Moreover, the simultaneous bifurcation and distribution of limit cycles from two-period annuli are considered.

Keywords: Abelian integral; limit cycle; Chebyshev system; quadratic reversible system; simultaneous bifurcation and distribution

2020 *Mathematics subject classification:* Primary: 34C07
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1. Introduction and statement of the main result

As is well known, Żołądek [**32**] classified the integrable quadratic systems with at least one centre into four families: Hamiltonian Q_3^H, codimension four Q_4, generalized Lotka–Volterra Q_3^{LV} and reversible system Q_3^R. By using the terminology from

[32], Iliev [15] classified them into the following five classes in the complex form:

(1) $\dot{z} = -iz - z^2 + 2|z|^2 + (b + ic)\bar{z}^2$, Hamiltonian (Q_3^H),

(2) $\dot{z} = -iz + az^2 + 2|z|^2 + b\bar{z}^2$, reversible (Q_3^R),

(3) $\dot{z} = -iz + 4z^2 + 2|z|^2 + (b + ic)\bar{z}^2$, $|b + ic| = 2$, codimension four (Q_4),

(4) $\dot{z} = -iz + z^2 + (b + ic)\bar{z}^2$, generalized Lotka–Volterra (Q_3^{LV}),

(5) $\dot{z} = -iz + \bar{z}^2$, Hamiltonian triangle,

where a, b, $c \in \mathbb{R}$ and $z = x + iy$. A classic problem is to give the cyclicity of the period annulus of above five systems, here the cyclicity of a period annulus is the maximal number of limit cycles bifurcating from any compact region of this period annulus under quadratic perturbations, see [15]. For completeness, we will make a brief introduction. If the quadratic centres belong to Q_3^H, then the cyclicity of period annulus is two, see for instance [9, 10, 13, 20, 29]. If the quadratic centres belong to Q_4, Gavrilov and Iliev [11] prove that the cyclicity of period annulus of the systems is no more than eight. Later, Zhao [31] develops the method in [11] and improves this number from eight to five. The cyclicity of period annulus of the system Q_3^{LV}, having two or three real invariant lines, has been studied by Żołądek in [32]. For the case of Hamiltonian triangle, the cyclicity of period annulus has been proved to be three in [14]. As for the case of Q_3^R, owing to the rich dynamics of the systems, the studies on the cyclicity of period annulus are difficult and the results are limited [1–3, 15, 16, 18, 21, 23, 24, 27].

An integrable quadratic system is called generic, if it belongs to one of the first four classes and does not belong to other classes of the classification given above. Otherwise, it is degenerate. Obviously, a quadratic reversible system is generic if and only if $a \neq -1$, 4 or $a = 4$, $b \neq \pm 2$.

In this paper, we will investigate the cyclicity of period annulus for a class of generic quadratic reversible systems. When $a \neq b$, taking $z = x + iy$ and doing the coordinate and time scaling changes ($\tilde{x} = -2(a - b)y + 1$, $\tilde{y} = -2(a - b)x$, $\tilde{t} = -t/2$, $\tilde{a} = -(a + b + 2)/(a - b)$, $\tilde{b} = (a + b - 2)/(a - b)$, and rewrite ($\tilde{x}$, $\tilde{y}$, $\tilde{t}$, $\tilde{a}$, $\tilde{b}$) as (x, y, t, a, b)), we obtain from the class (2) for reversible system that

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= 2xy, \\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -a\,y^2 - b\,x^2 + 2(b - 1)x + 2 - b. \end{aligned} \tag{1.1}$$

System (1.1) has an invariant straight line $\{x = 0\}$, and a centre at (1, 0). The singularity at $((b - 2)/b, 0)$ is also a centre if $0 < b < 2$, and is a saddle if $b < 0$ or $b > 2$. In addition, when $a\,(2 - b) > 0$, the system has two saddle points at $(0, \sqrt{(2 - b)/a})$ and $(0, -\sqrt{(2 - b)/a})$.

When $a = 0$, -1 and -2, the first integral of system (1.1) is not a rational function, which brings much more difficulty to the study of the cyclicity of period annulus. Hence there are few results in this regard. To our knowledge, only the case of $a = -2$, $b = 1$ has been solved in [24].

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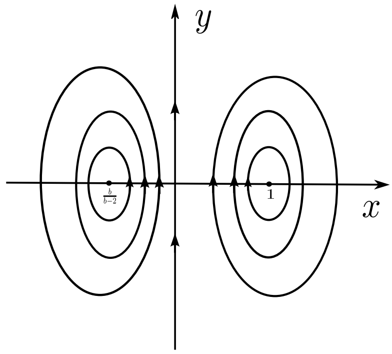

Figure 1. Phase portrait of system (1.2).

In this paper, we will discuss the generic case of $a = 0$, $0 < b < 2$. Substituting $a = 0$ into system (1.1), we have the following quadratic system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2xy, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx^2 + 2(b-1)x + 2 - b, \tag{1.2}$$

where $0 < b < 2$.

A first integral of system (1.2) is

$$H(x, y) = y^2 + \frac{bx^2}{2} - 2(b-1)x + (b-2)\ln|x| + \frac{3b}{2} - 2,$$

and the integrating factor is $1/x$. System (1.2) has two centres, one of which is at $(1, 0)$ and the other is at $((b-2)/b, 0)$, see the phase portrait of system (1.2) in figure 1. The continuous family of ovals surrounding the centre $(1, 0)$ is

$$\Gamma_h^1 = \{(x, y) \mid H(x, y) = h,\ h \in (0, +\infty)\},$$

and the continuous family of ovals surrounding the centre $((b-2)/b, 0)$ is

$$\Gamma_h^2 = \{(x, y) \mid H(x, y) = h,\ h \in \left(2 - \frac{2}{b} + (b-2)\ln\left(\frac{2}{b} - 1\right), +\infty\right)\}. \tag{1.3}$$

Obviously, besides considering the cyclicity of either period annulus, another interesting problem is to study the simultaneous bifurcation and distribution of limit cycles.

Consider the quadratic perturbations of system (1.2)

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2xy + \epsilon f(x, y), \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -bx^2 + 2(b-1)x + 2 - b + \epsilon g(x, y), \tag{1.4}$$

where $f(x, y) = \sum_{i+j\leqslant 2} a_{ij}x^i y^j$, $g(x, y) = \sum_{i+j\leqslant 2} b_{ij}x^i y^j$.

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As we know, for quadratic system (1.2), Iliev has given an essential perturbation in theorem 1 of [**15**] which can realize the maximum number of limit cycles produced by the whole class of quadratic systems (1.4) provided we consider bifurcations of any order in ε. And he has also shown that the first Melnikov functions can help us to obtain the cyclicity of period annulus of system (1.2) in theorem 2 of [**15**]. The following Melnikov functions can be obtained by the conclusions of [**15**], but to be clear, we will compute them directly in subsequent subsections. For the cyclicity on a period annulus of generic quadratic systems, the essential perturbations problem has also been clarified and justified by Françoise *et al.*, see [**8**] in which the components of the Nash space of arcs associated to a blow-up of Bautin ideal is used.

LEMMA 1.1. *The exact upper bound for the number of limit cycles produced by the compact region of the period annulus around the centre* $(1,0)$ *(resp.* $((b-2)/b,0)$*) of system* (1.2) *under quadratic perturbations is equal to the maximal number of zeros in* $(0,+\infty)$ *(resp.* $(2-2/b+(b-2)\ln(2/b-1),+\infty)$*) counting multiplicities, of the Melnikov integral (Abelian integral)*

$$I(h)=\alpha\oint_{\Gamma_h^1}\frac{y}{x^2}\mathrm{d}x+\beta\oint_{\Gamma_h^1}\frac{y}{x}\mathrm{d}x+\gamma\oint_{\Gamma_h^1}y\,\mathrm{d}x, \tag{1.5}$$

$$\left(resp.\ \ I(h)=\alpha\oint_{\Gamma_h^2}\frac{y}{x^2}\mathrm{d}x+\beta\oint_{\Gamma_h^2}\frac{y}{x}\mathrm{d}x+\gamma\oint_{\Gamma_h^2}y\,\mathrm{d}x\right), \tag{1.6}$$

where α, β *and* γ *are linear combinations of* a_{ij} *and* b_{ij}.

In general, the study of the number of zeros of Melnikov functions depending on the parameters is very difficult. Using the Chebyshev criterion [**24**] and some computational skills, our main results are obtained as follows.

THEOREM 1.2. *If* $a=0$ *and* $0<b<2$, *then the integral* $I(h)$ *given in* (1.5) *has at most two zeros in* $(0,+\infty)$ *counted with multiplicities, that is the cyclicity of period annulus of the generic quadratic reversible system* (1.2) *around the centre* $(1,0)$ *is equal to two. Similarly, the cyclicity of period annulus around the centre* $((b-2)/b,0)$ *is also two.*

THEOREM 1.3. *Under quadratic polynomial perturbations, all configurations* (u,v) *of limit cycles bifurcated from the two period annuli of system* (1.2) *can be realized, where* $0\leqslant u,v\leqslant 2$, $u+v\leqslant 2$. *That is, for* ε *small enough, exactly* u *(resp.* v*) limit cycles bifurcate from the periodic orbits surrounding the centre* $(1,0)$ *(resp.* $((b-2)/b,0)$*) simultaneously.*

The paper is organized as follows. In § 2, we give some preliminary definitions and known results about the Chebyshev criterion for Abelian integrals. We emphasize the advantage of the Chebyshev criterion we use in proposition 2.8 and give an example for application (see theorem 2.9). In § 3, we state the proof of lemma 1.1 and theorem 1.2. Finally, we prove theorem 1.3 in § 4, which shows the simultaneous bifurcation and distribution of limit cycles bifurcated from the two-period annuli around the centres (1, 0) and $((b-2)/b, 0)$.

2. Preliminary results

In this section, we introduce some definitions and compare two interesting criteria to estimate the lowest upper bound of the number of isolated zeros of Abelian integrals.

Definition 2.1. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an open interval J of $\mathbb{R}$.*

(a) *$\{f_0, f_1, \ldots, f_{n-1}\}$ is a Chebyshev system (for short, a T-system) on J if any nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{n-1} f_{n-1}(x)$$

has at most $n-1$ isolated zeros on J.

(b) *$\{f_0, f_1, \ldots, f_{n-1}\}$ is a complete Chebyshev system (for short, a CT-system) on J if $\{f_0, f_1, \ldots, f_{k-1}\}$ is a T-system for all $k = 1, 2, \ldots, n$.*

(c) *$\{f_0, f_1, \ldots, f_{n-1}\}$ is an extended complete Chebyshev system (for short, an ECT-system) on J, if for all $k = 1, 2, \ldots, n$, any nontrivial linear combination*

$$\alpha_0 f_0(x) + \alpha_1 f_1(x) + \cdots + \alpha_{k-1} f_{k-1}(x)$$

has at most $k-1$ isolated zeros on J counted with multiplicities.

Note that if $\{f_0, f_1, \ldots, f_{n-1}\}$ is an ECT-system on J, then $\{f_0, f_1, \ldots, f_{n-1}\}$ is a CT-system on J. However, the reverse implication is not true.

Definition 2.2. *Let $f_0, f_1, \ldots, f_{n-1}$ be analytic functions on an open interval J of $\mathbb{R}$. The continuous Wronskian of $\{f_0, f_1, \ldots, f_{k-1}\}$ at $x \in J$ is*

$$W[\mathbf{f_k}](x) = W[f_0, f_1, \ldots, f_{k-1}](x) = \begin{vmatrix} f_0(x) & \ldots & f_{k-1}(x) \\ f_0'(x) & \ldots & f_{k-1}'(x) \\ \vdots & \ddots & \vdots \\ f_{k-1}^{(k-1)}(x) & \ldots & f_{k-1}^{(k-1)}(x) \end{vmatrix}.$$

A well-known result is as follows (see [17, 25] for instance).

Lemma 2.3. *$\{f_0, f_1, \ldots, f_{n-1}\}$ is an ECT-system on J if and only if for all $k = 1, 2, \ldots, n$,*

$$W[\mathbf{f_k}](x) \neq 0, \quad x \in J.$$

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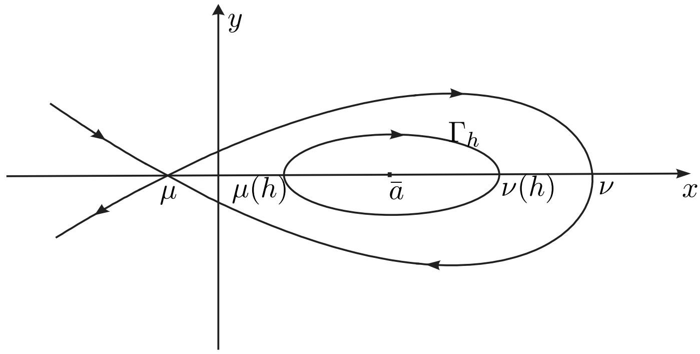

Figure 2. Phase portrait of system (2.1).

Consider the Newtonian mechanical system

$$\dot{x} = 2y, \quad \dot{y} = -\Psi'(x), \tag{2.1}$$

which has the first integral

$$H(x, y) = y^2 + \Psi(x),$$

with $\Psi(x)$ being analytic on the open interval (μ, ν). Assume that there exists $\bar{a} \in (\mu, \nu)$, such that the following hypothesis is satisfied:

$$(\mathrm{H}_1) : \Psi'(x)(x - \bar{a}) > 0, \quad x \in (\mu, \nu) \setminus \{\bar{a}\}.$$

It is easy to verify that, under above hypothesis (H_1), $(\bar{a}, 0)$ is a centre. Denote $h_c = H(\bar{a}, 0)$. Without loss of generality, we assume that $\Psi(\mu) = \Psi(\nu) = h_s$ and $\Psi(\bar{a}) = 0$. Thus using (H_1) we get $h_s > h_c = 0$. For each $h \in (h_c, h_s)$, let $\mu(h)$, $\nu(h)$ be the abscissas of intersections of the closed orbit $\Gamma_h = \{(x, y) \mid H(x, y) = h\}$ with the x-axis, then $\mu < \mu(h) < \bar{a} < \nu(h) < \nu$ (see figure 2).

It is easy to see from hypothesis (H_1) that for $x \in (\bar{a}, \nu(h))$, there exists a one to one mapping $x \longmapsto \sigma(x) \in (\mu(h), \bar{a})$, such that $\Psi(x) = \Psi(\sigma(x))$. For simplicity, we denote $z = \sigma(x)$.

By using lemma 2.3, Grau *et al.* [**12**] prove the following theorem.

Theorem 2.4. *Consider the Abelian integrals*

$$J_i(h) = \oint_{\Gamma_h} g_i(x) y^{2s-1} \mathrm{d}x, \quad i = 1, 2, \ldots, n. \tag{2.2}$$

Assume that $g_i(x)$ is analytic on the interval (μ, ν). For $x \in (\bar{a}, \nu)$, define the functions as follows

$$G_i(x) = \frac{g_i(x)}{\Psi'(x)} - \frac{g_i(z)}{\Psi'(z)}. \tag{2.3}$$

If $\{G_1, G_2, \ldots, G_n\}$ is a CT-system on $(\bar{a}, \nu)$ and $s > n - 2$, then $\{J_1(h), J_2(h), \ldots, J_n(h)\}$ is an ECT-system on (h_c, h_s).

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It is difficult to verify CT-system, so the verification of ECT-system is often considered in the application.

For the integral $I(h)$ in (1.5) or (1.6), using the notations in theorem 2.4, we have $n = 3$, $s = 1$. Obviously, theorem 2.4 cannot be used directly. In order to apply theorem 2.4, we need to promote the power of y in the integrand of Abelian integrals $\oint_{\Gamma_h^k}(y/x^{i-1})\mathrm{d}x$ ($i = 1, 2, 3$, $k = 1, 2$) from 1 to 3. To deal with this difficulty, we need the following lemma (see [**12**]).

Lemma 2.5. *Let Γ_h be a closed orbit contained in the level curve $\{(x, y) \mid H(x, y) = h,\ h \in (h_c, h_s)\}$, the following statements hold:*

(1) *Assume that $g(x)/\Psi'(x)$ is analytic at $x = \bar{a}$, then for any $s \in \mathbb{N}$,*

$$\oint_{\Gamma_h} g(x)y^{2s-1}\mathrm{d}x = \oint_{\Gamma_h} f(x)y^{2s+1}\mathrm{d}x,$$

where $f(x) = \frac{2}{2s+1}\left(\frac{g(x)}{\Psi'(x)}\right)'$.

(2) *Assume that $f(x)$ is analytic at $x = \bar{a}$, then for any $s \in \mathbb{N}$,*

$$\oint_{\Gamma_h} f(x)y^{2s+1}\mathrm{d}x = \oint_{\Gamma_h} g(x)y^{2s-1}\mathrm{d}x,$$

where $g(x) = \frac{2s+1}{2}\Psi'(x)\int_{\bar{a}}^{x} f(t)\mathrm{d}t$.

By lemma 2.5, after some calculations, we get

$$h\oint_{\Gamma_h^k} \frac{y}{x^{i-1}}\mathrm{d}x = \oint_{\Gamma_h^k}\left(\frac{1}{x^{i-1}} + \left(\frac{2\Psi(x)}{3x^{i-1}\Psi'(x)}\right)'\right)y^3\ \mathrm{d}x, \quad i = 1, 2, 3,\ k = 1, 2,$$

where $\Psi(x) = bx^2/2 - 2(b-1)x + (b-2)\ln|x| + 3b/2 - 2$. Since both of the integrands and the first integral contain the logarithmic function $\ln|x|$, it is very difficult to estimate the number of isolated zeros of $I(h)$. Thus we need to find other ways.

Consider the following Abelian integral

$$I(h) = \alpha\oint_{\Gamma_h} f_1(x)y\ \mathrm{d}x + \beta\oint_{\Gamma_h} f_2(x)y\ \mathrm{d}x + \gamma\oint_{\Gamma_h} f_3(x)y\ \mathrm{d}x, \tag{2.4}$$

where $f_i(x)\,(i = 1, 2, 3)$ are analytic in (μ, ν) and $\alpha, \beta, \gamma \in \mathbb{R}$, and Γ_h is a continuous family of level curves surrounding the centre $(\bar{a}, 0)$ as $h \in (h_c, h_s)$. For convenience, we denote

$$I_i(h) = \oint_{\Gamma_h} f_i(x)y\ \mathrm{d}x, \quad i = 1, 2, 3. \tag{2.5}$$

Define three functions

$$F_i(x) = \frac{f_i(x)}{\Psi'(x)} - \frac{f_i(z)}{\Psi'(z)}, \quad x \in (\bar{a}, \nu),\ i = 1, 2, 3. \tag{2.6}$$

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Assume that

$$(\mathrm{H}_2): \quad F_1(x) > 0 \quad \text{for all } x \in (\bar{a},\, \nu),$$

and let

$$\xi(x) = \frac{F_2(x)}{F_1(x)}, \quad \eta(x) = \frac{F_3(x)}{F_1(x)}. \tag{2.7}$$

Then Liu and Xiao prove the following theorem [**24**].

THEOREM 2.6. *Assume that* (H_1) *and* (H_2) *are satisfied. Furthermore, if the following hypotheses also hold:*

(H_3): $F_1'(x) > 0$, *for* $x \in (\bar{a},\, \nu)$;

(H_4): $\xi'(x) \neq 0$ *and* $(\eta'(x)/\xi'(x))' \neq 0$, *for* $x \in (\bar{a},\, \nu)$,

then for any real parameters α, β *and* γ, *the Abelian integral* $I(h)$ *in* (2.4) *has at most two zeros in* $(h_c,\, h_s)$ *counted with multiplicities.*

REMARK 2.7. In fact, theorem 2.6 and theorem 2.3 in [**22**] imply that $\{I_1,\, I_2,\, I_3\}$ is an ECT-system on $(h_c,\, h_s)$. In addition, the conditions (H_2) and (H_4) are equivalent to that $\{F_1(x),\, F_2(x),\, F_3(x)\}$ is an ECT-system on $(\bar{a},\, \nu)$ and $F_1(x) > 0$. This is easily obtained by the following formulas:

$$W[F_1](x) = F_1(x), \quad W[F_1, F_2](x) = F_1^2(x)\xi'(x),$$
$$W[F_1, F_2, F_3](x) = F_1^3(x) W[1, \xi, \eta](x) = F_1^3(x)\left(\frac{\eta'(x)}{\xi'(x)}\right)'.$$

Now, let's discuss the relationship between theorems 2.6 and 2.4 with the same system (2.1) and the same integral $\alpha J_1(h) + \beta J_2(h) + \gamma J_3(h)$, where $J_i(h) = \oint_{\Gamma_h} g_i(x) y^{2s-1} \mathrm{d}x,\ i = 1,\, 2,\, 3$ and $\alpha,\, \beta,\, \gamma$ are arbitrary constants. Here we assume that $g_i(x),\ \ i = 1,\, 2,\, 3$ is analytic at $x = \bar{a}$.

The following proposition implies that for $n = 3$, the criterion of theorem 2.6 is weaker than the criterion of theorem 2.4.

PROPOSITION 2.8. *If* $\{G_1,\, G_2,\, G_3\}$ *given in* (2.3) *is an ECT-system on* $(\bar{a},\, \nu)$, *then* $f_1,\, f_2,\, f_3$ *defined in* (2.4) *satisfy* (H_2)–(H_4).

Proof. Firstly, consider $s = 2$. Substituting $s = 2$ into $J_i(h)\,(i = 1,\, 2,\, 3)$, we get from lemma 2.4 that

$$J_i(h) = \frac{3}{2} \oint_{\Gamma_h} f_i(x) y \ \mathrm{d}x, \quad i = 1, 2, 3, \tag{2.8}$$

where $f_i(x) = \Psi'(x) \int_{\bar{a}}^{x} g_i(s) \mathrm{d}s$. Since $\{G_1,\, G_2,\, G_3\}$ is an ECT-system, using lemma 2.3 we deduce that $W[G_1]$, $W[G_1,\, G_2]$ and $W[G_1,\, G_2,\, G_3]$ are non-vanishing on $(\bar{a},\, \nu)$. Now let's verify the hypotheses (H_2), (H_3) and (H_4).

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- Verify hypotheses (H_2) and (H_3): Without loss of generality, suppose that $W[G_1] = G_1(x) > 0$ for $x \in (\bar{a}, \nu)$ (otherwise, we use $-G_1(x)$ instead of $G_1(x)$). Since $f_i(x) = \Psi'(x) \int_{\bar{a}}^{x} g_i(s)\mathrm{d}s$, we get

$$F_i(x) = \frac{f_i(x)}{\Psi'(x)} - \frac{f_i(z)}{\Psi'(z)} = \int_z^x g_i(s)\mathrm{d}s.$$

  Notice that $F_i(\bar{a}) = 0$ and

$$F_i'(x) = g_i(x) - g_i(z)\frac{\mathrm{d}z}{\mathrm{d}x} = \Psi'(x)\, G_i(x), \quad i = 1, 2, 3.$$

  Then we have $F_1(x) > 0$ and $F_1'(x) > 0$ for all $x \in (\bar{a}, \nu)$. Thus hypotheses (H_2) and (H_3) hold.

- Verify hypothesis (H_4): Firstly, we prove that $\alpha F_1(x) + \beta F_2(x) + \gamma F_3(x)$ has at most two zeros on $(\bar{a}, \nu)$. Since $\{G_1, G_2, G_3\}$ is an ECT-system on $(\bar{a}, \nu)$, we deduce that $\alpha G_1 + \beta G_2 + \gamma G_3$ has at most two zeros on $(\bar{a}, \nu)$ counted with multiplicities. Combining

$$\alpha F_1(x) + \beta F_2(x) + \gamma F_3(x) = \int_{\bar{a}}^{x} \Psi'(s)\, \left(\alpha G_1(s) + \beta G_2(s) + \gamma G_3(s)\right) \mathrm{d}s$$

  and Rolle's theorem, we obtain that $\alpha F_1(x) + \beta F_2(x) + \gamma F_3(x)$ has at most three zeros on $[\bar{a}, \nu)$ counted with multiplicities. Since $\alpha F_1(\bar{a}) + \beta F_2(\bar{a}) + \gamma F_3(\bar{a}) = 0$, it follows that $\alpha F_1(x) + \beta F_2(x) + \gamma F_3(x)$ has at most two zeros on $(\bar{a}, \nu)$ counted with multiplicities.

  Similarly, we have that $\alpha F_1(x) + \beta F_2(x)$ has at most one zero on $(\bar{a}, \nu)$ counted with multiplicities.

  Summarizing the above analysis, it follows from the definition of ECT-system and $F_1(x) > 0$ that $\{F_1, F_2, F_3\}$ is an ECT-system on $(\bar{a}, \nu)$. By lemma 2.3 and remark 2.7, we get that $W[F_1, F_2] \neq 0$ and $W[F_1, F_2, F_3] \neq 0$. So we can see that hypothesis (H_4) also holds.

When $s > 2$, the same result can be obtained by using lemma 2.5 repeatedly. □

Thus theoretically each problem which can be solved by theorem 2.4 can be solved by theorem 2.6. More importantly, usually the Abelian integrals we meet have the form

$$I(h) = \alpha \oint_{\Gamma_h} f_1(x)y\ \mathrm{d}x + \beta \oint_{\Gamma_h} f_2(x)y\ \mathrm{d}x + \gamma \oint_{\Gamma_h} f_3(x)y\ \mathrm{d}x,$$

which do not satisfy the conditions in theorem 2.4. So, we have to consider $hI_i(h)$, $i = 1, 2, 3$, see (2.5) for the form of $I_i(h)$. By (1) in lemma 2.5 with $s = 1$, we have

$$hI_i(h) = \oint_{\Gamma_h} \left(f_i(x) + \frac{2}{3}\left(\frac{f_i(x)\Psi(x)}{\Psi'(x)}\right)' \right) y^3\mathrm{d}x,\ \ i = 1, 2, 3.$$

The integrand functions become much more complicated, thus the verification of the conditions in theorem 2.4 will be more difficult.

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Now we will give a concrete example. In the progress of tackling the infinitesimal Hilbert's 16th problem, many scholars focus on the generalized Liénard system of type (m, n)

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -p(x) + \epsilon q(x) y,\end{aligned} \tag{2.9}$$

where $p(x)$ and $q(x)$ are polynomials of degree m and n, respectively. In this respect, scholars have done a lot of work (see [**4**–**7**, **28**]). Recently some scholars consider the small perturbations of system (2.9) with some symmetric properties, that is $p(x)$ is an odd function and $q(x)$ is an even function. In this setting the Liénard system of type (5, 4) [**26**] are stated as follows:

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x\,(a + bx^2 + cx^4) + \epsilon(\alpha + \beta x^2 + \gamma x^4) y,\end{aligned} \tag{2.10}$$

where $0 < |\epsilon| \ll 1$, $(a, b, c) \in \Omega \subseteq \mathbb{R}^3$ and $c \neq 0$, Ω is a compact set. When $a > 0$ and $c < 0$, under a coordinate transformation and time scaling, system (2.10) becomes

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= y,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -x\,(1 + bx^2 - x^4) + \epsilon(\alpha + \beta x^2 + \gamma x^4) y,\end{aligned} \tag{2.11}$$

where $0 < |\epsilon| \ll 1$.

Consider system (2.11). When $\epsilon = 0$, a Hamiltonian first integral is

$$H(x, y) = \frac{y^2}{2} + \Psi(x), \quad \Psi(x) = \frac{x^2}{2} + \frac{b\,x^4}{4} - \frac{x^6}{6}.$$

Denote the closed orbit enclosing the centre (0, 0) by Γ_h, which is given by

$$\Gamma_h = \{(x, y) \mid H(x, y) = h,\ h \in (0,\ h_s)\}.$$

Here $h_s = \Psi(\nu(b)) = (2\sqrt{b^2+4} - b)(\sqrt{b^2+4} + b)^2/48 > 0$ with $\nu(b) = (\sqrt{2}/2)\sqrt{\sqrt{b^2+4} + b}$.

According to Poincaré–Pontryagin theorem, to estimate the maximum number of limit cycles of system (2.11), we only need to estimate the number of zeros of the following integral

$$I(h) = \alpha \oint_{\Gamma_h} y\ \mathrm{d}x + \beta \oint_{\Gamma_h} x^2 y\ \mathrm{d}y + \gamma \oint_{\Gamma_h} x^4 y\ \mathrm{d}x. \tag{2.12}$$

In [**30**], Zhao and Li concluded that when b is a sufficiently small negative number, system (2.11) has at most two limit cycles, counted with multiplicities. Here, applying theorem 2.6, we can reach the same conclusion for arbitrary $b \in \mathbb{R}$ easily.

THEOREM 2.9. *For Liénard system* (2.11), *suppose that* $0 < |\epsilon| \ll 1$, *for any* $b \in \mathbb{R}$, *then system* (2.11) *has at most two limit cycles bifurcating from the compact period annulus, counted with multiplicities.*

Proof. Following the notations in theorem 2.6, let

$$f_1(x) = x^4, \quad f_2(x) = x^2, \quad f_3(x) = 1.$$

Notice that $\Psi(x)$ is an even function, thus the involution is $z = \sigma(x) = -x$. Then by direct calculations, we have

$$
\begin{aligned}
&(\mathrm{H}_1) : \Psi'(x)x = x^2(1 + bx^2 - x^4) > 0, \quad x \in (-\nu(b), \nu(b)) \setminus \{0\}, \\
&(\mathrm{H}_2) : F_1(x) = \frac{2\,x^3}{1 + bx^2 - x^4} > 0, \quad x \in (0, \nu(b)), \\
&(\mathrm{H}_3) : F_1'(x) = \frac{2x^2(x^4 + bx^2 + 3)}{(1 + bx^2 - x^4)^2} = \frac{4x^6 + 4x^2 + 2x^2\,(1 + bx^2 - x^4)}{(1 + bx^2 - x^4)^2} > 0, \\
&\qquad x \in (0, \nu(b)).
\end{aligned}
$$

Furthermore, one has (H_4):

$$\xi(x) = \frac{1}{x^2}, \quad \xi'(x) = -\frac{2}{x^3} \neq 0,$$

and

$$\eta(x) = \frac{1}{x^4}, \quad \eta'(x) = -\frac{4}{x^5}.$$

Finally it follows that

$$\left(\frac{\eta'(x)}{\xi'(x)}\right)' = -\frac{4}{x^3} \neq 0.$$

Thus, according to theorem 2.6, we obtain that the integral $I(h)$ in (2.12) has at most two zeros in $(0,\, h_s)$ counted with multiplicities.

Theorem 2.9 follows. □

Now, let's try using theorem 2.4 to solve this problem. Since the Abelian integrals defined in (2.12) do not fulfil the hypothesis $s > n - 2$ in theorem 2.4, one needs to promote the power of y in $\oint_{\Gamma_h} x^{2k-2}\, y \, \mathrm{d}x \; (k = 1,\, 2,\, 3)$ using lemma 2.5.

To apply lemma 2.5, we consider $h \oint_{\Gamma_h} x^{2k-2}\, y \mathrm{d}x \; (k = 1,\, 2,\, 3)$. Since $h = y^2/2 + x^2/2 + b\,x^4/4 - x^6/6$, it follows from lemma 2.5 that

$$h \oint_{\Gamma_h} x^{2k-2}\, y \, \mathrm{d}x = \oint_{\Gamma_h} g_k(x)\, y^3 \, \mathrm{d}x,$$

where $k = 1,\, 2,\, 3$ and

$$
\begin{aligned}
g_k(x) = {} & \frac{1}{36\,(1 + bx^2 - x^4)^2} \big(12(k+1)x^{2k-2} + 3b(6k+7)x^{2k} \\
& + \left(6b^2k + 15b^2 - 16k - 12\right) x^{2k+2} - b(10k + 29)x^{2k+4} + 4(k+4)x^{2k+6}\big)\,.
\end{aligned}
$$

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Let

$$G_k(x) = \frac{g_k(x)}{\Psi'(x)} - \frac{g_k(-x)}{\Psi'(-x)} = \frac{2\,g_k(x)}{\Psi'(x)}, \quad k = 1, 2, 3.$$

By theorem 2.4, we need to show that $W[G_1](x), W[G_1, G_2](x)$ and $W[G_1, G_2, G_3](x)$ all do not vanish on $(0, \nu(b))$. Obviously, it is much more difficult to deal with this problem by theorem 2.4 than by theorem 2.6. Furthermore, when $b > 0$, it can be judged that $W[G_i, G_j](x)\,(i \neq j,\ \ i, j = 1, 2, 3)$ has at least one zero on the interval $(0, \nu(b))$. In other words, for $b > 0$, this problem can't be solved by theorem 2.4 directly.

3. Proof of theorem 1.2

This section is devoted to proving theorem 1.2 by using theorem 2.6. Before this, we give a proof of lemma 1.1 first.

Proof of lemma 1.1. By [**15**], we just need compute the Abelian integrals associated to system (1.4). By Poincaré–Pontryagin theorem, one has

$$I(h) = \oint_{\Gamma_h^k} \frac{1}{x}\left(f(x,y)\ \mathrm{d}y - g(x,y)\ \mathrm{d}x\right), \quad k = 1, 2. \tag{3.1}$$

Since $f(x,\,y)$ and $g(x,\,y)$ are quadratic polynomials, we calculate each item directly. Obviously,

$$\oint_{\Gamma_h^k} \frac{x^i}{x}\ \mathrm{d}x = 0, \quad i = 0, 1, 2, \qquad \oint_{\Gamma_h^k} \frac{xy^j}{x}\ \mathrm{d}y = 0, \quad j = 0, 1.$$

Note that both the ovals Γ_h^1 and Γ_h^2 are symmetry with respect to the x-axis, which locate in the half plane $x > 0$ and $x < 0$, respectively. It follows that

$$\oint_{\Gamma_h^k} \frac{y^2}{x}\ \mathrm{d}x = (-1)^k \iint_{H \leqslant h} -\frac{2y}{x}\ \mathrm{d}x\mathrm{d}y = 0,$$
$$\oint_{\Gamma_h^k} \frac{y}{x}\ \mathrm{d}y = (-1)^k \iint_{H \leqslant h} -\frac{y}{x^2}\ \mathrm{d}x\mathrm{d}y = 0.$$

Besides, we have

$$\oint_{\Gamma_h^k} \frac{1}{x}\ \mathrm{d}y = -\oint_{\Gamma_h^k} y\ \mathrm{d}\frac{1}{x} = \oint_{\Gamma_h^k} \frac{y}{x^2}\ \mathrm{d}x, \quad \oint_{\Gamma_h^k} \frac{x^2}{x}\ \mathrm{d}y = -\oint_{\Gamma_h^k} y\ \mathrm{d}x,$$
$$\oint_{\Gamma_h^k} \frac{y^2}{x}\ \mathrm{d}y = -\oint_{\Gamma_h^k} \left(bx - 2(b-1) + \frac{b-2}{x}\right)\frac{y}{2x}\ \mathrm{d}x$$
$$= -\frac{b}{2}\oint_{\Gamma_h^k} y\ \mathrm{d}x + (b-1)\oint_{\Gamma_h^k} \frac{y}{x}\ \mathrm{d}x - \frac{b-2}{2}\oint_{\Gamma_h^k} \frac{y}{x^2}\ \mathrm{d}x.$$

Lemma 1.1 is completed by all the computations above and (3.1).

Let $\Psi(x) = bx^2/2 - 2(b-1)x + (b-2)\ln|x| + 3b/2 - 2$, then $\Psi'(x) = b(x-1)(1 + r/x)$, where $r = (2-b)/b > 0$. □

Proof of theorem 1.2. Consider the centre (1, 0) first. There exists an involution $z = \sigma(x)$ satisfying that $\Psi(x) = \Psi(z)$, where $0 < z < 1 < x < +\infty$.

We claim that $x + z > 2$, $0 < xz < 1$ hold for all $x \in (1, +\infty)$. Because $\Psi'(x) = b(x-1)(1 + r/x)$, we have that $\Psi(x) - \Psi(1) > 0$ for $x \in (0, +\infty) \setminus \{1\}$. Write $\Psi(x) - \Psi(1) = \Psi(z) - \Psi(1) = \lambda^2$ $(\lambda > 0)$. When $0 < x - 1 \ll 1$, a straightforward calculation yields to the inverse transformation

$$\begin{aligned} x - 1 &= \lambda + \frac{2-b}{6}\lambda^2 + O(\lambda^3), \\ z - 1 &= -\lambda + \frac{2-b}{6}\lambda^2 + O(\lambda^3). \end{aligned} \tag{3.2}$$

According to (3.2), we have

$$\begin{aligned} x + z &= 2 + \frac{2-b}{3}\lambda^2 + O(\lambda^3), \\ xz &= 1 - \frac{1+b}{3}\lambda^2 + O(\lambda^3). \end{aligned} \tag{3.3}$$

So, when $0 < x - 1 \ll 1$, we obtain that $x + z > 2$ and $0 < xz < 1$.

Next, we prove that $x + z > 2$ and $0 < xz < 1$ hold for all $x \in (1, +\infty)$.

Since $\mathrm{d}z/\mathrm{d}x = \Psi'(x)/\Psi'(z)$, some calculation shows that

$$\frac{\mathrm{d}(x+z)}{\mathrm{d}x} = \frac{2(r-1) - (x+z)(r - xz)/(xz)}{(z-1)\,(1 + r/z)} \tag{3.4}$$

and

$$\frac{\mathrm{d}(xz)}{\mathrm{d}x} = \frac{(x+z)(x+z+r-1) - 2(xz+r)}{(z-1)\,(1 + r/z)}. \tag{3.5}$$

Let $\mathbb{A} = \{x : x + z \leqslant 2,\ x > 1\}$ and $\mathbb{B} = \{x : xz \geqslant 1,\ x > 1\}$.

(I) If $\mathbb{A} = \mathbb{B} = \varnothing$, then our claim is established.

(II) If $\mathbb{A} \neq \varnothing$, $\mathbb{B} = \varnothing$, denote $x_1 = \inf\{x : x \in \mathbb{A}\}$. It is easy to check that

$$\left.\frac{\mathrm{d}(x+z)}{\mathrm{d}x}\right|_{x=x_1} \leqslant 0, \quad x_1 z_1 < 1,\ x_1 + z_1 = 2, \tag{3.6}$$

where $z_1 = \sigma(x_1)$. Substituting (3.6) into (3.4), we have $2(r-1) - 2(r - x_1 z_1)/(x_1 z_1) \geqslant 0$. A direct calculation shows that $x_1 z_1 \geqslant 1$, which contradicts with $x_1 z_1 < 1$.

(III) If $\mathbb{A} = \varnothing$, $\mathbb{B} \neq \varnothing$, denote $x_2 = \inf\{x : x \in \mathbb{B}\}$. Using this and (3.3) we have

$$\left.\frac{\mathrm{d}(xz)}{\mathrm{d}x}\right|_{x=x_2} \geqslant 0, \quad x_2 z_2 = 1,\ x_2 + z_2 > 2, \tag{3.7}$$

where $z_2 = \sigma(x_2)$. Substituting (3.7) into (3.5), we have $(x_2 + z_2)(x_2 + z_2 + r - 1) - 2(1 + r) \leqslant 0$, further we obtain $x_2 + z_2 \leqslant 2$. This leads to a contradiction.

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(IV) If $\mathbb{A} \neq \varnothing$, $\mathbb{B} \neq \varnothing$, denote $x_1 = \inf\{x : x \in \mathbb{A}\}$, $x_2 = \inf\{x : x \in \mathbb{B}\}$.
If $x_1 < x_2$, similar to the proof in (II), we will arrive at a contradiction.
If $x_2 < x_1$, similar to the proof in (III), we will obtain the contradiction.
If $x_1 = x_2$, then we have $x_1 + z_1 = x_2 + z_2 = 2$ and $x_1 z_1 = x_2 z_2 = 1$. According to mean value inequality, we obtain $x_1 = z_1 = x_2 = z_2 = 1$, which contradicts with $x > 1 > z > 0$.

From all the discussions above, we get $\mathbb{A} = \mathbb{B} = \varnothing$. Hence $x + z > 2$, $0 < xz < 1$ hold for all $x \in (1, +\infty)$.

After a linear combination, the integral $I(h)$ given in (1.5) can be rewritten as

$$I(h) = -\frac{\gamma}{b} \oint_{\Gamma_h^1} b\left(\frac{1}{x} - 1\right) y \, dx - \frac{\gamma+\beta}{b} \oint_{\Gamma_h^1} \frac{b}{x}\left(\frac{1}{x} - 1\right) y \, dx + \frac{\alpha+\beta+\gamma}{b} \oint_{\Gamma_h^1} \frac{b}{x^2} y \, dx.$$

For the sake of shortness, we denote

$$f_1(x) = b\left(\frac{1}{x} - 1\right), \quad f_2(x) = \frac{b}{x}\left(\frac{1}{x} - 1\right), \quad f_3(x) = \frac{b}{x^2}.$$

Now, we verify the hypotheses (H_1)–(H_4).

- Verify the hypothesis (H_1)

$$\Psi'(x)(x-1) = b(x-1)^2\left(1 + \frac{r}{x}\right) > 0, \quad x \in (0, +\infty) \setminus \{1\}$$

.

- Verify the hypothesis (H_2)
  Since $f_1(x) = b(1/x - 1)$, it is easy to get that $F_1(x) = 1/(z + r) - 1/(x + r) > 0$.

- Verify the hypothesis (H_3)
  After some calculation, we get

$$F_1'(x) = \frac{(x-z)\left(3r^2xz + 3rx^2z + 3rxz^2 + x^3z + x^2z^2 + xz^3 + r^3 - 3rxz - x^2z - xz^2\right)}{(z+r)^3 (x+r)^2 x (1-z)}.$$

  Let $x + z = u$, $xz = v$, then we have $u > 2$, $0 < v < 1$, and

$$F_1'(x) = \frac{(x-z)\left(r^3 + 3r^2v + (3uv - 3v)r + u^2v - uv - v^2\right)}{(z+r)^3 (x+r)^2 x (1-z)}.$$

  It is easy to check that $3uv - 3v > 0$ and $u^2v - uv - v^2 > 0$ for all $u > 2$, $0 < v < 1$. Thus, we get that $F_1'(x) > 0$ for all $x \in (0, +\infty)$.

- Verify the hypothesis (H_4)

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Since $F_2(x) = 1/(z(z+r)) - 1/(x(x+r))$, after some calculation, we obtain

$$\xi'(x) = \left(\frac{F_2(x)}{F_1(x)}\right)' = \frac{r^2x+r^2z+2rx^2+2rz^2+x^3+z^3-2r^2-2xr-2zr-x^2-z^2}{z\,x^2\,(1-z)\,(z+r)}.$$

Taking $x+z=u,\ xz=v$ into $\xi'(x)$, we get

$$\xi'(x) = \frac{(u-2)\,r^2 + \left(2\,u^2-2\,u-4\,v\right)r + u^3 - u^2 - 3\,uv + 2\,v}{z\,x^2\,(1-z)\,(z+r)}.$$

It is easy to check that all the coefficients of r^i are positive for $u > 2$ and $0 < v < 1$, thus $\xi'(x) > 0$.

Furthermore, by (2.7)

$$\left(\frac{\eta'(x)}{\xi'(x)}\right)' = \frac{A_5(u,v)r^5 + A_4(u,v)r^4 + A_3(u,v)r^3 + A_2(u,v)r^2 + vA_1(u,v)r + vA_0(u,v)}{\xi'^2(x)(x-1)^3x^4(z-1)^6z(r+z)^3},$$

where

$$\begin{aligned}
A_5(u,v) =&2\,u^4v + u^4 - 15\,u^3v - 4\,u^2v^2 - 3\,u^3 + 34\,u^2v + 30\,uv^2 - 4\,v^3\\
&+ 6\,u^2 - 48\,uv - 32\,v^2 - 6\,u + 36\,v,\\
A_4(u,v) =&\ 10\,u^5v + 3\,u^5 - 65\,u^4v - 38\,u^3v^2 - 6\,u^4 + 135\,u^3v\\
&+ 208\,u^2v^2 + 22\,uv^3 + 9\,u^3 - 174\,u^2v - 288\,uv^2 - 92\,v^3 - 6\,u^2\\
&+ 114\,uv + 192\,v^2 - 36\,v,\\
A_3(u,v) =&\ 20\,u^6v + 3\,u^6 - 115\,u^5v - 108\,u^4v^2 - 6\,u^5 + 242\,u^4v + 491\,u^3v^2\\
&+ 162\,u^2v^3 + 6\,u^4 - 276\,u^3v - 808\,u^2v^2 - 424\,uv^3 - 80\,v^4 - 3\,u^3\\
&+ 174\,u^2v + 600\,uv^2 + 368\,v^3 - 48\,uv - 192\,v^2,\\
A_2(u,v) =&\ 20\,u^7v + u^7 - 105\,u^6v - 136\,u^5v^2 - 3\,u^6 + 226\,u^5v + 567\,u^4v^2\\
&+ 293\,u^3v^3 + 3\,u^5 - 242\,u^4v - 992\,u^3v^2 - 798\,u^2v^3\\
&- 216\,uv^4 - u^4 + 135\,u^3v + 808\,u^2v^2\\
&+ 888\,uv^3 + 240\,v^4 - 34\,u^2v - 288\,uv^2 - 368\,v^3 + 32\,v^2,\\
A_1(u,v) =&\ 10\,u^8 - 50\,u^7 - 80\,u^6v + 105\,u^6 + 334\,u^5v + 211\,u^4v^2 - 115\,u^5\\
&- 567\,u^4v - 686\,u^3v^2 - 186\,u^2v^3 + 65\,u^4 + 491\,u^3v + 798\,u^2v^2\\
&+ 450\,uv^3 - 12\,v^4 - 15\,u^3 - 208\,u^2v - 424\,uv^2 - 240\,v^3\\
&+ 30\,uv + 92\,v^2,\\
A_0(u,v) =&\ 2\,u^9 - 10\,u^8 - 18\,u^7v + 20\,u^7 + 80\,u^6v + 54\,u^5v^2 - 20\,u^6\\
&- 136\,u^5v - 211\,u^4v^2 - 53\,u^3v^3 + 10\,u^5 + 108\,u^4v + 293\,u^3v^2\\
&+ 186\,u^2v^3 - 6\,uv^4 - 2\,u^4 - 38\,u^3v\\
&- 162\,u^2v^2 - 216\,uv^3 + 12\,v^4 + 4\,u^2v + 22\,uv^2 + 80\,v^3 + 4\,v^2.
\end{aligned}$$

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In order to verify that $A_i(u,\, v) > 0\,(i = 0,\, 1,\, 2,\, 3,\, 4,\, 5)$ for all $(u,\, v) \in \{(u,\, v) \mid u > 2,\, 0 < v < 1\}$, we will consider $A_i(u,\, v)$ in two regions $\{(u,\, v) \mid u \geqslant 200,\, 0 < v < 1\}$ and $\{(u,\, v) \mid 2 < u < 200,\, 0 < v < 1\}$. Firstly, choose some functions $B_i(u)$ such that

$$A_i(u, v) > B_i(u) \quad (i = 0, 1, 2, 3, 4, 5),$$

for $u > 2$ and $0 < v < 1$, where

$$\begin{aligned}
B_0(u) &= 2\,u^9 - 10\,u^8 - 18\,u^7 - 20\,u^6 - 136\,u^5 - 213\,u^4 - 91\,u^3 - 162\,u^2 - 222\,u,\\
B_1(u) &= 10\,u^8 - 50\,u^7 - 80\,u^6 - 115\,u^5 - 567\,u^4 - 701\,u^3 - 394\,u^2 - 424\,u - 252,\\
B_2(u) &= u^7 - 108\,u^6 - 136\,u^5 - 243\,u^4 - 992\,u^3 - 832\,u^2 - 504\,u - 368,\\
B_3(u) &= 3\,u^6 - 121\,u^5 - 108\,u^4 - 279\,u^3 - 808\,u^2 - 472\,u - 272,\\
B_4(u) &= 3\,u^5 - 71\,u^4 - 38\,u^3 - 180\,u^2 - 288\,u - 128,\\
B_5(u) &= u^4 - 18\,u^3 - 4\,u^2 - 54\,u - 36.
\end{aligned}$$

Then, it is easy to verify that $B_i(u) > 0$ $(i = 0,\, 1,\, 2,\, 3,\, 4,\, 5)$ for all $u \geqslant 200$ by the estimate step by step. For example,

$$\begin{aligned}
B_5(u) &\geqslant 200\,u^3 - 18\,u^3 - 4\,u^2 - 54\,u - 36\\
&\geqslant 182 \times 200\,u^2 - 4\,u^2 - 54\,u - 36\\
&\geqslant 36\,396 \times 200\,u - 54u - 36 > 0, \quad u \geqslant 200.
\end{aligned}$$

Hence, $A_i(u,\, v) > 0\,(i = 0,\, 1,\, 2,\, 3,\, 4,\, 5)$ for $u \geqslant 200,\ 0 < v < 1$.

On the other hand, according to the polynomial-ring method in algebraic geometry in Maple, it is concluded that $A_i(u,\, v)\,(0 \leqslant i \leqslant 5)$ has no extreme points in $\{(u,\, v) \mid 2 < u < 200,\, 0 < v < 1\}$. And it is easy to check that $A_i(2,\, v) \geqslant 0$, $A_i(u,\, 1) \geqslant 0$, $A_i(u,\, 0) \geqslant 0$ and $A_i(200,\, v) > 0$. This implies that all the coefficients of r^i are positive for $2 < u < 200$ and $0 < v < 1$. Therefore, we deduce that $A_i(u,\, v) > 0\,(i = 0,\, 1,\, 2,\, 3,\, 4,\, 5)$ for all

$$(u,\, v) \in \{(u,\, v) \mid u > 2,\, 0 < v < 1\}.$$

Summarizing the above analysis, we get $(\eta'(x)/\xi'(x))' \neq 0$.

To sum up, by theorem 2.6, the integral $I(h)$ in (1.5) has at most two zeros in $(0, +\infty)$ counted with multiplicities.

Finally, we consider the number of limit cycles bifurcated from the centre $((b-2)/b,\, 0)$. Use the transformation $\tilde{x} = b/(b-2)x$, $\tilde{y} = \sqrt{b/(2-b)}y$, $\tilde{t} = \sqrt{(2-b)/b}t$, $c = 2 - b$, and rewrite $(\tilde{x},\, \tilde{y},\, \tilde{t})$ as $(x,\, y,\, t)$, then system (1.2) has the

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following form

$$\begin{aligned}\frac{\mathrm{d}x}{\mathrm{d}t} &= 2xy,\\ \frac{\mathrm{d}y}{\mathrm{d}t} &= -cx^2 + 2(c-1)x + 2 - c.\end{aligned} \tag{3.8}$$

A first integral of system (3.8) is

$$\tilde{H}(x, y) = y^2 + \frac{cx^2}{2} - 2(c-1)x + (c-2)\ln|x| + \frac{3c}{2} - 2. \tag{3.9}$$

It is easy to verify that the centre $((b-2)/b,\, 0)$ of system (1.2) becomes the centre $(1,\, 0)$ of system (3.8) and the continuous family of ovals Γ_h^2 surrounding the centre $((b-2)/b,\, 0)$ defined in (1.3) becomes $\Gamma_h = \{(x,\, y) \mid \tilde{H}(x,\, y) = h,\ \ h \in (0,\, +\infty)\}$ if we take $\tilde{h} = ((2-c)/c)h + (2-c)\ln(c/(2-c)) + 2 - 2/c$ in (1.3) and rewrite $\tilde{h}$ as h. Similarly, the cyclicity of period annulus around the centre $((b-2)/b,\, 0)$ is equal to the number of zeros of the following Abelian integrals in $(0,\, +\infty)$ by lemma 1.1

$$I(h) = -\alpha\sqrt{\frac{2-c}{c}}\oint_{\Gamma_h}\frac{y}{x^2}\,\mathrm{d}x + \beta\sqrt{\frac{c}{2-c}}\oint_{\Gamma_h}\frac{y}{x}\,\mathrm{d}x - \gamma\left(\frac{c}{2-c}\right)^{3/2}\oint_{\Gamma_h} y\,\mathrm{d}x,$$

where Γ_h is the periodic orbit surrounding the centre $(1,\, 0)$ of system (3.8) defined by $\tilde{H}(x,\, y) = h$ with $\tilde{H}(x,\, y)$ given in (3.9). Using the conclusion on the centre $(1,\, 0)$, thus we can see that at most two limit cycles generate from the period annulus around the centre $((b-2)/b,\, 0)$ counted with multiplicities.

The proof is finished. □

4. Proof of theorem 1.3

In this section, we mainly consider all the possible configurations of limit cycles from two period annuli for system (1.4). For convenience, denote $I_i(h) = \oint_{\Gamma_h^1}(y/x^{i-1})\mathrm{d}x\,(i = 1,\, 2,\, 3)$ and define the centroid curve $\sum = \{(P,\, Q) \mid P(h) = I_3(h)/I_2(h),\, Q(h) = I_1(h)/I_2(h),\quad h \in (0,\, +\infty)\}$, where $\Gamma_h^1 = \{(x,\, y) \mid H(x,\, y) = h,\, x > 0\}$ is the closed curve around the centre $(1,\, 0)$. Let $l_i(x) = 1/x^{i-1}$ and $L_i(x) = l_i(x)/\Psi'(x) - l_i(z)/\Psi'(z)\,(i = 1,\, 2,\, 3)$, where $z = \sigma(x)$ is the involution satisfying that $\Psi(x) = \Psi(z)$.

First, we will study some geometric properties of the centroid curve $\sum$.

Proposition 4.1. *For* $h \in (0,\, +\infty)$, $P(h) > 0$, $Q(h) > 0$, $P'(h) > 0$ *and* $Q'(h) < 0$.

Proof. It is easy to check that $P(h) > 0$, $Q(h) > 0$ and we only prove that $P'(h) > 0$ and $Q'(h) < 0$.

Since $l_1(x) = 1$, $l_2(x) = 1/x$ and $l_3(x) = 1/x^2$, it is easy to obtain that

$$L_1(x) = \frac{(1+r)(z-x)(xz+r)}{2(x-1)(x+r)(z-1)(z+r)},$$
$$L_2(x) = \frac{(1+r)(z-x)(x+r-1+z)}{2(x-1)(x+r)(z-1)(z+r)}$$

and

$$L_3(x) = \frac{(1+r)(z-x)\left(xr+zr+x^2+xz+z^2-r-x-z\right)}{2x(x-1)(x+r)z(z-1)(z+r)}.$$

After some calculation, we get

$$\left(\frac{L_1(x)}{L_2(x)}\right)' = \frac{1}{(x+r-1+z)^2 x(z-1)(z+r)} \left(r^2x^2z+r^2xz^2+2rx^3z+2rxz^3+x^4z\right.$$
$$+xz^4-4r^2xz-4rx^2z-4rxz^2-2x^3z-2xz^3+r^2x+r^2z+4rxz$$
$$\left.+x^2z+xz^2\right).$$

Let $x+z=u$, $xz=v$, then

$$\left(\frac{L_1(x)}{L_2(x)}\right)' = \frac{(uv+u-4v)r^2+2\left(u^2-2u-2v+2\right)vr+\left(u^3-2u^2-3uv+u+4v\right)v}{(x+r-1+z)^2 x(z-1)(z+r)}.$$

Similar to the proof of $(\eta'(x)/\xi'(x))' \neq 0$ in theorem 1.2, we obtain $(L_1(x)/L_2(x))' < 0$. Using theorem 1 in [**19**], we know $Q'(h) = (I_1(h)/I_2(h))' < 0$.

As for $L_3(x)/L_2(x)$, some computations show that

$$\left(\frac{L_3(x)}{L_2(x)}\right)' = \frac{M(x,z,r)}{zx^2(x+r-1+z)^2(1-z)(z+r)},$$

where

$$M(x,z,r) = \left(u^2-2u-2v+2\right)r^3+\left(3u^3-7u^2-7uv+7u+6v-2\right)r^2$$
$$+\left(3u^4-8u^3-8u^2v+7u^2+16uv-2v^2-2u-6v\right)r$$
$$+u^5-3u^4-3u^3v+3u^3+8u^2v-uv^2-u^2-7uv+2v^2+2v.$$

Similarly, we have $M(x,z,r) > 0$. This implies that $(L_3(x)/L_2(x))' > 0$. Therefore, by theorem 1 in [**19**], we get $P'(h) = (I_3(h)/I_2(h))' > 0$. The proof is finished. □

PROPOSITION 4.2. *When* $h \to 0^+$, $P(h) \to 1$, $Q(h) \to 1$. *When* $h \to +\infty$, $P(h) \to +\infty$, $Q(h) \to 0$.

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Proof. By Green formula and mean value theorem, there exist $\rho_1,\ \rho_2$ such that

$$Q(h) = \frac{\iint_{\{(x,y)|H(x,y)\leqslant h,x>0\}} 1\ \mathrm{d}x\mathrm{d}y}{\iint_{\{(x,y)|H(x,y)\leqslant h,x>0\}} (1/x)\ \mathrm{d}x\mathrm{d}y} = \rho_1$$

and

$$P(h) = \frac{\iint_{\{(x,y)|H(x,y)\leqslant h,x>0\}} (1/x^2)\ \mathrm{d}x\mathrm{d}y}{\iint_{\{(x,y)|H(x,y)\leqslant h,x>0\}} (1/x)\ \mathrm{d}x\mathrm{d}y} = \rho_2.$$

When $h \to 0^+$, $\{(x,\, y) \mid H(x,\, y) \leqslant h,\, x > 0\}$ tends to the point $(1,\, 0)$, then

$$\rho_1,\ \rho_2 \to 1,\ P(h),\ Q(h) \to 1.$$

When $h \gg 1$, from $\Psi(\nu(h)) = h$, we obtain

$$\sqrt{\frac{b}{2}}\nu(h)\left[1 - \frac{2(b-1)}{\nu(h)}\frac{2}{b} + O\left(\frac{\ln \nu(h)}{\nu^2(h)}\right)\right]^{1/2} = \sqrt{h},$$

which implies that

$$\nu(h) = \sqrt{\frac{2}{b}}h^{1/2} + \frac{2(b-1)}{b} + o(1), \quad h \gg 1. \tag{4.1}$$

On the other hand, by $\Psi(\mu(h)) = h$, we get $0 < \mu(h) \ll 1$ and

$$\exp\left\{\frac{b\mu^2(h)}{2} - 2(b-1)\mu(h) + (b-2)\ln \mu(h) + \frac{3b}{2} - 2\right\} = \exp(h).$$

After some calculation

$$\mu(h)\exp\left\{\frac{b\mu^2(h)}{2b-4} - \frac{2(b-1)\mu(h)}{b-2}\right\} = \exp\left\{\frac{3b-4}{4-2b}\right\}\exp\left\{\frac{h}{b-2}\right\}. \tag{4.2}$$

Combining (4.2) and $\exp(z) = 1 + z + o(z)$ for $|z| \ll 1$, we have

$$\mu(h)\left(1 - \frac{2(b-1)\mu(h)}{b-2} + o(\mu(h))\right) = \exp\left\{\frac{3b-4}{4-2b}\right\}\exp\left\{\frac{h}{b-2}\right\}.$$

Further, we get that

$$\mu(h) = \exp\left\{\frac{3b-4}{4-2b}\right\}\exp\left\{\frac{h}{b-2}\right\} + o\left(\exp\left\{\frac{h}{b-2}\right\}\right), \quad h \gg 1. \tag{4.3}$$

In the following we estimate the order of $I_1(h)$, $I_2(h)$ and $I_3(h)$ for $h \to +\infty$.

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Let us consider the integral $I_1(h)$ firstly. Since $y^2 + \Psi(x) = h$, we get

$$I_1(h) = \oint_{\Gamma_h^1} y \, \mathrm{d}x = 2\int_{\mu(h)}^{\nu(h)} \sqrt{h - \Psi(x)} \, \mathrm{d}x.$$

Then

$$I_1(h) \leqslant 2\int_{\mu(h)}^{\nu(h)} \sqrt{h} \, \mathrm{d}x = 2\sqrt{\frac{2}{b}} h + o(h). \tag{4.4}$$

Secondly, consider the integral $I_2(h)$. Because

$$I_2(h) = 2\int_{\mu(h)}^{\nu(h)} \frac{\sqrt{h - \Psi(x)}}{x} \mathrm{d}x = 2\int_{\mu(h)}^{1} \frac{\sqrt{h - \Psi(x)}}{x} \mathrm{d}x + 2\int_{1}^{\nu(h)} \frac{\sqrt{h - \Psi(x)}}{x} \mathrm{d}x,$$

we have

$$I_2(h) > 2\int_{\mu(h)}^{1} \frac{\sqrt{h - \Psi(x)}}{x} \mathrm{d}x = 2\int_{0}^{\bar{\mu}(h)} \sqrt{h - \Psi(\exp\{-w\})} \, \mathrm{d}w,$$

where

$$\bar{\mu}(h) = -\ln(\mu(h)) = \frac{h}{2-b} + o(h)$$

follows from (4.3). Let $w = -\ln x$, one has

$$\Phi(w) := \Psi(\exp\{-w\}) = \frac{b}{2} \exp\{-2w\} - 2(b-1)\exp\{-w\} + (2-b)w + \frac{3b}{2} - 2.$$

Then for $0 < w < \bar{\mu}(h)/2$,

$$\Phi(w) < \frac{b}{2} + 2 + (2-b)w + \frac{3b}{2} - 2 = (2-b)w + 2b < (2-b)\frac{\bar{\mu}(h)}{2} + 2b = \frac{h}{2} + o(h).$$

Consequently

$$\begin{aligned} I_2(h) &> 2\int_{0}^{\bar{\mu}(h)} \sqrt{h - \Phi(w)} \, \mathrm{d}w \\ &> 2\int_{0}^{\bar{\mu}(h)/2} \sqrt{h - \Phi(w)} \, \mathrm{d}w \\ &> 2\int_{0}^{\bar{\mu}(h)/2} \left(\sqrt{\frac{h}{2}} + o\left(\sqrt{h}\right)\right) \, \mathrm{d}w \\ &= \frac{\sqrt{2}h^{3/2}}{2(2-b)} + o\left(h^{3/2}\right). \end{aligned} \tag{4.5}$$

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On the other hand, we have

$$\begin{aligned} I_2(h) &= 2\int_0^{\bar{\mu}(h)} \sqrt{h-\Phi(w)}\ \mathrm{d}w + 2\int_1^{\nu(h)} \frac{\sqrt{h-\Psi(x)}}{x}\ \mathrm{d}x \\ &\leqslant 2\int_0^{\bar{\mu}(h)} \sqrt{h}\ \mathrm{d}w + 2\int_1^{\nu(h)} \sqrt{h}\ \mathrm{d}x \\ &= \frac{2h^{3/2}}{2-b} + o\left(h^{3/2}\right). \end{aligned}$$

To sum up, we obtain

$$I_2(h) \sim h^{3/2}, \quad h \to +\infty. \tag{4.6}$$

Thereby, by (4.4) and (4.6),

$$Q(h) = \frac{I_1(h)}{I_2(h)} \to 0,\ \text{when}\ h \to +\infty.$$

Similar as the estimate in (4.5), we have

$$\begin{aligned} I_3(h) &> 2\int_0^{\bar{\mu}(h)} \exp\{w\}\sqrt{h-\Phi(w)}\ \mathrm{d}w \\ &> 2\int_0^{\bar{\mu}(h)/2} \exp\{w\}\left(\sqrt{\frac{h}{2}} + o\left(\sqrt{h}\right)\right)\ \mathrm{d}w \\ &= \sqrt{2h}\exp\left\{\frac{h}{2}\right\} + o\left(\sqrt{h}\exp\left\{\frac{h}{2}\right\}\right). \end{aligned}$$

Thus,

$$\lim_{h\to+\infty} P(h) = \lim_{h\to+\infty} \frac{I_3(h)}{I_2(h)} = +\infty.$$

The proof is finished. □

Now we consider the simultaneous bifurcation and distribution of limit cycles from the period annuli around $(1, 0)$ and $((b-2)/b, 0)$. Let $\bar{\Sigma} = \{(\bar{P}, \bar{Q}) \mid \bar{P}(h) = I_3(h)/I_2(h),\ \bar{Q}(h) = I_1(h)/I_2(h),\ \ h \in (2-2/b+(b-2)\ln(2/b-1), +\infty)\}$, where $I_i(h) = \oint_{\Gamma_h^2} (y/x^{i-1})\mathrm{d}x\ (i = 1, 2, 3)$ with Γ_h^2 given in (1.3) being the ovals around the centre $((b-2)/b, 0)$. Similar to the proof of theorem 1.2, we use the transformation $\tilde{x} = b/(b-2)x$, $\tilde{y} = \sqrt{b/(2-b)}y$, $\tilde{t} = \sqrt{(2-b)/b}t$, $c = 2-b$, and rewrite $(\tilde{x}, \tilde{y}, \tilde{t})$ as (x, y, t), then the system (1.2) is changed to system (3.8), which has the same form as system (1.2) with the parameter c instead of b. Taking $\tilde{h} = ((2-c)/c)h + (2-c)\ln(c/(2-c)) + 2 - 2/c$, the closed curve Γ_h^2 around the centre $((b-2)/b, 0)$ becomes $\{(x, y) \mid \tilde{H}(x, y) = \tilde{h},\ \ x > 0,\ \ \tilde{h} \in (0, +\infty)\}$ with $\tilde{H}$ given in (3.9). Notice that $c \in (0, 2)$, too, thus by proposition 4.1, we have

$$\bar{P}(h) < 0, \quad \bar{Q}(h) < 0, \quad \bar{P}'(h) < 0, \quad \bar{Q}'(h) > 0. \tag{4.7}$$

Notice that the number of limit cycles bifurcating from the period annulus around $(1, 0)$ (resp. $((b-2)/b, 0)$) is equal to the number of intersection points of the curve

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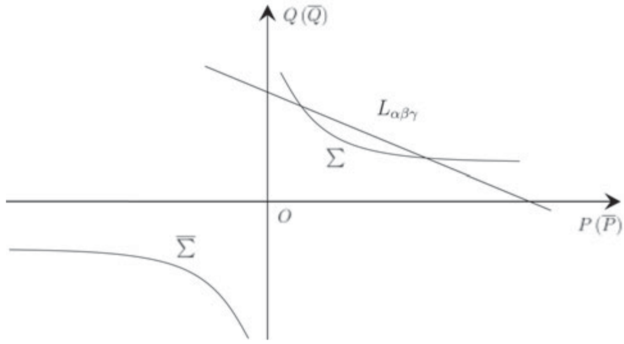

Figure 3. Number of intersection points of the curve Σ ($\bar{\Sigma}$) with the line $L_{\alpha\beta\gamma}$ ($\bar{L}_{\alpha\beta\gamma}$).

Σ (resp. $\bar{\Sigma}$) with the line $L_{\alpha\beta\gamma}$ (resp. $\bar{L}_{\alpha\beta\gamma}$), taking into account their multiplicities, where

$$L_{\alpha\beta\gamma}: \ \beta + \alpha P + \gamma Q = 0 \quad (\text{resp. } \bar{L}_{\alpha\beta\gamma}: \ \beta + \alpha \bar{P} + \gamma \bar{Q} = 0).$$

Proof of theorem 1.3. Put two centroid curves Σ and $\bar{\Sigma}$ in the same plane, thus the lines $L_{\alpha\beta\gamma}$ and $\bar{L}_{\alpha\beta\gamma}$ are the same. The graphs of Σ and $\bar{\Sigma}$ are shown in figure 3 by propositions 4.1 and 4.2 and (4.7). Obviously, the configurations of limit cycles (0, 0), (1, 0), (0, 1) and (1, 1) are trivial (see figure 3), where (u, v) represents that exactly u and v limit cycles simultaneously bifurcate from the periodic orbits surrounding the centres (1, 0) and $((b-2)/b, 0)$ respectively. Without loss of generality, suppose that $L_{\alpha\beta\gamma}$ has two intersections with the centroid curve Σ. As is shown in figure 3, the slope of $L_{\alpha\beta\gamma}$ must be negative, thus $\bar{L}_{\alpha\beta\gamma}$ has no intersection with $\bar{\Sigma}$, and the configuration of limit cycles must be (2, 0). Similarly, the configuration of limit cycles (0, 2) can be achieved.

□

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