1
Introduction

Classical special functions are a traditional field of mathematics. As particular solutions of singular boundary eigenvalue problems of linear ordinary differential equations of second order, they are by definition functions that can be represented as the product of an asymptotic factor and a (finite or infinite) Taylor series. The coefficients of these series are by definition solutions of two-term recurrence relations, from which an algebraic boundary eigenvalue criterion can be formulated. This method is called the *Sommerfeld polynomial method* (Rubinowicz, 1972); thus, one can say that the boundary eigenvalue condition is by definition algebraic in nature. It is the central message in this book that one can resolve this restriction and it is shown how to do this methodically, and what the fundamental mathematical principle underlying this method is.

Now, of course, the method developed for this also applies to problems that can be solved with classical methods. So, in order to present the newly developed methods in the light of what is known, and to be able to understand the new perspective more easily (and also measure the results obtained against what is already known), this new method is applied in this chapter to the already known solutions. This makes it possible to classify the new according to the well known. Accordingly, it is a 'phenomenological' introduction, where the focus is not on definitions, theorems and their proofs, but on the ad hoc introduction of the relevant quantities. The systematic introduction based on definitions follows in the subsequent chapters, this time with respect to differential equations, which are no longer accessible by classical methods.

1.1 Historical Remarks

The history of the subject treated here goes back to the middle of the nineteenth century, to a time when the German mathematician **Lazarus Fuchs** (1833–1902) wrote down the local solution of a linear, ordinary differential equation with polynomial coefficients in the vicinity of a singularity. This solution consists of a product of an (in general) irrational power of the independent variable and an (in general)

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infinite Taylor, thus one-sided, series. Yet, the explicit form of the solution was only one aspect of the great importance of Fuchs' approach. It was just as important to have recognised the significance of the singularity of a differential equation in the first place. Singularities are something ubiquitous; they cannot be avoided, not in the equation and certainly not in the solution. And not only that: singularities of differential equations determine the behaviour of their solutions everywhere, even where the equations are holomorphic. This is perhaps the most significant peculiarity of differential equations. Therefore, it is more than justified to bring singularities to the centre of consideration if one wants to deal with differential equations in depth. This insight, which is now about 150 years old, is the basis of this work.

It is well known that the basic approach to solving a differential equation is based on Weierstrass' approximation theorem for integer powers, according to which powers can be used to approximate any holomorphic function. Based on this, the French mathematician **Paul Painlevé** (1863–1933) developed a basic existence theorem for local solutions, the so-called 'calcul des limites' or majorant method. In turn, all fundamental questions – such as those concerning analytical continuations or the uniqueness of solutions – are based on this. One can say that this complex of fundamental questions was largely settled at the beginning of the twentieth century.

The full significance of Fuchs' work was revealed in the fact that there are two types of singularities in differential equations: those that are now called *regular* and those that are called *irregular*. This distinction is triggered by the nature of their local solutions. For the irregular singularities of differential equations, the Fuchsian approach did not provide local solutions, only for regular ones. As the German mathematician **Meyer Hamburger** (1838–1903) showed a little later, this requires generalised, two-sided infinite power series, i.e., Laurent series. This makes for a significant technical complication. So, it remained to search for a one-sided series approach for the case of irregular singularities. Although the Fuchsian approach could not do this, it showed the way: a one-sided replacement for the Laurent series was then concretely worked out about 20 years later by the French mathematician **Achille Marie Gaston Floquét** (1847–1920) and, at about the same time, by the German mathematician **Ludwig Wilhelm Thomé** (1841–1910). (That is why these approaches are now called Thomé solutions in Germany and Floquét solutions in France.) The crucial idea was to write down the asymptotic factors, explicitly. The problem with Thomé's solutions, however, was that they did not converge in all cases. Thomé put a lot of effort into showing under what conditions his series approaches converged. It was finally the French mathematician **Henri Poincaré** (1854–1912) who showed the significance of Thomé's series: they are asymptotic solutions of the differential equation that represent their solutions within sectors, at the top of which is placed the singularity. The lateral lines delimiting these sectors are called *Stokes lines*, after the Irish mathematician and physicist **Sir George Gabriel Stokes** (1819–1903).

Another important insight into linear differential equations was that it was understood that differential equations with exclusively regular singularities could serve as

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the cornerstone of a mathematical theory, because the irregular singularities could be generated by merging regular singularities. Equations with exclusively regular singularities were given the name *Fuchsian differential equations* and the simplest forms among them were in turn given their own names. Thus, the simplest form of the Fuchsian differential equation with only one singularity is called *Laplace's equation*, the one with two singularities is called *Euler's equation* and the one with three singularities is called *Gauss' equation*.

The Fuchsian approach always leads to difference equations for its coefficients and thus to asymptotic questions of their solution for large values of the index. It has been found that significant differences in the asymptotic behaviour of the solutions can exist, although the ratio of two successive terms of the solutions tends towards one and the same value for large values of the index. Depending on whether this is the case with a difference equation or not, it is called *regular* or *irregular*.

As far as the investigation of regular difference equations is concerned, Henri Poincaré and the German mathematician **Oskar Perron** (1880–1975) excelled. For the study of irregular difference equations, in turn, the Canadian mathematician **George David Birkhoff** (1884–1944), the Russian mathematician **Waldemar Juliet Trjitzinsky** (1901–1973) and the American mathematician **Clarence Raymond Adams** (1898–1965) made lasting contributions in the 1920s (see, e.g., Birkhoff–Trjitzinsky theory and the Birkhoff–Adams theorem in Aulbach et al., 2004). Thus, while Henri Poincaré recognised the importance of Thomé's approaches to linear differential equations by proving their asymptotic character, Birkhoff and Trjitzinsky carried out the analogous investigations for linear difference equations.

The importance of all this work for the development of modern physics at the beginning of the twentieth century should be undisputed, namely relativity and quantum theory, just as this development in turn had an effect on mathematics. The realisation that nature is essentially linear on small scales, as evidenced by the *Schrödinger equation*, has given rise to this retroactive influence. Singular boundary eigenvalue problems of linear differential equations of second order were henceforth an important object of research. It turned out that it is not the order of the differential equation that determines the order of the difference equation resulting from a Fuchsian approach, but the number of its singularities.

In order for the solution of a linear differential equation to behave in a prescribed manner not only at one point, but at two, this solution must have a parameter. In quantum theory, this is the energy parameter. In order for the solution to behave in a prescribed manner at two different points, this energy parameter must assume certain values. Calculating these means solving the boundary eigenvalue problem. The condition on which this determination is based is called the boundary eigenvalue condition.

The mathematical form of the boundary eigenvalue condition is of central importance for the solution of the problem. It turns out that this condition is generally only algebraic in nature if the underlying differential equation is either of the Laplacian, Eulerian or Gaussian type, i.e., originates from a Fuchsian equation that has

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at most three singularities. If this number of singularities is larger than three, then the boundary eigenvalue condition becomes transcendental. And for such equations, there was only one method to write the boundary eigenvalue condition at all, and this is only if the order of the difference equation which the coefficients of Fuchs' solution approaches is at most two: the method of infinite continued fractions.

Transcendental boundary eigenvalue conditions thus occur systematically for Fuchsian differential equations that have more than three singularities. The simplest of these equations is called *Heun's differential equation*, because the German mathematician **Karl Heun** (1859–1929) studied it systematically for the first time in a paper from 1889.

Today, a distinction is made between algebraic and transcendental boundary eigenvalue conditions in two respects: the singular boundary eigenvalue problems that lead to transcendental eigenvalue conditions are called *central two-point connection problems* (CTCPs) and the resulting solution functions are called *higher special functions*; all others are called *classical special functions*.

To be precise, local solution approaches such as those of Lazarus Fuchs are no longer sufficient for solving singular boundary eigenvalue problems. This required a large-scale scientific development that began when **Arnold Sommerfeld** (1868–1951) had his great era as a professor at the Ludwig Maximilian University of Munich. In 1919, **Wolfgang Ernst Pauli** (1900–1958) asked Sommerfeld for a topic for a doctoral thesis. Pauli received from Sommerfeld the task of applying the rules of quantum mechanics established by **Niels Bohr** (1885–1962) in 1913 in order to show that the ionised hydrogen molecule is stable under the conditions of the then new quantum mechanics. Since the quantisation rules of Bohr and Sommerfeld (for which Bohr received the Nobel Prize) were not quite correct (they were what are now called semiclassical quantisation rules), he did not quite succeed, but obtained the result that the ionised hydrogen molecule was at least metastable. Thus the problem of calculating quantum mechanical energy levels became the most important in quantum physics, as it was supposed to explain the chemical bonding between two protons by a single electron.

In 1933 **George Cecil Jaffé** (1888–1965) took up the problem anew, then having available Schrödinger's differential equation and thus the correct quantisation rules. So, Jaffé was eventually able to show the stability of the ionised hydrogen molecule. However, what became much more important was that Jaffé applied a marvellous transformation of the underlying differential equation that enabled him to apply a solution ansatz that solved the problem exactly. This idea is the basis of the method of solution of the specific singular boundary eigenvalue problem, called the *central two-point connection problem* in this book. It consists of a series ansatz and a transformation that I call, in honour and commemoration of George Jaffé, the *Jaffé ansatz* and the *Jaffé transformation*, respectively. The particular solution of the underlying differential equation represented by the Jaffé ansatz is called the *Jaffé solution* (see Jaffé, 1933).

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Unfortunately, George Jaffé, as a Jewish professor at the Universität Gießen, was dismissed in 1933 and in 1934 had to abandon his position as a professor, eventually going into exile in 1939, leaving Germany for ever. He went to Bâton Rouge in the US state of Louisiana but did not pick up his ideas there any more.

Sixty years after this great publication, in spring 1993, I visited the elderly **Friedrich Hund** (1896–1997) – one of the founders of quantum theory – in Göttingen. He told me the sad story of George Jaffé's fate. So this book was also written as a protest against oblivion, against the forgetting of a great scientist and his difficult life.

In 1989, Karl Heun's publication about the differential equation that nowadays bears his name celebrated its centennial appearance. In order to mark this date, an international conference was organised that brought together experts from all over the world who worked in the field. The result of this conference was several commitments to promote the topic, and even several notations that have been agreed. It was common opinion that the step from classical special functions to higher special functions was ripe to be made, and the participants were in agreement that this was a fundamental step.

Subsequently, following this conference, an unpublished conference booklet was written (Seeger and Lay, 1990), then a book written by several participants of the conference (Ronveaux, 1995), then a monograph on the topic by the Russian expert **Serguei Yuriewitsch Slavyanov** (1942–2019) and the present author (Slavyanov and Lay, 2000), and eventually a successor to the famous *Handbook of Mathematical Functions* by **Milton Abramowitz** (1915–1958) and **Irene Stegun** (1919–2008) that was edited by **Frank William John Olver** (1924–2013), entitled *The NIST Handbook of Mathematical Functions*, published (within the DLMF) by the National Institute of Standards and Technologies (NIST; Olver et al., 2010). In this book, the Heun differential equation has been dealt with as a new differential equation, not incorporated within the set of differential equations, the solutions of which belong to the classical special functions of mathematical physics (cf. Chapter 31). The authors of this part on Heun's differential equation were **Brian D. Sleeman** (1939–2021), a British professor and participant of the famous conference introduced above, and the Russian mathematician **Vadim B. Kuznetsov** (*1963).

However, the Heun equation has not been treated in an exhaustive manner, since several important topics had not been tackled or settled up to that time. These circumstances were the impetus for me to undertake a large-scale investigation in order to treat the main remaining problem: the central two-point connection problem for differential equations beyond Gaussian type. The solution of this problem in a satisfactory manner, i.e., theoretically as well as calculatory, yields the possibility of raising the whole field on a new level: singular boundary eigenvalue problems of differential equations beyond Gaussian type, stemming from applications that are solvable. Now, these mathematical problems lead to new functions (higher special functions) and show up new phenomena not yet seen, based on the fact that the differential equations

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have a sort of parameter that differential equations yielding classical special functions do not have.

1.2 Classical Special Functions: Testing the New in a Well-Known Area

In this section, those differential equations are described whose singular boundary eigenvalue problems lead to the *classical special functions*. It may seem superfluous to rewrite what has already been sufficiently described in books such as Whittaker and Watson (1927), Coddington and Levinson (1955), Ince (1956), Bieberbach (1965), Olver (1974) or Hille (1997). However, this is an inaccurate impression. Here, in order to get beyond the results of classical theory, one must adopt a somewhat different viewpoint than the one which developed the classical theory. This slightly different viewpoint, which allows a generalisation, must of course also be applicable to the classical theory. Thus, the known terrain is useful because it allows the new theory to be proven on the basis of known results. Anyone who picks up pencil and paper and starts calculating will appreciate being able to direct the new along the lines of the old.

1.2.1 Differential Equations

The Gauss Equation

Basically, there is one differential equation whose singular eigenvalue problems produce classical special functions: that is, the *Gauss differential equation*. It is a differential equation (1), where $P_0(z)$ is the fourth-order polynomial

$$P_0(z) = z^2\,(z-1)^2,$$

$P_1(z)$ is the third-order polynomial

$$P_1(z) = (A_0 + A_1)\,z^3 - (2\,A_0 + A_1)\,z^2 + A_0\,z$$

and $P_2(z)$ is the second-order polynomial

$$P_2(z) = (B_0 + B_1 - C)\,z^2 - (2\,B_0 - C)\,z + B_0.$$

By means of

$$P(z) = \frac{P_1(z)}{P_0(z)}, \quad Q(z) = \frac{P_2(z)}{P_0(z)}, \tag{1.2.1}$$

equation (1) on page ix becomes

$$\frac{d^2y}{dz^2} + P(z)\,\frac{dy}{dz} + Q(z)\,y(z) = 0, \quad z \in \mathbb{C} \tag{1.2.2}$$

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with

$$P(z) = \frac{P_1(z)}{P_0(z)} = \frac{A_0}{z} + \frac{A_1}{z-1},$$
$$Q(z) = \frac{P_2(z)}{P_0(z)} = \frac{B_0}{z^2} + \frac{B_1}{(z-1)^2} + \frac{C}{z} - \frac{C}{z-1}. \tag{1.2.3}$$

As is seen, the coefficient $P(z)$ has a first-order pole at the singularities of the differential equation, located at $z = 0$ and at $z = 1$, and the coefficient $Q(z)$ has a second-order pole there. This is the crucial significance of the singularity of the differential equation at hand. Astonishingly, this differential equation has not just two but three singularities, two of which are placed at finite points, being called *finite singularities* for short, namely at $z = 0$ and at $z = 1$; the third one is located at infinity, being called *infinite singularity* for short. Since infinity is neither a proper point nor a number, this improper point of the differential equation (1.2.2) is dealt with by means of inverting it, viz. applying the transformation

$$\zeta = \frac{1}{z}, \tag{1.2.4}$$

thus getting

$$\frac{d^2y}{d\zeta^2} + \left[\frac{2}{\zeta} - \frac{1}{\zeta^2} P(\zeta)\right] \frac{dy}{d\zeta} + \frac{1}{\zeta^4} Q(\zeta)\, y = 0, \quad \zeta \in \mathbb{C}, \tag{1.2.5}$$

with $P(\zeta) = P(1/z)$ and $Q(\zeta) = Q(1/z)$, and then considering the point $\zeta = 0$. With equation (1.2.3) this means

$$\tilde{P}(\zeta) = \frac{2}{\zeta} - \frac{1}{\zeta^2} P(\zeta) = \frac{\tilde{A}_0}{\zeta} + \frac{A_1}{\zeta - 1},$$
$$\tilde{Q}(\zeta) = \frac{1}{\zeta^4} Q(\zeta) = \frac{\tilde{B}_0}{\zeta^2} + \frac{B_1}{(\zeta-1)^2} + \frac{\tilde{C}}{\zeta} - \frac{\tilde{C}}{\zeta - 1}. \tag{1.2.6}$$

As may be seen, this differential equation is just the same as (1.2.2), (1.2.3), except for the coefficients A_0, B_0, C. These do have other values, namely

$$\tilde{A}_0 = 2 - A_0 - A_1,$$
$$\tilde{B}_0 = B_0 + B_1 - C, \tag{1.2.7}$$
$$\tilde{C} = 2\,B_1 - C,$$

while A_1 and B_1 are not affected. In particular, the order of the pole at $\zeta = 0$ is the same as at $z = 0$ in (1.2.3).

All the properties of singularities of equation (1.2.2), (1.2.3) at infinity are transferred to the singularity at zero of equation (1.2.5), (1.2.6) by means of the transformation (1.2.4).

Turning to the *local solutions* at the finite singularities of the differential equation (1.2.2), (1.2.3) it was the great discovery of Lazarus Fuchs that an ansatz for the local solutions at the singularity $z = 0$ has the form (Fuchs, 1866)

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$$y_0(z) = \sum_{n=0}^{\infty} a_n\, z^{n+\alpha_0} = z^{\alpha_0} \sum_{n=0}^{\infty} a_n\, z^n . \tag{1.2.8}$$

The term z^{α_0} is denoted the *asymptotic factor*. Inserting the ansatz into the differential equation and equating each power in z to zero yields

$$\begin{aligned}
&[\alpha_0\,(\alpha_0 - 1) + A_0\,\alpha_0 + B_0\,]\; a_0 = 0,\\
&[(A_0 + \alpha_0)\,(\alpha_0 + 1) + B_0]\;\alpha_1\\
&\qquad - [2\,\alpha_0\,(\alpha_0 - 1) + 2\,(A_0 + B_0) + A_1 - C]\; a_0 = 0,\\
&[1]\; a_{n+1} + [0]\; a_n + [-1]\; a_{n-1} = 0, \qquad n = 1, 2, 3, \ldots,
\end{aligned}$$

with

$$\begin{aligned}
[1] &= (n + 1 + \alpha_0)\;[(n + \alpha_0) + A_0] + B_0,\\
[0] &= -2\,(n + \alpha_0)\;[(n - 1 + \alpha_0) - (2\,A_0 + A_1)] + (C - 2\,B_0),\\
[-1] &= (n - 1 + \alpha_0)\;[(n - 2 + \alpha_0) + A_0 + A_1] + B_0 + B_1 - C.
\end{aligned}$$

This may also be written in the form

$$[\alpha_0\,(\alpha_0 - 1) + A_0\,\alpha_0 + B_0\,]\; a_0 = 0, \tag{1.2.9}$$

$$\begin{aligned}
&[(A_0 + \alpha_0)\,(\alpha_0 + 1) + B_0]\; a_1\\
&\qquad - [2\,\alpha_0\,(\alpha_0 - 1) + 2\,(A_0 + B_0) + A_1 - C]\; a_0 = 0,
\end{aligned} \tag{1.2.10}$$

$$\begin{aligned}
&\left(1 + \frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) a_{n+1} - \left(2 + \frac{\bar{\alpha}_0}{n} + \frac{\beta_0}{n^2}\right) a_n\\
&\qquad + \left(1 + \frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) a_{n-1} = 0, \quad n = 1, 2, 3, \ldots,
\end{aligned} \tag{1.2.11}$$

with

$$\begin{aligned}
\alpha_1 &= A_0 + 2\,\alpha_0 + 1, \quad \beta_1 = (A_0 + \alpha_0)\,(\alpha_0 + 1) + B_0,\\
\bar{\alpha}_0 &= 2\,A_0 + A_1 + 2\,(2\,\alpha_0 - 1),\\
\beta_0 &= (2\,A_0 + A_1)\,\alpha_0 + 2\,\alpha_0\,(\alpha_0 - 1) + 2\,B_0 - C,\\
\alpha_{-1} &= A_0 + A_1 + 2\,\alpha_0 - 3,\\
\beta_{-1} &= (A_0 + A_1)\,(\alpha_0 - 1) + \alpha_0\,(\alpha_0 - 3) + B_0 + B_1 - C + 2.
\end{aligned}$$

With $a_0 \neq 0$ ($a_0 = 0$ would mean getting the trivial solution $y(z) \equiv 0$), two values of α_0 result from (1.2.9), namely

$$\begin{aligned}
\alpha_{01} &= \frac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\,B_0}}{2},\\
\alpha_{02} &= \frac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}.
\end{aligned} \tag{1.2.12}$$

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Thus, fixing the value of the first coefficient a_0 of the series in (1.2.8) [that actually may be used for normalising the particular solution $y(z)$] allows us to calculate the value of the second coefficient a_1 from the initial condition (1.2.10). This, in turn, allows us to recursively calculate as many coefficients a_n from (1.2.11) as are needed.

As can be seen, there appear two particular solutions of the differential equation (1.2.2) from the ansatz (1.2.8). However, these are only two linearly independent solutions, i.e., a fundamental system of the differential equation, if the difference between the two characteristic exponents (1.2.12) is not an integer. Otherwise, there may appear a logarithmic term in one of the two particular solutions (see, e.g., Bieberbach, 1965, pp. 128, 136). These particular solutions are called *Frobenius solutions*. The quantity α_0 is called the *characteristic exponent*. $y_{01}(z)$ and $y_{02}(z)$ are linearly independent, thus, the pair $\{y_{01}(z),\ y_{02}(z)\}$ may also serve as a fundamental system of solutions of the differential equation (1.2.2), (1.2.3), i.e., the general solution $y^{(g)}(z)$ of (1.2.2), (1.2.3) may be written in the form

$$y^{(g)}(z) = C_1\, y_{01}(z) + C_2\, y_{02}(z),$$

where C_1 and C_2 are the totality of complex-valued constants in z. The series in (1.2.8) is convergent with radius of convergence r ranging to the neighbouring singularity of the differential equation (1.2.2), (1.2.3), thus $r = 1$. This is a consequence of the fact that for linear differential equations, all the particular solutions at ordinary points of the differential equations are holomorphic there (cf. Bieberbach, 1965, p. 5).

There are also two local solutions at the singularity $z = 1$, given by

$$y_{1i}(z) = \sum_{n=0}^{\infty} a_n\,(z-1)^{n+\alpha_{1i}} = (z-1)^{\alpha_{1i}} \sum_{n=0}^{\infty} a_n\,(z-1)^n, \quad i = 1,2,$$

with

$$\alpha_{11} = \frac{1 - A_1 + \sqrt{(1-A_1)^2 - 4\,B_1}}{2},$$
$$\alpha_{12} = \frac{1 - A_1 - \sqrt{(1-A_1)^2 - 4\,B_1}}{2}.$$

$y_{11}(z)$ and $y_{12}(z)$ are linearly independent, thus the pair $\{y_{11}(z),\ y_{12}(z)\}$ may also serve as a fundamental system of solutions of the differential equation (1.2.2), (1.2.3), i.e., the general solution $y^{(g)}(z)$ of (1.2.2), (1.2.3) may be written in the form

$$y^{(g)}(z) = C_1\, y_{11}(z) + C_2\, y_{12}(z),$$

where C_1 and C_2 are not specific constants but are the totality of complex-valued constants in z.

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Last but not least, the characteristic exponents of the Frobenius solutions at infinity are given by

$$\alpha_{\infty 1} = \frac{\tilde{A}_0 + \tilde{A}_1 - 1 + \sqrt{[\tilde{A}_0 + \tilde{A}_1 - 1]^2 - 4\,(\tilde{B}_0 + \tilde{B}_1 + \tilde{C}_1)}}{2},$$
$$\alpha_{\infty 2} = \frac{\tilde{A}_0 + \tilde{A}_1 - 1 - \sqrt{[\tilde{A}_0 + \tilde{A}_1 - 1]^2 - 4\,(\tilde{B}_0 + \tilde{B}_1 + \tilde{C}_1)}}{2},$$

with $\tilde{A}_0$ and $\tilde{B}_0$ from (1.2.7) and

$$\tilde{A}_1 = A_1, \tilde{B}_1 = B_1, \tilde{C}_1 = C.$$

Eventually it should be mentioned that the sum of all the characteristic exponents is given by

$$\alpha_{01} + \alpha_{02} + \alpha_{12} + \alpha_{12} + \alpha_{\infty 2} + \alpha_{\infty 2} = 1.$$

It may be seen from (1.2.12) that, if B_0 vanishes, thus $B_0 = 0$, one of the two characteristic exponents vanishes as well, namely

$$\alpha_{02} = 0.$$

This, in turn, means that one of the two Frobenius solutions is holomorphic at the singularity $z = 0$ of the differential equation (1.2.2), (1.2.3), and thus may be represented by a pure Taylor series. The analogous happens with respect to the singularity at $z = 1$ in the case when $B_1 = 0$. This is an important mathematical mechanism for dealing with the singular boundary eigenvalue problem in §1.2.4.

The Single Confluent Case of the Gauss Equation

The differential equation

$$\frac{d^2 y}{dz^2} + \left[\frac{A_0}{z} + G_0\right]\frac{dy}{dz} + \left[\frac{B_0}{z^2} + \frac{C}{z} + D_0\right] y(z) = 0, \quad z \in \mathbb{C}, \tag{1.2.13}$$

is called the *single confluent case of the Gauss equation* since it may be derived from the Gauss differential equation (1.2.2), (1.2.3)

$$\frac{d^2 y}{dz^2} + \left[\frac{A_0}{z} + \frac{A_1}{z-1}\right]\frac{dy}{dz} + \left[\frac{B_0}{z} + \frac{B_1}{(z-1)^2} + \frac{C}{z} - \frac{C}{z-1}\right] y(z) = 0, \quad z \in \mathbb{C},$$

in such a way that the singularity at $z = 1$ is considered to be located at an arbitrary point $z = z_0$ that is driven to infinity:

$$\frac{d^2 y}{dz^2} + \left[\frac{A_0}{z} + \frac{A_1}{z-z_0}\right]\frac{dy}{dz} + \left[\frac{B_0}{z} + \frac{B_1}{(z-z_0)^2} + \frac{C}{z} - \frac{C}{z-z_0}\right] y(z) = 0, \quad z \in \mathbb{C},$$

with $z_0 \to \infty$, thus by carrying out a limiting process. However, the result of this limiting process depends on how it is carried out, for it has to be recognised that in order to get the most general result, the coefficients A_0, A_1, B_0, B_1, C have to be dependent on z_0. Taking the constants A_0, A_1, B_0, B_1, C as independent of z_0 does not lead to the most general result. To correctly carry out this limiting process, one has

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to dive deeper into the topic and give some more definitions. Therefore, the technical carrying out of this process is postponed to Chapter 2.

The single confluent case of the Gauss differential equation (1.2.13) has two singularities, one located at the origin $z = 0$ and the other positioned at infinity, as seen by the inversion $\zeta = \frac{1}{z}$:

$$\frac{d^2 y}{d\zeta^2} + \tilde{P}(\zeta)\,\frac{dy}{d\zeta} + \tilde{Q}(\zeta)\,y = 0, \quad \zeta \in \mathbb{C},$$

with

$$\tilde{P}(\zeta) = \left[\frac{2}{\zeta} - \frac{1}{\zeta^2}\,P(\zeta)\right]\frac{dy}{d\zeta} = -\frac{G_0}{\zeta^2} + \frac{2 - A_0}{\zeta},$$
$$\tilde{Q}(\zeta) = \frac{1}{\zeta^4}\,Q(\zeta) = \frac{D_0}{\zeta^4} + \frac{C}{\zeta^3} + \frac{B_0}{\zeta^2}.$$

As seen, the nature of this singularity is different in that the order of the pole of $\tilde{P}(\zeta)$ at the origin $\zeta = 0$ is two and the order of the pole of $\tilde{Q}(\zeta)$ at the origin $\zeta = 0$ is four. This makes a significant difference with respect to the local solution, as may be seen below.

Local solutions at the origin $z = 0$ are given by

$$y(z) = z^{\alpha_0} \sum_{n=0}^{\infty} a_n\,z^n. \tag{1.2.14}$$

The quantity α_0 is called the *characteristic exponent*. As above, it is given by inserting the ansatz (1.2.14) into (1.2.13), (1.2.1) and setting the equation of zeroth power to zero, resulting in the quadratic algebraic equation called the *indicial equation*:

$$\alpha_0^2 + (A_0 - 1)\,\alpha_0 + B_0 = 0, \tag{1.2.15}$$

generally yielding the two values

$$\alpha_{01,02} = \begin{cases} \dfrac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}, \\ \dfrac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}. \end{cases} \tag{1.2.16}$$

The coefficients a_n of the ansatz (1.2.14) are given by means of the above-mentioned insertion and equating higher powers of z to zero, resulting in the difference equation

$$\begin{aligned} &\alpha_0^2 + (A_0 - 1)\,\alpha_0 + B_0 = 0, \\ &G_0\,(1 + \alpha_0)\,a_1 + (B_0 + D_0)\,a_0 = 0, \\ &(n + \alpha_0)(n - 1 + \alpha_0)\,a_{n+1} + [(n + \alpha_0)\,A_0 + B_0 + D_0]\,a_n \\ &\qquad + C\,a_{n-1} = 0, \qquad n = 1, 2, 3, \ldots, \end{aligned} \tag{1.2.17}$$

which may be turned into a recurrence equation by fixing a_0.

Now, turning to the local solutions at infinity, the ansatz has to change such that – according to the German mathematician **Ludwig Wilhelm Thomé** (1841–1910) –

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an exponential term occurs in the asymptotic factor. He wrote this down for the first time in 1883:

$$y(z) = \exp\left(\alpha_{1\infty}\, z\right)\, z^{\alpha_{0\infty}} \sum_{n=0}^{\infty} C_n\, z^{-n}. \tag{1.2.18}$$

It is plausible that the series is to be written in descending powers, since the singularity is located at infinity. It is to be remarked here that even in this case, the infinite series in the ansatz (1.2.18) are one-sided infinite, that is by no means to be taken for granted. The quantity α_1 – calculated by the same procedure as above – is called the *characteristic exponent of the second kind* and is determined by an algebraic second-order equation

$$\alpha_{1\infty j}^2 + G_0\, \alpha_{1\infty j} + D_0 = 0, \quad j = 1, 2, \tag{1.2.19}$$

generally yielding the two values

$$\begin{aligned} \alpha_{1\infty 1} &= \frac{-G_0 + \sqrt{G_0^2 - 4\, D_0}}{2}, \\ \alpha_{1\infty 2} &= \frac{-G_0 - \sqrt{G_0^2 - 4\, D_0}}{2}. \end{aligned} \tag{1.2.20}$$

The *characteristic exponent of the first kind* $\alpha_{0\infty j}$ obeys the first-order rational equation

$$\alpha_{0\infty j}\left(G_0 + 2\, \alpha_{1\infty j}\right) + A_0\, \alpha_{1\infty j} + C = 0, \quad j = 1, 2,$$

which depends on the characteristic exponents of the second kind $\alpha_{1\infty j}$, thus generally yielding two values:

$$\alpha_{0\infty j} = -\frac{A_0\, \alpha_{1\infty j} + C}{G_0 + 2\, \alpha_{1\infty j}}, \quad j = 1, 2. \tag{1.2.21}$$

The Biconfluent Case of the Gauss Equation

For the sake of completeness, it has to be remarked that there is still one further confluent case of the Gauss equation, namely by moving both of the finite singularities to infinity. The result is the *biconfluent case of the Gauss equation*. It has no finite singularity any more and looks like

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left(G_0 + G_1\, z\right) \frac{\mathrm{d}y}{\mathrm{d}z} + \left(D_0 + D_1\, z + D_2\, z^2\right)\, y = 0, \quad z \in \mathbb{C}. \tag{1.2.22}$$

As may be seen from (1.2.5) and (1.2.6), at infinity there is a singularity of this differential equation: $P(\zeta)$ has a third-order pole while $Q(\zeta)$ has a sixth-order one.

The special choice

$$G_0 = G_1 = 0$$

of its parameters does not touch the fundamental character of the differential equation with respect to its local solutions at its singularity, but yields a biconfluent case of

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the Gauss equation, a pair of fundamental solutions of which are the so-called and well-known *parabolic cylinder functions* (cf. Slavyanov, 1996, §3).

The local solutions of (1.2.22) at the infinite singularity (which is actually a Thomé one, see above) are given by

$$y(z) = \exp\left(\frac{\alpha_2}{2}\, z^2 + \alpha_1\, z\right) z^{\alpha_0} \sum_{n=0}^{\infty} \frac{C_n}{z^n},$$

where the *characteristic exponents of second kind and second order* α_2 are given by

$$\begin{aligned} \alpha_{21} &= -\frac{1}{2}\left(G_1 + \sqrt{G_1^2 - 4\,D_2}\right), \\ \alpha_{22} &= -\frac{1}{2}\left(G_1 - \sqrt{G_1^2 - 4\,D_2}\right), \end{aligned} \tag{1.2.23}$$

and the *characteristic exponents of second kind and first order* α_1 are given by

$$\begin{aligned} \alpha_{11} &= -\frac{G_0\,\alpha_{21} + D_1}{2\,\alpha_{21} + G_1}, \\ \alpha_{12} &= -\frac{G_0\,\alpha_{22} + D_1}{2\,\alpha_{22} + G_1}. \end{aligned} \tag{1.2.24}$$

Eventually, the characteristic exponents of first kind α_0 are given by

$$\begin{aligned} \alpha_{01} &= -\frac{\alpha_{21} + \alpha_{11} + G_0 + D_0}{G_0 + 2\,\alpha_{21}}, \\ \alpha_{02} &= -\frac{\alpha_{22} + \alpha_{12} + G_0 + D_0}{G_0 + 2\,\alpha_{22}}. \end{aligned} \tag{1.2.25}$$

1.2.2 Linear Transformations

There is a linear transformation of the independent variable of equation (1.2.2) as well as of the dependent variable, the characteristic property of which is that it leaves the essential properties of the differential equation untouched. These linear transformations play an important role in the theory. In the former case, it is a specific Moebius transformation, which changes the position of the singularities of a differential equation in the complex number plane without touching the essentials of the equation itself, and the other changes in asymptotic factors of the local solutions, also without changing the essential properties of the equation. Because of their importance, they should be explicitly mentioned here.

Moebius Transformations

To have a transformation of the independent variable of the differential equation (1.2.2) that changes the locations of its singularities without touching the essential properties is of calculatory importance. In the actual context this means that the transformed differential equation has the same number of singularities, keeping the pole orders of the coefficients (1.2.3). The only change allowed is in the locations of the singularities of the differential equation.

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It is to be expected that such a transformation is not of substantial significance, since it does not touch the structure or character of the differential equation, or the local solutions in the neighbourhood of its singularities, wherever these are located. However, as will be seen below, placing the singularities in an appropriate manner results in calculatory simplifications that often make the theory clearer and more comprehensible. Therefore, it is not surprising that the Moebius transformation is frequently applied in the following.

The access to Moebius transformations may be alleviated by means of purely geometric considerations. It is well known that the Eulerian complex plane of numbers may be compactified by means of a stereographic mapping, i.e., a mapping of the plane onto a sphere, the south pole of which is placed at the origin of the Eulerian plane (see Behnke and Sommer, 1976, pp. 13–19). The result is a sphere-shaped space of the complex numbers, with point of infinity the north pole. This is to be kept in mind in the following considerations. If the compactified plane of complex numbers is meant in the following, it is spoken of as the 'complex sphere'.

The transformation

$$x = \frac{a\,z + b}{c\,z + d}, \quad a\,d \neq b\,c, \tag{1.2.26}$$

of the independent variable z is called the *Moebius transformation* (cf. Gutzwiller, 1990, p. 345). This transformation is named after the German mathematician **August Ferdinand Moebius** (1790–1868). It is the only bijective conformal mapping of the closed complex sphere onto itself (sometimes also called *isomorphic transformation* or *homographic transformation*; cf. Behnke and Sommer, 1976, p. 324). It is just this mapping, interpreted as a transformation, that meets the condition formulated above. A Moebius transformation is nothing other than a rotation, as well as a stretching of the complex sphere, irrespective of what is done mathematically on this sphere.

The inverse mapping of (1.2.26) is of the same sort:

$$z = \frac{d\,x - b}{-c\,x + a}. \tag{1.2.27}$$

This mapping can put three singularities of equation (1.2.2) onto three other singularities without touching the substance of the equation, i.e., the number and s-ranks, and thus the characteristics, of any of its singularities. This allows us always to put three arbitrary points of equation (1.2.2) lying in the complex plane to the points 0, 1, ∞ (cf. Figure 1.1). This will become important in the following.

These are the locations where the differential equations have the most simple formulae, whatever is to be calculated.

With the help of the derivatives

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}z} &= \frac{a\,d - b\,c}{(c\,z + d)^2} = \frac{(c\,x - a)^2}{a\,d - b\,c},\\
\frac{\mathrm{d}^2 x}{\mathrm{d}z^2} &= -2\,c\,\frac{a\,d - b\,c}{(c\,z + d)^3} = 2\,c\,\frac{(c\,x - a)^3}{(a\,d - b\,c)^2},
\end{aligned}$$

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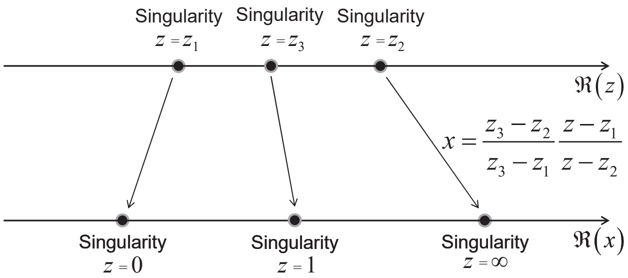

Figure 1.1 Moebius transformation placing three arbitrary points.

the coefficients $\tilde{P}(x)$ and $\tilde{Q}(x)$ of the transformed differential equation

$$\frac{d^2 y}{dx^2} + \tilde{P}(x) \frac{dy}{dx} + \tilde{Q}(x) y = 0$$

become

$$\tilde{P}(x) = \frac{P(x)}{\frac{dx}{dz}} + \frac{\frac{d^2 x}{dz^2}}{\left(\frac{dx}{dz}\right)^2} = P(x) \frac{a d - b c}{(c x - a)^2} + \frac{2 c}{c x - a} \tag{1.2.28}$$

and

$$\tilde{Q}(x) = \frac{Q(x)}{\left(\frac{dx}{dz}\right)^2} = Q(x) \frac{(a d - b c)^2}{(c x - a)^4}. \tag{1.2.29}$$

s-Homotopic Transformations

The ansatz (1.2.8)

$$y(z) = z^{\alpha_0} \sum_{n=0}^{\infty} a_n z^n$$

for the solution of the differential equation (1.2.2), (1.2.3), may also be written in the form

$$y(z) = f(z) v(z) \tag{1.2.30}$$

with

$$\begin{aligned} f(z) &= z^{\alpha_0}, \\ v(z) &= \sum_{n=0}^{\infty} a_n z^n. \end{aligned} \tag{1.2.31}$$

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Equation (1.2.30) may be considered as a transformation of the underlying differential equation (1.2.2), (1.2.3) to a differential equation for $v(z)$. This equation has the form

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \tilde{P}(z)\,\frac{\mathrm{d}v}{\mathrm{d}z} + \tilde{Q}(z)\,v(z) = 0, \quad z \in \mathbb{C},$$

with

$$\begin{aligned} \tilde{P}(z) &= P(z) + 2\,\frac{f'}{f} = \frac{\tilde{A}_0}{z} + \frac{\tilde{A}_1}{z-1}, \\ \tilde{Q}(z) &= Q(z) + P(z)\,\frac{f'}{f} + \frac{f''}{f} = \frac{\tilde{B}_0}{z^2} + \frac{\tilde{B}_1}{(z-1)^2} + \frac{\tilde{C}}{z} - \frac{\tilde{C}}{z-1}, \end{aligned} \tag{1.2.32}$$

whereupon

$$\begin{aligned} f' &= \frac{\mathrm{d}f}{\mathrm{d}z}, \quad f' = \frac{\mathrm{d}^2 f}{\mathrm{d}z^2}, \\ \tilde{A}_0 &= A_0 + 2\,\alpha_0, \\ \tilde{A}_1 &= A_1, \\ \tilde{B}_0 &= \alpha_0\,(\alpha_0 + A_0 - 1) + B_0, \\ \tilde{B}_1 &= B_1, \\ \tilde{C} &= C - \alpha_0\,A_1. \end{aligned} \tag{1.2.33}$$

The crucial point now is that for $\alpha_0 = \alpha_{01}$ from (1.2.12), as well as for $\alpha_0 = \alpha_{02}$, it holds that

$$\begin{aligned} \alpha_{01}\,(\alpha_{01} + A_0 - 1) + B_0 &= 0, \\ \alpha_{02}\,(\alpha_{02} + A_0 - 1) + B_0 &= 0. \end{aligned}$$

This means, because of (1.2.33), that for α_0 in (1.2.30), (1.2.31) being one of the characteristic exponents of the singularity of the equation (1.2.2), (1.2.3) at $z = 0$, the coefficent $\tilde{B}_0$ vanishes. Moreover, for $\alpha_0 = \alpha_{01}$ in (1.2.30), (1.2.31) it holds that

$$\begin{aligned} \tilde{\alpha}_{01} &= \frac{1 - \tilde{A}_0 + \sqrt{(1-\tilde{A}_0)^2 - 4\,\tilde{B}_0}}{2} = \alpha_{01} - \alpha_{01} = 0, \\ \tilde{\alpha}_{02} &= \frac{1 - \tilde{A}_0 - \sqrt{(1-\tilde{A}_0)^2 - 4\,\tilde{B}_0}}{2} = \alpha_{02} - \alpha_{01}, \end{aligned} \tag{1.2.34}$$

and for $\alpha_0 = \alpha_{02}$ in (1.2.30), (1.2.31), it holds that

$$\begin{aligned} \tilde{\alpha}_{01} &= \frac{1 - \tilde{A}_0 + \sqrt{(1-\tilde{A}_0)^2 - 4\,\tilde{B}_0}}{2} = \alpha_{01} - \alpha_{02}, \\ \tilde{\alpha}_{02} &= \frac{1 - \tilde{A}_0 - \sqrt{(1-\tilde{A}_0)^2 - 4\,\tilde{B}_0}}{2} = \alpha_{02} - \alpha_{02} = 0. \end{aligned} \tag{1.2.35}$$

Thus, in any case, one of the characteristic exponents vanishes under the transformation (1.2.30), (1.2.31).

As a result, it may be seen first that, as might have been expected, the coefficients $\tilde{A}_1$ and $\tilde{B}_1$ in (1.2.33) are not affected by the transformation (1.2.30), (1.2.31). Second, as

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may be seen from (1.2.34), (1.2.35), the transformed characteristic exponents $\tilde{\alpha}_{01}$ and $\tilde{\alpha}_{02}$ are the differences between the original α_{01}, α_{02}, respectively, and those taken in the transformation (1.2.30), (1.2.31). As may be seen from (1.2.34), (1.2.35), this means that in any case one of the characteristic exponents vanishes. This, in turn, means that one of the two Frobenius solutions becomes holomorphic at the origin, and thus has a pure Taylor series representation.

The transformation (1.2.30), (1.2.31) is called an s-*homotopic transformation*, a linear transformation of the dependent variable y with respect to one or more singularities that changes the characteristic exponents of the related singularity of a differential equation (1.2.2), (1.2.3) but leaves the location of the singularity unchanged.

Two subsequent s-homotopic transformations at the singularities $z = 0$ and $z = 1$ may be put together by applying

$$f(z) = z^{\alpha_0}\,(z-1)^{\alpha_1}.$$

In this case, the final result is that $\tilde{Q}(z)$ does not have a second-order pole any more, but only a first-order one. Or, in other words, it holds that $\tilde{B}_0 = 0$ and $\tilde{B}_1 = 0$ in (1.2.32):

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} + \left(\frac{\tilde{A}_0}{z} + \frac{\tilde{A}_1}{z-1}\right)\frac{\mathrm{d}v}{\mathrm{d}z} + \left(\frac{\tilde{C}}{z} - \frac{\tilde{C}}{z-1}\right) v(z) = 0, \quad z \in \mathbb{C}. \tag{1.2.36}$$

This means that there may be solutions of the differential equation that are holomorphic at $z = 0$ as well as at $z = 1$. Or, said another way, these particular solutions $y(z)$ are entire functions. Since the coefficient $P(z)$ in (1.2.3) at the singularity at infinity has a first-order pole, and the coefficient $Q(z)$ in (1.2.3) at the singularity at infinity has a second-order pole, as may be seen from (1.2.5), (1.2.6), this entire function must be a polynomial.

Traditionally, it is just this form that is called a *Gauss differential equation*, although – for didactical reasons – I use this notation for the slightly more general equation (1.2.2), (1.2.3).

It is clear that the differential equation (1.2.36) has a local solution at the origin $z = 0$ that may be written as in a Taylor series:

$$y(z) = \sum_{n=0}^{\infty} a_n\, z^n. \tag{1.2.37}$$

Inserting (1.2.37) into (1.2.36) yields a first-order linear difference equation

$$a_{n+1} = f(n)\, a_n, \quad n = 0, 1, 2, 3, \ldots, \tag{1.2.38}$$

with

$$f(n) = \frac{n^2 + (\tilde{A}_0 + \tilde{A}_1 - 1)\, n - \tilde{C}}{n^2 + (\tilde{A}_0 + 1)\, n + \tilde{A}_0} = 1 + \frac{\alpha}{n + \tilde{A}_0} + \frac{\beta}{n+1}, \tag{1.2.39}$$

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whereby

$$\alpha = \frac{\tilde{A}_0\,(\tilde{A}_1 - 1) + \tilde{C}}{\tilde{A}_0 - 1},$$
$$\beta = -\frac{\tilde{A}_0 + \tilde{A}_1 + \tilde{C} - 2}{\tilde{A}_0 - 1}. \tag{1.2.40}$$

Determining a_0 makes a linear two-term recurrence relation out of the first-order difference equation (1.2.38), whereas this determination means a normalisation of the solution (1.2.37) of (1.2.36). The resulting solution $y(z)$ via (1.2.37) is called a *hypergeometric function* (cf. Abramowitz and Stegun, 1970, Chapter 15).

Since (1.2.38) is a first-order equation, its general solution is given by the totality of values of a_0. All the particular solutions of (1.2.38) differ just by a factor:

$$a_n^{(g)} = L\,a_n, \quad n = 0, 1, 2, 3, \ldots,$$

where L is the totality of complex-valued contants in n, a_n is given by (1.2.38) and a_0 takes any fixed value, for example $a_0 = 1$.

Classical analysis tells us that the ratio of two consecutive terms a_{n+1} and a_n tends to the inverse of the radius of convergence of the Taylor series in (1.2.37), thus to unity. Here,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{r} = 1.$$

This is quite clear since the solution, represented by this ansatz, has the form

$$y_1(z) = (z-1)^{\alpha_0}\sum_{n=0}^{\infty} a_n\,(z-1)^n$$

at the singularity at $z = 1$. This, in general, is a function that is not holomorphic at $z = 1$. Generally, series in the ansatzes of local solutions of linear differential equations do converge up to the neighbouring singularity of the underlying differential equation.

The totality of particular solutions a_n of the difference equation (1.2.38) only comprises the non-trivial solutions. However, there are also trivial particular solutions, defined by a truncation of the sequence a_n:

$$a_{n+1} = f(n)\,a_n, \quad a_0 = 1, n = 0, 1, 2, 3, \ldots, m,$$

where

$$f(m) = 0, \quad m \in \mathbb{N}^0. \tag{1.2.41}$$

This means

$$m^2 + (\tilde{A}_0 + \tilde{A}_1 - 1)\,m - \tilde{C} = 0, \quad m = 0, 1, 2, \ldots. \tag{1.2.42}$$

In this case, $a_n = 0$ for $n > m$ holds. The corresponding series in (1.2.37) in this case become finite and the function $y(z)$ becomes a polynomial, which is actually called a *hypergeometric polynomial*.

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Examples

- Consider the condition $f(1) = 0$, i.e., $m = 1$. Then

$$\tilde{C} = \tilde{A}_0 + \tilde{A}_1.$$

As a result, the particular solution of the difference equation (1.2.38) is a trivial one, namely

$$\begin{aligned} a_0 &= 1, \\ a_1 &= f(0)\, a_0 = -\frac{\tilde{C}}{\tilde{A}_0} = -\frac{\tilde{A}_0 + \tilde{A}_1}{\tilde{A}_0} = -1 - \frac{\tilde{A}_1}{\tilde{A}_0}, \\ a_2 &= f(1)\, a_1 = 0 \cdot a_1 = 0, \\ a_3 &= f(2)\, a_2 = 0, \\ &\ldots \end{aligned}$$

The solution is the first-order hypergeometric polynomial

$$y(z) = 1 + a_1\, z = 1 - \left(1 + \frac{\tilde{A}_1}{\tilde{A}_0}\right) z.$$

- Consider the condition $f(2) = 0$, i.e., $m = 2$. Then

$$\tilde{C} = 2\,\left[\tilde{A}_0 + \tilde{A}_1 + 1\right].$$

As a result, the particular solution of the difference equation (1.2.38) is a trivial one, namely

$$\begin{aligned} a_0 &= 1, \\ a_1 &= f(0)\, a_0 = -\frac{\tilde{C}}{\tilde{A}_0} = -2\,\frac{\tilde{A}_0 + \tilde{A}_1 + 1}{\tilde{A}_0}, \\ a_2 &= f(1)\, a_1 = -\frac{\tilde{A}_0 + \tilde{A}_1 + 2}{2\,\tilde{A}_0 + 2}\, a_1 = 2\,\frac{\tilde{A}_0 + \tilde{A}_1 + 1}{\tilde{A}_0}\,\frac{\tilde{A}_0 + \tilde{A}_1 + 2}{2\,\tilde{A}_0 + 2}, \\ a_3 &= f(2)\, a_2 = 0, \\ a_4 &= f(3)\, a_3 = 0, \\ &\ldots \end{aligned}$$

The solution is the second-order hypergeometric polynomial

$$y(z) = 1 + a_1\, z + a_2\, z^2.$$

1.2.3 Forms of Differential Equations

As may be seen from what is written above, it is crucial how many singularities there are of the differential equation (1.2.2), and of what sort these are. On the other hand, it is not essential for a differential equation where these singularities are located. This is nothing but a matter of appropriateness. Moreover, the concrete values of the characteristic exponents at the singularities of equation (1.2.2)

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are not substantially important, since these values might always be changed by an s-homotopic transformation. What is significant in this respect is the difference in the pair of characteristic exponents at a singularity of equation (1.2.2) (these are actually invariants of the s-homotopic transformations). Therefore, two differential equations (1.2.2) may look completely different, although they are essentially the same. Such a (trivial) distinction between differential equations (1.2.2) is called their *form*.

The *form of a differential equation* distinguishes equation (1.2.2) according to the locations of its singularities, as well as the concrete values of the pairs of characteristic exponents belonging to the singularities. As may be understood, this open up a rich variety of forms of equation (1.2.2). It is not reasonable to display them all here explicitly; the totality of forms of a differential equation (1.2.2) is generated by means of a Moebius as well as s-homotopic transformations. Both of these are linear transformations, the former of the independent variable and the latter of the dependent variable.

It is a matter of historical evolution that there are a few forms of equation (1.2.2) which have gained certain attention and thus are introduced below. The most important ones among them are determined by the following properties:

- The point at infinity is a singularity of the differential equation (1.2.2) with pole order the highest (or one of the highest) among all appearing singularities.
- At least one of the characteristic exponents at each of the finite singularities is zero.
- If the differential equation (1.2.2) has at least two singularities, then one of them is located at the origin $z = 0$.
- If the differential equation (1.2.2) has three or more singularities, then one of them is located at unity $z = 1$.

Such a differential equation is said to be in *standard form*. There are several other forms of equation (1.2.2) bearing a similar notation, the most important of which are mentioned in the following (cf. Slavyanov and Lay, 2000, pp. 4, 5, 13, 21):

- *General form*: No parameter of the differential equation is specified.
- *Natural form*: The singularity (or one of the singularities) having the highest pole order is located at infinity.
- *Normal form*: The coefficient in front of the first derivative vanishes.
- *Self-adjoint form*: The differential equation (1) is characterised by the condition

$$\frac{\mathrm{d}P_0(z)}{\mathrm{d}z} = P_1(z)$$

  (see Courant and Hilbert, 1968, p. 238).
- *Canonical form*: All zeros of $P_0(z)$ in equation (1) are simple ones.

The totality of forms of a differential equation (1.2.2) is the totality of equations (1.2.2) that emerge from one another by means of Moebius, as well as s-homotopic, transformations. This totality is an uncountably infinite set.

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1.2.4 Singular Boundary Eigenvalue Problems

In §1.2.2 we gave a criterion (cf. 1.2.41) for the parameters of the Gauss differential equation (1.2.36) that determines a particular solution that is holomorphic simultaneously at both of its finite singularities. Moreover, it was given an explicit method for its calculation. This is the most simple singular boundary eigenvalue problem.

Now we are going to treat the singular boundary eigenvalue problem of the single confluent case of the Gauss differential equation, a typical singular boundary eigenvalue problem, in order to display the aspect under which this mathematical problem is considered here, in this book, and, moreover, in order to give the method that is based on this aspect. The main point is that the method works not only for those differential equations the solutions of whose singular boundary eigenvalue problems yield classical special functions, but also for the full generality of the differential equation (1.2.2), (1.2.3):

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + P(z)\,\frac{\mathrm{d}y}{\mathrm{d}z} + Q(z)\,y(z) = 0, \quad z \in \mathbb{C}, \tag{1.2.43}$$

with

$$P(z) = \frac{P_1(z)}{P_0(z)},\ Q(z) = \frac{P_2(z)}{P_0(z)} \tag{1.2.44}$$

and with $P_i(z)$, $i = 0, 1, 2$, being polynomials in z, from which result higher special functions by formulating singular boundary eigenvalue problems.

As already mentioned in the Introduction, the basic idea of this method for coping technically with this problem comes from George Cecil Jaffé, who, in connection with the calculation of the quantum mechanical energy levels of the hydrogen molecule ion, in 1933 applied this method for the first time (cf. Jaffé, 1933).

Jaffé did not do this with the single confluent case of the Gauss differential equation, as is done here for the sake of showing this method for a more simple equation, such that the principle comes out clearly without being bothered with complex calculations. The single confluent case of the Fuchsian differential equation with three singularities is the simplest non-trivial equation for which the method of singular boundary eigenvalue problems, presented in this book, applies. In the following this method is presented in detail for this differential equation that may be considered as exemplary.

Formulation

As was shown in §1.2.1, the single confluent case (1.2.13)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[\frac{A_0}{z} + G_0\right]\frac{\mathrm{d}y}{\mathrm{d}z} + \left[\frac{B_0}{z^2} + \frac{C}{z} + D_0\right] y(z) = 0, \quad z \in \mathbb{C}, \tag{1.2.45}$$

of the Gauss differential equation has the characteristic exponents (1.2.16)

$$\begin{aligned} \alpha_{01} &= \frac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}, \\ \alpha_{02} &= \frac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\,B_0}}{2} \end{aligned} \tag{1.2.46}$$

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at the singularity of the differential equation at the origin $z = 0$, from which result two different asymptotic behaviours of its particular solutions

$$\begin{aligned} y_{01}(z) &\sim z^{\alpha_{01}}, \\ y_{02}(z) &\sim z^{\alpha_{02}} \end{aligned} \tag{1.2.47}$$

on approaching the singularity, radially, on the positive real axis.

Moreover, as shown in §1.2.1, the single confluent case of the Gauss differential equation has the characteristic exponents

$$\begin{aligned} \alpha_{1\infty 1} &= \frac{-G_0 + \sqrt{G_0^2 - 4\,D_0}}{2}, \\ \alpha_{1\infty 2} &= \frac{-G_0 - \sqrt{G_0^2 - 4\,D_0}}{2} \end{aligned} \tag{1.2.48}$$

and

$$\begin{aligned} \alpha_{0\infty 1} &= -\frac{A_0\,\alpha_{1\infty 1} + C}{G_0 + 2\,\alpha_{1\infty 1}}, \\ \alpha_{0\infty 2} &= -\frac{A_0\,\alpha_{1\infty 2} + C}{G_0 + 2\,\alpha_{1\infty 2}} \end{aligned} \tag{1.2.49}$$

[cf. (1.2.20), (1.2.1)] at the infinite singularity of the differential equation, from which result two different asymptotic behaviours of its particular solutions, on approaching the singularity, radially, on the positive real axis:

$$\begin{aligned} y_{\infty 1}(z) &\sim \exp\left(\alpha_{1\infty 1}\, z\right)\, z^{\alpha_{0\infty 1}}, \\ y_{\infty 2}(z) &\sim \exp\left(\alpha_{1\infty 2}\, z\right)\, z^{\alpha_{0\infty 2}} \end{aligned} \tag{1.2.50}$$

as $z \longrightarrow \infty$.

Suppose now that we are looking for a particular solution of the differential equation (1.2.45) that behaves like

$$y(z) = y_{02}(z) \tag{1.2.51}$$

and simultaneously like

$$y(z) = y_{\infty 2}(z) \tag{1.2.52}$$

on approaching the singularity at $z = 0$ and at infinity on the positive real axis, radially. Because of the linearity of the differential equation, it holds that

$$y_{02}(z) = c_1\, y_{\infty 1}(z) + c_2\, y_{\infty 2}(z). \tag{1.2.53}$$

Thus, the condition for this to happen is given by

$$c_1 = 0,$$

whereby this quantity c_1 is dependent on the parameters of the underlying differential equation but not dependent on the independent variable z.

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This is the formulation of a boundary eigenvalue problem in searching for particular solutions that behave in a prescribed manner, simultaneously, on approaching the finite as well as the infinite singularity, radially, on the positive real axis.

The formulation of a criterion in dependence on the parameters of the differential equation is the solution of this problem. This criterion is called the *eigenvalue condition*. Such a condition is developed in the following.

One of the two Frobenius solutions at the finite singularity of the differential equation is to be connected with one of the two Thomé solutions at the infinite singularity of the differential equation. Hereby, it is assumed that the Thomé solution to be connected decreases exponentially as $z \to \infty$ along the positive real axis, and thus is a recessive particular solution of the underlying differential equation.

Jaffé Ansatz I

The particular solution of the singular eigenvalue problem immediately above is a rather special one. The eigensolutions, viz. the solutions of the differential equation (1.2.13)

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + P(z)\,\frac{\mathrm{d}y}{\mathrm{d}z} + Q(z)\,y = 0 \tag{1.2.54}$$

with

$$\begin{aligned} P(z) &= \frac{A}{z} + G_0, \\ Q(z) &= \frac{B}{z^2} + \frac{C}{z} + D_0 \end{aligned} \tag{1.2.55}$$

that meet the boundary condition (i.e., the specific asymptotic behaviour as $z \to \infty$ on the positive real axis), consist of an exponential factor, a power term, and an entire function $w(z)$:

$$y(z) = \exp\left(\alpha_{1\infty 2}\, z\right)\, z^{\alpha_{0\infty 2}}\, w(z) \tag{1.2.56}$$

with $\Re(\alpha_{1\infty 2}) < 0$. The differential equation for $w(z)$ is

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \tilde{P}(z)\,\frac{\mathrm{d}w}{\mathrm{d}z} + \tilde{Q}(z)\,w = 0,$$

the coefficients $\tilde{P}(z)$ and $\tilde{Q}(z)$ of which are given by

$$\begin{aligned} \tilde{P}(z) &= \tilde{g}_0 + \frac{\tilde{g}_{-10}}{z}, \\ \tilde{Q}(z) &= \frac{\tilde{d}_{-10}}{z} \end{aligned} \tag{1.2.57}$$

with

$$\begin{aligned} \tilde{g}_0 &= G_0 + 2\,\alpha_{1\infty 2}, \\ \tilde{g}_{-10} &= A + 2\,\alpha_{0\infty 2}, \\ \tilde{d}_{-10} &= 2\,\alpha_{1\infty 2}\,\alpha_{0\infty 2} + A\,\alpha_{1\infty 2} + \alpha_{0\infty 2}\,G_0 + C. \end{aligned}$$

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In order to get the eigenvalue condition, these particular solutions are to be connected to those, that meet the boundary condition at the origin. This is to be investigated in the following.

Jaffé Transformation

The singular boundary eigenvalue problem of the differential equation (1.2.45), and the differential equation (1.2.4), (1.2.57) under the ansatz (1.2.56), is considered. We apply the specific Moebius transformation

$$x = \frac{z}{z+1} \tag{1.2.58}$$

to the independent variable z of (1.2.4), (1.2.57), the crucial point of which is that the positive z-axis, i.e., the relevant interval of the singular boundary eigenvalue problem dealt with here, is compressed such that the points on the z-axis are shifted according to the following table:

$$\begin{array}{cccccc} z: & +\infty & +1 & 0 & -\frac{1}{2} & -1 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ x: & +1 & +\frac{1}{2} & 0 & -1 & -\infty. \end{array}$$

As a result, the relevant interval of the boundary eigenvalue problem changes from

$$\{z \in \mathbb{R} | 0 \le z < \infty\}$$

to

$$\{x \in \mathbb{R} | 0 \le x \le 1\}.$$

In order to carry out the transformation, it is necessary to write the coefficients $\tilde{P}(z)$ and $\tilde{Q}(z)$ in (1.2.57) in dependence on x, yielding, with the help of

$$\frac{1}{z} = -\frac{x-1}{x} = \frac{1}{x} - 1,$$

the coefficients

$$\tilde{P}(x) = \tilde{g}_0 - \tilde{g}_{-10} + \frac{\tilde{g}_{-10}}{x},$$

$$\tilde{Q}(x) = -\tilde{d}_{-10} + \frac{\tilde{d}_{-10}}{x}$$

and the differential equation for $w(x)$:

$$\frac{d^2 w}{dx^2} + \tilde{\tilde{P}}(x)\,\frac{dw}{dx} + \tilde{\tilde{Q}}(x)\,w = 0$$

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with [cf. (1.2.28)]

$$\tilde{\tilde{P}}(x) = \frac{\tilde{P}(x)}{(x-1)^2} + \frac{2}{x-1}$$
$$= \frac{\tilde{g}_0 - \tilde{g}_{-10} + \frac{\tilde{g}_{-10}}{x}}{(x-1)^2} + \frac{2}{x-1}$$
$$= \frac{\tilde{g}_0 - \tilde{g}_{-10}}{(x-1)^2} + \frac{\tilde{g}_{-10}}{x\,(x-1)^2} + \frac{2}{x-1}.$$

By means of

$$\frac{1}{x\,(x-1)^2} = \frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x},$$

we can eventually write

$$\tilde{\tilde{P}}(x) = \frac{\tilde{g}_0 - \tilde{g}_{-10}}{(x-1)^2} + \tilde{g}_{-10}\left(\frac{1}{(x-1)^2} - \frac{1}{x-1} + \frac{1}{x}\right) + \frac{2}{x-1}$$
$$= \frac{\tilde{g}_0}{(x-1)^2} + \frac{2-\tilde{g}_{-10}}{x-1} + \frac{\tilde{g}_{-10}}{x}$$
$$= \frac{\tilde{\tilde{g}}_{-2+1}}{(x-1)^2} + \frac{\tilde{\tilde{g}}_{-1+1}}{x-1} + \frac{\tilde{\tilde{g}}_{-10}}{x} \tag{1.2.59}$$

with

$$\tilde{\tilde{g}}_{-2+1} = \tilde{g}_0,$$
$$\tilde{\tilde{g}}_{-1+1} = 2 - \tilde{g}_{-10},$$
$$\tilde{\tilde{g}}_{-10} = \tilde{g}_{-10}.$$

The coefficient $\tilde{\tilde{Q}}(x)$ [cf. (1.2.29)] is

$$\tilde{\tilde{Q}}(x) = \frac{\tilde{Q}(x)}{(x-1)^4}$$
$$= \frac{-\tilde{d}_{-10} + \frac{\tilde{d}_{-10}}{x}}{(x-1)^4}$$
$$= -\frac{\tilde{d}_{-10}}{(x-1)^4} + \frac{\tilde{d}_{-10}}{x\,(x-1)^4}$$

and with the help of

$$\frac{1}{x\,(x-1)^4} = \frac{1}{x} + \frac{1}{(x-1)^4} - \frac{1}{(x-1)^3} + \frac{1}{(x-1)^2} - \frac{1}{x-1}$$

it becomes eventually

$$\tilde{\tilde{Q}}(x) = \frac{\tilde{\tilde{d}}_{-10}}{x} + \frac{\tilde{\tilde{d}}_{-3+1}}{(x-1)^3} + \frac{\tilde{\tilde{d}}_{-2+1}}{(x-1)^2} + \frac{\tilde{\tilde{d}}_{-1+1}}{x-1}$$

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with

$$\begin{aligned}\tilde{\tilde{d}}_{-3+1} &= -\tilde{d}_{-10},\\ \tilde{\tilde{d}}_{-2+1} &= \tilde{d}_{-10},\\ \tilde{\tilde{d}}_{-1+1} &= -\tilde{d}_{-10},\\ \tilde{\tilde{d}}_{-10} &= \tilde{d}_{-10}.\end{aligned}$$

Jaffé Ansatz II

In order to take into account the power term

$$z^{\alpha_{02}}$$

of the asymptotic factor of the Frobenius solution (1.2.8), we make a rather sophisticated ansatz, that I refer to as the *Jaffé ansatz*, since it was George Jaffé who seems to have applied it for the first time in Jaffé (1933, p. 539):

$$w(x) = f(x)\, v(x) \tag{1.2.60}$$

with

$$f(x) = (1-x)^{\beta} \tag{1.2.61}$$

and where $v(x)$ is to be holomorphic at the origin $x = 0$. This ansatz completes the solution method.

The resulting differential equation in $v(x)$ is

$$\frac{d^2v}{dx^2} + \bar{P}(x)\frac{dv}{dx} + \bar{Q}(x)\, v = 0 \tag{1.2.62}$$

with [cf. (1.2.32)]

$$\begin{aligned}\bar{P}(x) &= \tilde{\tilde{P}}(x) + 2\,\frac{f'}{f},\\ \bar{Q}(x) &= \tilde{\tilde{Q}}(x) + \tilde{\tilde{P}}(x)\,\frac{f'}{f} + \frac{f''}{f},\end{aligned}$$

whereby

$$f' = \frac{df}{dx},\; f'' = \frac{d^2f}{dx^2}$$

and

$$\begin{aligned}f'(x) &= -\beta\,(1-x)^{\beta-1} = -\beta\,(1-x)^{-1}\, f = \frac{\beta}{x-1}\, f,\\ f''(x) &= \beta\,(\beta-1)\,(1-x)^{\beta-2} = \beta\,(\beta-1)\,(1-x)^{-2}\, f\\ &= \frac{\beta\,(\beta-1)}{(x-1)^2}\, f.\end{aligned}$$

The calculation of $\bar{P}(x)$ results in

$$\bar{P}(x) = \frac{\bar{g}_{-2+1}}{(x-1)^2} + \frac{\bar{g}_{-1+1}}{x-1} + \frac{\bar{g}_{-10}}{x}$$

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and the calculation of $\bar{Q}(x)$ results in[1]

$$\begin{aligned}\bar{Q}(x) = & \frac{\tilde{d}_{-10}}{x} - \frac{\tilde{d}_{-10}}{(x-1)^3} + \frac{\tilde{d}_{-10}}{(x-1)^2} - \frac{\tilde{d}_{-10}}{x-1} \\ & + \left(\frac{\tilde{g}_0}{(x-1)^2} + \frac{2-\tilde{g}_{-10}}{x-1} + \frac{\tilde{g}_{-10}}{x} \right) \frac{\beta}{x-1} \\ & + \frac{\beta\,(\beta-1)}{(x-1)^2} \\ = & \frac{\bar{d}_{-3+1}}{(x-1)^3} + \frac{\bar{d}_{-2+1}}{(x-1)^2} + \frac{\bar{d}_{-1-1}}{x+1} + \frac{\bar{d}_{-10}}{x}\end{aligned}$$

with

$$\begin{aligned}\bar{g}_{-2+1} &= \tilde{g}_0, \\ \bar{g}_{-1+1} &= 2 + 2\,\beta - \tilde{g}_{-10}, \\ \bar{g}_{-10} &= \tilde{g}_{-10}, \\ \bar{d}_{-3+1} &= \tilde{g}_0\,\beta - \tilde{d}_{-10}, \\ \bar{d}_{-2+1} &= \tilde{d}_{-10} + (2-\tilde{g}_{-10})\,\beta + \beta\,(\beta-1), \\ \bar{d}_{-1+1} &= \tilde{g}_{-10}\,\beta - \tilde{d}_{-10}, \\ \bar{d}_{-10} &= -(\tilde{g}_{-10}\,\beta - \tilde{d}_{-10}) = -\bar{d}_{-1+1}.\end{aligned}$$

The reason for applying the Jaffé transformation (1.2.60), (1.2.61) is that it is possible now to put

$$\bar{d}_{-3+1} = 0$$

by choosing β accordingly:

$$\beta = \frac{\tilde{d}_{-10}}{\tilde{g}_0}. \tag{1.2.63}$$

The final forms of $\bar{P}(x)$ and of $\bar{Q}(x)$ are thus

$$\begin{aligned}\bar{P}(x) &= \frac{\bar{g}_{-2+1}}{(x-1)^2} + \frac{\bar{g}_{-1+1}}{x-1} + \frac{\bar{g}_{-10}}{x}, \\ \bar{Q}(x) &= \frac{\bar{d}_{-2+1}}{(x-1)^2} + \frac{\bar{d}_{-1+1}}{x-1} - \frac{\bar{d}_{-1+1}}{x} = \frac{\bar{d}_{-2+1}}{(x-1)^2} + \frac{\bar{d}_{-1+1}}{x\,(x-1)}\end{aligned}$$

and the differential equation for $v(x)$ is given by

$$x\,(x-1)^2\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \sum_{i=0}^{2} \Gamma_i\, x^i\, \frac{\mathrm{d}v}{\mathrm{d}x} + \sum_{i=0}^{1} \Delta_i\, x^i\, v = 0 \tag{1.2.64}$$

[1] It should be recognised that $\frac{1}{x\,(x-1)} = \frac{1}{x-1} - \frac{1}{x}$.

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with

$$\begin{aligned}
\Gamma_2 &= \bar{g}_{-1+1} + \bar{g}_{-10},\\
\Gamma_1 &= -(\bar{g}_{-2+1} + \bar{g}_{-1+1} + 2\,\bar{g}_{-10}),\\
\Gamma_0 &= \bar{g}_{-10},\\
\Delta_1 &= \bar{d}_{-2+1} + \bar{d}_{-1+1},\\
\Delta_0 &= -\bar{d}_{-1+1}.
\end{aligned}$$

The Meaning of β

It is quite clear that the exponent β in (1.2.61) reflects the connection of the two relevant local solutions of the differential equation (1.2.60), (1.2.61) at $z = 0$ and at infinity. This is to be substantiated in the following.

The power terms of the asymptotic factors of the two Jaffé ansatzes (1.2.56) and (1.2.60), (1.2.61) are

$$\begin{aligned}
z^{\alpha_{02}} \left(\frac{2}{z+1}\right)^{\beta} &= z^{\alpha_{02}} (z+1)^{-\beta}\, 2^{\beta}\\
&= z^{\alpha_{02}-\beta} \underbrace{\left(1 + \frac{1}{z}\right)^{-\beta}}_{\to 1 \text{ as } z\to\infty} 2^{\beta}.
\end{aligned} \tag{1.2.65}$$

According to (1.2.52), the power term of the asymptotic factor of the Thomé solution (1.2.18) is

$$z^{\alpha_{0\infty 2}} \tag{1.2.66}$$

as $z \longrightarrow \infty$. Thus, it has to be

$$\beta = \alpha_{02} - \alpha_{0\infty 2} \tag{1.2.67}$$

if the power of z in (1.2.65) has to meet the power of z in (1.2.66). That this is the case may be seen from (1.2.63):

$$\begin{aligned}
\beta &= \frac{\tilde{d}_{-10}}{\tilde{g}_0}\\
&= \frac{2\,\alpha_{1\infty 2}\,\alpha_{02} + A\,\alpha_{1\infty 2} + G_0\,\alpha_{02} + C}{G_0 + 2\,\alpha_{1\infty 2}}\\
&= \frac{A\,\alpha_{1\infty 2} + C}{G_0 + 2\,\alpha_{1\infty 2}} + \frac{G_0 + 2\,\alpha_{1\infty 2}}{G_0 + 2\,\alpha_{1\infty 2}}\,\alpha_{02}\\
&= -\alpha_{0\infty 2} + \alpha_{02},
\end{aligned}$$

which is to be compared with (1.2.67):

$$\alpha_{0\infty 2} = -\frac{A\,\alpha_{1\infty 2} + C}{G_0 + 2\,\alpha_{1\infty 2}}$$

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that is just the same and explains its meaning. The choice (1.2.63) of β guarantees that the two-step Jaffé ansatz (1.2.56) and (1.2.60), (1.2.61) admits the correct and complete asymptotic factor of the Thomé solution (1.2.18) when approaching the irregular singularity of the differential equation (1.2.22) (being located at infinity) along the positive real axis.

Difference Equation

Inserting the ansatz

$$v(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{1.2.68}$$

into the differential equation (1.2.64) results in the following calculation.

Taking the derivatives

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \sum_{n=0}^{\infty} a_n\, n\, x^{n-1},$$
$$\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} = \sum_{n=0}^{\infty} a_n\, n\,(n-1)\, x^{n-2}$$

and inserting these as well as (1.2.68) into the differential equation

$$x\,(x-1)^2\,\frac{\mathrm{d}^2 v}{\mathrm{d}x^2} + \sum_{i=0}^{2} \Gamma_i\, x^i\, \frac{\mathrm{d}v}{\mathrm{d}x} + \sum_{i=0}^{1} \Delta_i\, x^i\, v = 0,$$

results in

$$\sum_{n=0}^{\infty} a_n\, \left[n\,(n-1) + \Gamma_0\, n\right]\, x^{n-1} + \sum_{n=0}^{\infty} a_n\, \left[-2\,n\,(n-1) + \Gamma_1\, n + \Delta_0\right]\, x^n + \sum_{n=0}^{\infty} a_n\, \left[n\,(n-1) + \Gamma_2\, n + \Delta_1\right]\, x^{n+1} = 0.$$

Equating coefficients of equal powers to zero yields the following second-order linear difference equations:

$$x^0:\ (1\cdot 0 + 1\,\Gamma_0)\,a_1 + (-2\cdot 0 + 0\cdot\Gamma_1 + \Delta_0)\,a_0 = 0,$$
$$x^1:\ (2\cdot 1 + 2\,\Gamma_0)\,a_2 + (-2\cdot 1\cdot 0 + 1\cdot\Gamma_1 + \Delta_0)\,a_1 + \Delta_1\,a_0 = 0,$$
$$x^2:\ (3\cdot 2 + 3\,\Gamma_0)\,a_3 + (-2\cdot 2\cdot 1 + 2\cdot\Gamma_1 + \Delta_0)\,a_2 + (1\cdot 0 + 1\cdot\Gamma_2 + \Delta_1)\,a_1 = 0,$$
$$x^3:\ (4\cdot 3 + 4\,\Gamma_0)\,a_4 + (-2\cdot 3\cdot 2 + 3\cdot\Gamma_1 + \Delta_0)\,a_3 + (2\cdot 1 + 2\cdot\Gamma_2 + \Delta_1)\,a_2 = 0,$$
$$x^4:\ (5\cdot 4 + 5\,\Gamma_0)\,a_5 + (-2\cdot 4\cdot 3 + 4\cdot\Gamma_1 + \Delta_0)\,a_4 + (3\cdot 2 + 3\cdot\Gamma_2 + \Delta_1)\,a_3 = 0,$$
$$\ldots$$

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Second-order difference equations of Poincaré–Perron type (cf. §1.3) for the coefficients

$$
\begin{aligned}
&\Gamma_0\, a_1 + \Delta_0\, a_0 = 0,\\
&(n+1)\ [n+\Gamma_0]\ a_{n+1}\\
&\quad + [-2\,(n-1)\,n + n\,\Gamma_1 + \Delta_0]\ a_n\\
&\qquad + [(n-2)\,(n-1) + (n-1)\,\Gamma_2 + \Delta_1]\ a_{n-1} = 0,\ n = 1,2,3,4,\ldots,
\end{aligned}
$$

may be written in the form of a three-term recurrence relation

$$
\begin{aligned}
&a_0 \text{ arbitrary},\\
&\Gamma_0\, a_1 + \Delta_0\, a_0 = 0,\\
&\left(1 + \frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) a_{n+1} + \left(-2 + \frac{\alpha_0}{n} + \frac{\beta_0}{n^2}\right) a_n\\
&\qquad + \left(1 + \frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) a_{n-1} = 0, \qquad n \geq 1,
\end{aligned}
\tag{1.2.69}
$$

where the coefficients of the difference equation are

$$
\begin{aligned}
\alpha_1 &= \Gamma_0 + 1, & \beta_1 &= \Gamma_0,\\
\alpha_0 &= \Gamma_1 + 2, & \beta_0 &= \Delta_0,\\
\alpha_{-1} &= \Gamma_2 - 3, & \beta_{-1} &= \Delta_1 - \Gamma_2 + 2.
\end{aligned}
\tag{1.2.70}
$$

The term ‘recurrence relation’ is justified because of the initial condition in (1.2.69): after having fixed the value of a_0, all the subsequent terms a_n, $n \geq 1$ may be calculated, recursively. Thus, the partial solutions of (1.2.69) may be calculated numerically, as precisely as needed.

In the following we discuss which specific particular solutions of the linear difference equation (1.2.69) are to be calculated in order to solve the singular boundary eigenvalue problem. Since the particular solutions of the difference equation (1.2.69), solving the singular boundary eigenvalue problem, are characterised by its index asymptotic behaviour as $n \to \infty$, equation (1.2.69) is investigated in this respect.

Summary

The solution path shown above is paradigmatic and applicable to any differential equation (1) on page ix. It is the central method (cf. §1.4.2 below) for the realisation of the mathematical principle (cf. §1.4.1) proposed in this book. Therefore, in the following, the method is summarised in a comprehensive form.

Differential equation:

$$
\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + P(z)\,\frac{\mathrm{d}y}{\mathrm{d}z} + Q(z)\,y(z) = 0, \quad z \in \mathbb{C}, \tag{1.2.71}
$$

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with

$$P(z) = \frac{A_0}{z} + G_0, \qquad Q(z) = \frac{B_0}{z^2} + \frac{C}{z} + D_0. \tag{1.2.72}$$

Characteristic exponents of the singularity at zero:

$$\alpha_{01} = \frac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}, \qquad \alpha_{02} = \frac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\,B_0}}{2}. \tag{1.2.73}$$

Characteristic exponents of the singularity at infinity:

$$\alpha_{1\infty 1} = \frac{-G_0 + \sqrt{G_0^2 - 4\,D_0}}{2}, \qquad \alpha_{1\infty 2} = \frac{-G_0 - \sqrt{G_0^2 - 4\,D_0}}{2} \tag{1.2.74}$$

and

$$\alpha_{0\infty 1} = -\frac{A_0\,\alpha_{1\infty 1} + C}{G_0 + 2\,\alpha_{1\infty 1}}, \qquad \alpha_{0\infty 2} = -\frac{A_0\,\alpha_{1\infty 2} + C}{G_0 + 2\,\alpha_{1\infty 2}}. \tag{1.2.75}$$

Jaffé ansatz I:

$$y(z) = \exp\left(\alpha_{1\infty 2}\, z\right)\, z^{\alpha_{0\infty 2}}\, w(z) \tag{1.2.76}$$

with

$$\tilde{P}(z) = \tilde{g}_0 + \frac{\tilde{g}_{-10}}{z}, \qquad \tilde{Q}(z) = \frac{\tilde{d}_{-10}}{z}$$

and

$$\tilde{g}_0 = G_0 + 2\,\alpha_{1\infty 2},$$
$$\tilde{g}_{-10} = A + 2\,\alpha_{02},$$
$$\tilde{d}_{-10} = 2\,\alpha_{1\infty 2}\,\alpha_{02} + A\,\alpha_{1\infty 2} + \alpha_{02}\,G_0 + C.$$

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Jaffé transformation: The relation (1.2.58)

$$x = \frac{z}{z+1}$$

results in

$$\begin{aligned}\tilde{P}(x) &= \tilde{g}_0 - \tilde{g}_{-10} + \frac{\tilde{g}_{-10}}{x},\\ \tilde{Q}(x) &= -\tilde{d}_{-10} + \frac{\tilde{d}_{-10}}{x}.\end{aligned}$$

Thus we have

$$\begin{aligned}\tilde{\tilde{P}}(x) &= \frac{\tilde{\tilde{g}}_{-2+1}}{(x-1)^2} + \frac{\tilde{\tilde{g}}_{-1+1}}{x-1} + \frac{\tilde{\tilde{g}}_{-10}}{x},\\ \tilde{\tilde{Q}}(x) &= \frac{\tilde{\tilde{d}}_{-10}}{x} + \frac{\tilde{\tilde{d}}_{-3+1}}{(x-1)^3} + \frac{\tilde{\tilde{d}}_{-2+1}}{(x-1)^2} + \frac{\tilde{\tilde{d}}_{-1+1}}{x-1},\end{aligned}$$

with

$$\begin{aligned}\tilde{\tilde{g}}_{-2+1} &= \tilde{g}_0,\\ \tilde{\tilde{g}}_{-1+1} &= 2 - \tilde{g}_{-10},\\ \tilde{\tilde{g}}_{-10} &= \tilde{g}_{-10},\\ \tilde{\tilde{d}}_{-3+1} &= -\tilde{d}_{-10},\\ \tilde{\tilde{d}}_{-2+1} &= \tilde{d}_{-10},\\ \tilde{\tilde{d}}_{-1+1} &= -\tilde{d}_{-10},\\ \tilde{\tilde{d}}_{-10} &= \tilde{d}_{-10}.\end{aligned}$$

Jaffé ansatz II: The relation

$$w(x) = f(x)\,v(x) \tag{1.2.77}$$

with

$$f(x) = (1-x)^{\beta} \tag{1.2.78}$$

results in

$$\begin{aligned}\bar{P}(x) &= \frac{\bar{g}_{-2+1}}{(x-1)^2} + \frac{\bar{g}_{-1+1}}{x-1} + \frac{\bar{g}_{-10}}{x},\\ \bar{Q}(x) &= \frac{\bar{d}_{-3+1}}{(x-1)^3} + \frac{\bar{d}_{-2+1}}{(x-1)^2} + \frac{\bar{d}_{-1-1}}{x+1} + \frac{\bar{d}_{-10}}{x}\end{aligned}$$

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with

$$\begin{aligned}
\bar{g}_{-2+1} &= \tilde{g}_0,\\
\bar{g}_{-1+1} &= 2 + 2\beta - \tilde{g}_{-10},\\
\bar{g}_{-10} &= \tilde{g}_{-10},\\
\bar{d}_{-3+1} &= \tilde{g}_0\,\beta - \tilde{d}_{-10},\\
\bar{d}_{-2+1} &= \tilde{d}_{-10} + (2 - \tilde{g}_{-10})\,\beta + \beta\,(\beta - 1),\\
\bar{d}_{-1+1} &= \tilde{g}_{-10}\,\beta - \tilde{d}_{-10},\\
\bar{d}_{-10} &= -(\tilde{g}_{-10}\,\beta - \tilde{d}_{-10}) = -\bar{d}_{-1+1}.
\end{aligned}$$

A Taylor expansion of the particular solution (that is convergent in the unit disc) results in a linear second-order difference equation, the coefficients of which are given by

$$\begin{aligned}
\Gamma_2 &= \bar{g}_{-1+1} + \bar{g}_{-10},\\
\Gamma_1 &= -(\bar{g}_{-2+1} + \bar{g}_{-1+1} + 2\,\bar{g}_{-10}),\\
\Gamma_0 &= \bar{g}_{-10},\\
\Delta_1 &= \bar{d}_{-2+1} + \bar{d}_{-1+1},\\
\Delta_0 &= -\bar{d}_{-1+1}.
\end{aligned} \tag{1.2.79}$$

Index Asymptotics

As has been seen, the Jaffé transformation (1.2.58) and the Jaffé ansatz (1.2.60), (1.2.61) for getting the eigensolutions of the singular boundary eigenvalue problem, discussed above, lead to the irregular difference equation (1.2.69) of Poincaré–Perron type, that is symptomatic for this approach. With respect to the mathematical problem, these sort of difference equations have to be investigated in detail concerning their index-asymptotic behaviours, viz. as $n \longrightarrow \infty$.

We start by dealing with second-order equations, although third-order as well as fourth-order equations occur as well, later in the text. Higher than fourth-order equations do not occur in this book, concretely; however, the general underlying principle becomes clear and is addressed in several places, particularly on a general level in §§1.4 and 3.5.

Given the irregular second-order difference equation (1.2.69) of Poincaré–Perron type

$$\left(1 + \frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) a_{n+1} + \left(-2 + \frac{\alpha_0}{n} + \frac{\beta_0}{n^2}\right) a_n + \left(1 + \frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) a_{n-1} = 0 \tag{1.2.80}$$

with $n = 1, 2, 3, \ldots$, dividing this equation by a_n yields

$$\left(1 + \frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) t_n + \left(-2 + \frac{\alpha_0}{n} + \frac{\beta_0}{n^2}\right) + \left(1 + \frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) \frac{1}{t_{n-1}} = 0 \tag{1.2.81}$$

with

$$t_n = \frac{a_{n+1}}{a_n}.$$

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If the limit

$$t = \lim_{n\to\infty} t_n$$

exists, a carrying out of the limiting process yields a second-order algebraic equation for t:

$$t^2 - 2t + 1 = (t-1)^2 = 0 \qquad (1.2.82)$$

having a twofold solution

$$t_1 = t_2 = 1. \qquad (1.2.83)$$

Equation (1.2.82) is called the *characteristic equation* of zeroth order of the difference equation (1.2.80) (cf. Adams, 1928, pp. 507–541).

If all solutions of the characteristic equation of zeroth order of a linear, homogeneous, ordinary difference equation are finite values, then the underlying difference equation is said to be of *Poinaré–Perron type*. If, furthermore, the characteristic equation of a difference equation of Poincaré–Perron type does have multiple roots, it is said to be *irregular*; otherwise, it is called *regular*. Thus, (1.2.80) is an irregular difference equation of Poincaré–Perron type.

The two foldedness of the solution (1.2.83) of the characteristic equation (1.2.82) means that the ratio

$$\frac{a_{n+1}}{a_n}$$

of two consecutive terms of each particular solution of the difference equation (1.2.80) tends to unity.

As may be expected at this level of zeroth-order asymptotics, it is not obvious from the characteristic equation (1.2.82) that the general solution of the difference equation (1.2.80) is a two-dimensional vector space since it is an irregular one of Poincaré–Perron type. The distinction between the two fundamental solutions of (1.2.80) becomes clear only when carrying out a leading-order asymptotics. This is done in the following.

Take one more order term in equation (1.2.82):

$$\left(1 + \frac{\alpha_1}{n}\right) t_n^2 + \left(-2 + \frac{\alpha_0}{n}\right) t_n + 1 + \frac{\alpha_{-1}}{n} = 0, \qquad (1.2.84)$$

the asymptotic solutions of which are given by

$$t_{n1,2} = 1 \pm \sqrt{\frac{-\sum_{i=-1}^{i=+1} \alpha_i}{n}} + O\left(\frac{1}{n}\right) \quad \text{as } n \to \infty. \qquad (1.2.85)$$

The O symbol may be seen in Slavyanov (1996, §I.1.1, pp. 1, 2). This symbol had already been coined in the 1930s (cf. Erdélyi, 1956, Chapter I).

Equation (1.2.84) is called the *characteristic equation* of the first order of the difference equation (1.2.80) (cf. Adams, 1928, p. 509). Here it is seen that there are

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two particular solutions of equation (1.2.80), since the ratio of two consecutive terms tends to unity:

$$\frac{a_{n+1}}{a_n} \to 1$$

as $n \to \infty$ for each of them, albeit, from two different, namely opposite, directions in the complex t-plane at a rate

$$O\left(\frac{1}{\sqrt{n}}\right)$$

as $n \to \infty$. There are two possibilities. Either the discriminate in (1.2.46) is positive, then there is an exponentially increasing and an exponentially decreasing particular solution. This is shown in Figure 1.2. On the other hand, if the discriminant in (1.2.46) is a negative number, then both partial solutions are oscillating. In this case, the two arrows in Figure 1.2 are rotated by 90° at their arrowheads (cf. Figure 1.6). In the example given in §1.2.5 this case occurs and is considered in more detail there.

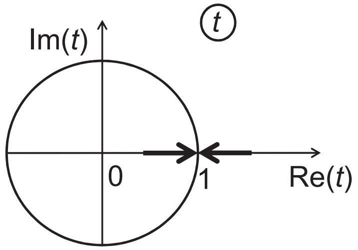

Figure 1.2 Solutions of the second-order characteristic equation.

Birkhoff Sets. Although the ratio of two consecutive terms

$$\frac{a_{n+1}}{a_n}$$

of all the particular solutions of the difference equation (1.2.80) tends to unity, there is a significant difference in their asymptotic behaviour as $n \to \infty$. Either

$$\frac{a_{n+1,1}}{a_{n,1}} \sim 1 + \sqrt{\frac{-\sum_{i=-1}^{i=+1} \alpha_i}{n}} \tag{1.2.86}$$

or

$$\frac{a_{n+1,2}}{a_{n,2}} \sim 1 - \sqrt{\frac{-\sum_{i=-1}^{i=+1} \alpha_i}{n}} \tag{1.2.87}$$

as $n \to \infty$. These ratios indicate the asymptotic behaviours of a pair of fundamental solutions of the difference equation (1.2.80) as $n \to \infty$, that is considered in the following.

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Proposition. *Suppose that two sequences of numbers* $a_{n1,2}$, $n = 1, 2, 3, \ldots$, *behave like*

$$a_{n1} \sim \exp\left(\gamma\, n^{\frac{1}{2}}\right), \tag{1.2.88}$$

$$a_{n2} \sim \exp\left(-\gamma\, n^{\frac{1}{2}}\right) \tag{1.2.89}$$

as $n \to \infty$, *then both of them fulfil the difference equation (1.2.80):*

$$\gamma = 2\sqrt{-\sum_{i=-1}^{i=+1} \alpha_i} \neq 0.$$

Proof Taking the first equation of (1.2.88) and adding the number 1 yields

$$\begin{aligned} a_{n+1} &= \exp\left[\gamma\,(n+1)^{\frac{1}{2}}\right] = \exp\left[\gamma\, n^{\frac{1}{2}}\left(1 + \frac{1}{n}\right)^{\frac{1}{2}}\right] \\ &\sim \exp\left[\gamma\, n^{\frac{1}{2}}\left(1 + \frac{1}{2\,n}\right)\right] = \exp\left[\gamma\, n^{\frac{1}{2}}\right] \exp\left[\frac{\gamma\, n^{\frac{1}{2}}}{2\,n}\right] \\ &= \exp\left[\gamma\, n^{\frac{1}{2}}\right] \exp\left[\frac{\gamma}{2\,n^{\frac{1}{2}}}\right] \quad \text{as } n \to \infty. \end{aligned}$$

Taking the first equation of (1.2.88) and subtracting the number 1 yields

$$\begin{aligned} a_{n-1} &= \exp\left[\gamma\,(n-1)^{\frac{1}{2}}\right] = \exp\left[\gamma\, n^{\frac{1}{2}}\left(1 - \frac{1}{n}\right)^{\frac{1}{2}}\right] \\ &\sim \exp\left[\gamma\, n^{\frac{1}{2}}\left(1 - \frac{1}{2\,n}\right)\right] = \exp\left[\gamma\, n^{\frac{1}{2}}\right] \exp\left[-\frac{\gamma\, n^{\frac{1}{2}}}{2\,n}\right] \\ &= \exp\left[\gamma\, n^{\frac{1}{2}}\right] \exp\left[-\frac{\gamma}{2\,n^{\frac{1}{2}}}\right] \quad \text{as } n \to \infty. \end{aligned}$$

Insertion of these expressions into the difference equation (1.2.80), which may be written in the form

$$\left(n^2 + \alpha_1\, n + \beta_1\right) a_{n+1} + \left(-2\,n^2 + \alpha_0\, n + \beta_0\right) a_n + \left(n^2 + \alpha_{-1}\, n + \beta_{-1}\right) a_{n-1} = 0,$$

leads to

$$\begin{aligned} &\left(n^2 + \alpha_1\, n + \beta_1\right) \exp\left(\frac{\gamma}{2}\, n\right) \exp\left[\frac{\gamma}{2\,n^{\frac{1}{2}}}\right] + \left(-2\,n^2 + \alpha_0\, n + \beta_0\right) \exp\left(\frac{\gamma}{2}\, n\right) \\ &\quad + \left(n^2 + \alpha_{-1}\, n + \beta_{-1}\right) \exp\left(\frac{\gamma}{2}\, n\right) \exp\left[-\frac{\gamma}{2\,n^{\frac{1}{2}}}\right] = 0 \end{aligned}$$

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or

$$
\begin{aligned}
&n^2\left(1+\frac{\gamma}{n^{\frac{1}{2}}}+\frac{\gamma^2}{2\,n^{\frac{2}{2}}}+\cdots\right)+\alpha_1\,n\left(1+\frac{\gamma}{n^{\frac{1}{2}}}+\frac{\gamma^2}{2\,n^{\frac{2}{2}}}+\cdots\right)\\
&\quad+\beta_1\left(1+\frac{\gamma}{n^{\frac{1}{2}}}+\frac{\gamma^2}{2\,n^{\frac{2}{2}}}+\cdots\right)-2\,n^2\left(1+\frac{\gamma}{2\,n^{\frac{1}{2}}}+\frac{\gamma^2}{8\,n^{\frac{2}{2}}}+\cdots\right)\\
&\quad+\alpha_0\,n\left(1+\frac{\gamma}{2\,n^{\frac{1}{2}}}+\frac{\gamma^2}{8\,n^{\frac{2}{2}}}+\cdots\right)+\beta_0\left(1+\frac{\gamma}{2\,n^{\frac{1}{2}}}+\frac{\gamma^2}{8\,n^{\frac{2}{2}}}+\cdots\right)\\
&\quad+n^2+\alpha_{-1}\,n+\beta_{-1}=0.
\end{aligned}
\tag{1.2.90}
$$

Thus, a decomposition of the difference equation (1.2.90) into half-integer powers yields

$$
\begin{aligned}
&n^{\frac{4}{2}}+\gamma\,n^{\frac{3}{2}}+\frac{1}{2}\gamma^2\,n^{\frac{2}{2}}+\cdots\\
&\quad+\alpha_1\,n^{\frac{2}{2}}+\alpha_1\,\gamma\,n^{\frac{1}{2}}+\frac{1}{2}\alpha_1\,\gamma^2\,n^{\frac{0}{2}}+\cdots\\
&\quad+\beta_1\,n^{\frac{0}{2}}+\beta_1\,\gamma\,n^{-\frac{1}{2}}+\frac{1}{2}\beta_1\,\gamma^2\,n^{-\frac{2}{2}}+\cdots\\
&\quad-2\,n^{\frac{4}{2}}-\gamma\,n^{\frac{3}{2}}-\frac{1}{4}\gamma^2\,n^{\frac{2}{2}}+\cdots\\
&\quad+\alpha_0\,n^{\frac{2}{2}}+\frac{1}{2}\alpha_0\,\gamma\,n^{\frac{1}{2}}+\frac{1}{8}\alpha_0\,\gamma^2\,n^{\frac{0}{2}}+\cdots\\
&\quad+\beta_0\,n^{\frac{0}{2}}+\frac{\beta_0\,\gamma}{2}\,n^{-\frac{1}{2}}+\frac{\beta_0\,\gamma^2}{8}\,n^{-\frac{2}{2}}+\cdots\\
&\quad+n^{\frac{4}{2}}+\alpha_{-1}\,n^{\frac{2}{2}}+\beta_{-1}\,n^{\frac{0}{2}}=0.
\end{aligned}
\tag{1.2.91}
$$

In order for the left-hand side of this equation to be zero, it is necessary that all of the coefficients in front of the powers of n have to be zero. Therefore, every power of equation (1.2.91) is to be considered separately.

The highest power term is $n^{\frac{4}{2}}$:

$$n^{\frac{4}{2}}\;:\;(1-2+1)\,n^{\frac{4}{2}}=0.$$

This equation is fulfilled by the difference equation (1.2.80) itself! Thus, a condition cannot be made out of this term.

The next lower power term is $n^{\frac{3}{2}}$:

$$n^{\frac{3}{2}}\;:\;(\gamma-\gamma)\,n^{\frac{3}{2}}=0.$$

This equation is also fulfilled by the difference equation (1.2.80) itself!

The next lower power term is $n^{\frac{2}{2}}$:

$$n^{\frac{2}{2}}\;:\;\left(\alpha_1+\alpha_0+\alpha_{-1}+\frac{1}{2}\gamma^2-\frac{1}{4}\gamma^2\right)n^{\frac{2}{2}}=0.$$

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From this equation result the two characteristic exponents $\gamma = \gamma_1$ and $\gamma = \gamma_2$:

$$\gamma_1 = +2\sqrt{-\sum_{i=-1}^{i=+1} \alpha_i}\,.$$

$$\gamma_2 = -2\sqrt{-\sum_{i=-1}^{i=+1} \alpha_i} = -\gamma_1\,.$$

Thus, both of the sequences in (1.2.88) fulfil the difference equation (1.2.80) as $n \longrightarrow \infty$ □

Thus, in dependence on the sign of the parameter γ, the sequence a_n, $n = 1, 2, 3, \ldots$, either tends to infinity or to zero as $n \longrightarrow \infty$. In both cases, the behaviour is of exponential type.

According to these lines, higher and higher orders may be incorporated into the calculations. The results are the coefficients of the so-called *Birkhoff set*, consisting of two formal expressions of two particular solutions $\{a_{n1}\}$ and $\{a_{n2}\}$ of the irregular difference equation (1.2.72) of Poincaré–Perron type and representing the full asymptotic behaviour of two linearly independent particular solutions of the difference equation as $n \longrightarrow \infty$ that may be considered exact. However, because of the considerable increase in calculatory expenditure, it is advisable to use an algebraic computer program. The Birkhoff set of the difference equation (1.2.72) is given by

$$\begin{aligned} a_{n1}(n) &= \exp\left(\gamma_1\, n^{\frac{1}{2}}\right)\, n^{r_1} \left[1 + \frac{C_{11}}{n^{\frac{1}{2}}} + \frac{C_{12}}{n^{\frac{2}{2}}} + \cdots\right], \\ a_{n2}(n) &= \exp\left(\gamma_2\, n^{\frac{1}{2}}\right)\, n^{r_2} \left[1 + \frac{C_{21}}{n^{\frac{1}{2}}} + \frac{C_{22}}{n^{\frac{2}{2}}} + \cdots\right] \end{aligned} \tag{1.2.92}$$

as $n \longrightarrow \infty$, with

$$\begin{aligned} \gamma_1 &= 2\sqrt{-\sum_{i=-1}^{i=+1} \alpha_i} \neq 0, \\ \gamma_2 &= -\gamma_1, \\ r_1 &= r_2 = -\frac{1}{4}\,(2\,\alpha_1 - 2\,\alpha_{-1} - 1)\,. \end{aligned} \tag{1.2.93}$$

The coefficients C_{ij}, $i = 1, 2$, $j = 1, 2, 3, \ldots$, may be calculated but do not play any role in the following. The solutions $\{a_{n1}\}$, $\{a_{n2}\}$ of the Birkhoff set (1.2.92), (1.2.93) are called *Birkhoff series solutions*.

This sort of formal solution of linear ordinary difference equations is the analogue to the Thomé solution (cf. §1.2) for linear ordinary differential equations. These analogous ansatzes were shown, by **George David Birkhoff** and **Waldemar Juliet Trjitzinsky** (Birkhoff and Trjitzinsky, 1933), in the late 1920s and early 1930s, to be asymptotic solutions as $n \longrightarrow \infty$. Henri Poincaré showed this with respect to Thomé's formal series solutions a few decades earlier for differential equations.

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Taking the difference equation (1.2.72) as a recurrence relation, the increase of the independent variable n is in a sense 'natural'. Thus, the index asymptotics here is the 'natural' behaviour of the independent variable n.

As a result of (1.2.93) it is now clear that if

$$\gamma = 2\sqrt{-\sum_{i=-1}^{i=+1} \alpha_i}$$

then (1.2.92) gives the possible asymptotic behaviours of the particular solutions a_n of the difference equation (1.2.80).

The question now appears which of the two particular solutions has to be chosen if the parameters adopt eigenvalues of the singular boundary eigenvalue problem, i.e., if the particular solution of the differential equation adopts the required asymptotic behaviour on approaching the singularity at $z = 0$ as well as at infinity along the positive real axis. This question is to be dealt with in the following.

A Proof

After having investigated the index-asymptotic behaviour of the particular solutions $\{a_n\}, n = 0, 1, 2, 3, \ldots$, of the irregular difference equation (1.2.69) of Poincaré–Perron type, the final step now is to be done in order to clarify the asymtotic behaviour of the function (1.2.68) as $z \to \infty$ or $x \to 1$.

General Remarks

It is quite clear from what is written above that the asymptotic behaviour of the function $v[x(z)]$ in (1.2.59) generally (viz. if the parameters of the differential equation $\underline{c}$ do not admit eigenvalues) is given by

$$v[x(z)] = \sum_{n=0}^{\infty} a_n\, x^n \sim \exp\left[(\alpha_{1\infty 1} - \alpha_{1\infty 2})\, z\right]$$

as $z \to \infty$, since only then is the asymptotic behaviour of $y(z)$ in (1.2.56) in accordance with the dominant behaviour given by the Thomé solutions in formula (1.2.18) as $z \to \infty$:

$$y(z) \sim \exp\left(\alpha_{1\infty 2}\, z\right)\, \exp\left[(\alpha_{1\infty 1} - \alpha_{1\infty 2})\, z\right] = \exp\left(\alpha_{1\infty 1}\, z\right).$$

On the other hand, it is clear that in the case when $\underline{c}$ admit eigenvalues, the asymptotic behaviour of the function $w(z)$ in (1.2.59) must be weaker than

$$\exp\left(\alpha\, z\right),\ |\alpha| = \alpha_{1\infty 2}$$

as $z \to \infty$, such that it is dominated by means of the asymptotic factor in (1.2.56):

$$\exp\left(\alpha_{1\infty 2}\, z\right)$$

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since only then is the asymptotic behaviour of $y(z)$ in (1.2.56) realised in formula (1.2.18) as $z \longrightarrow \infty$:

$$y(z) \sim \exp\left(\alpha_{1\infty 2}\, z\right)\, \exp\left[\alpha\, z^{\varepsilon}\right] \sim \exp\left(\alpha_{1\infty 2}\, z\right)$$

with $\{\varepsilon \in \mathbb{R}, -1 < \varepsilon < +1\}$ and α assumed to be $|\alpha| < \alpha_{1\infty 2}$. That this really happens is to be shown in detail in the following.

No-Eigenvalue Behaviour

In the following, these global aspects may be confirmed by a local consideration. We start with the non-eigenvalue case.

Proposition. *If the particular solution* $\{a_n\}$, $n = 0, 1, 2, 3, \ldots,$ *of the difference* equation (1.2.72) *is given by*

$$a_n \sim \exp\left(\gamma_1\, n^{\frac{1}{2}}\right) \quad \text{as } n \longrightarrow \infty$$

then the function $v[x(z)]$ *in* (1.2.60) *behaves like*

$$v(x(z)) = \sum_{n=0}^{\infty} a_n\, x^n \sim \exp\left[\left(\alpha_{11} - \alpha_{12}\right)\, z\right] \tag{1.2.94}$$

as $z \longrightarrow \infty$.

To prove this proposition, according to the Euler–Maclaurin formula (cf. also, e.g., deBruijn, 1961; Abramowitz and Stegun, 1970), the infinite sum in (1.2.94)

$$\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z+1}\right)^n$$

is replaced by the integral[2]

$$\int_{n=0}^{\infty} u_z(n)\, \mathrm{d}n$$

with

$$u_z(n) = \exp\left(\gamma_1\, n^{\frac{1}{2}} - \frac{n}{z}\right), \ \gamma_1,\ z > 0, \text{real-valued}, n \in \mathbb{N}$$

as $n \longrightarrow \infty$. We hereby make use of Auxiliary I below.

Auxiliary I. *If*

$$x = \frac{z}{z+1} \tag{1.2.95}$$

then it holds that

$$x^n \sim \exp\left(-\frac{n}{z}\right) \tag{1.2.96}$$

[2] This is correct modulo a constant, which does not matter in the following.

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as $z \longrightarrow \infty$. This means that the ratio

$$\frac{x^n}{\exp\left(-\frac{n}{z}\right)} \tag{1.2.97}$$

tends to unity as z tends to infinity for any natural value of n.

Proof of Auxiliary I We have[3]

$$\begin{aligned}\ln\left(\frac{z}{z+1}\right) &= \ln\left(\frac{z}{z+\frac{z}{z}}\right) = \ln\left(\frac{1}{1+\frac{1}{z}}\right)\\ &= \ln 1 - \ln\left(1+\frac{1}{z}\right) = -\ln\left(1+\frac{1}{z}\right)\\ &\sim -\frac{1}{z} \text{ as } z \longrightarrow \infty\end{aligned}$$

and thus

$$x^n = \exp\left[\ln\left(\frac{z}{z+1}\right)^n\right] = \exp\left[n\ \ln\left(\frac{z}{z+1}\right)\right] \sim \exp\left(-\frac{n}{z}\right)$$

as $z \longrightarrow \infty$. □

We formulate the following Auxiliary II.

Auxiliary II. *If the coefficients a_n in the Taylor series* (1.2.94)

$$v(x(z)) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z+1}\right)^n = \sum_{n=0}^{\infty} a_n\ \exp\left[n\ \ln\left(\frac{z}{z+1}\right)\right]$$

behave like

$$a_n \sim \exp\left(\gamma_1\, n^{\frac{1}{2}}\right)$$

as $n \longrightarrow \infty$, then the function $v(x(z))$ behaves like

$$v[x(z)] \sim \exp\left[\left(\frac{\gamma_1}{2}\right)^2\ z\right]$$

as $z \longrightarrow \infty$.

3 See Bronstein et al. (2001, p. 1043): $\ln(1+x) \sim x$ as $x \longrightarrow 0$, viz. $z \longrightarrow \infty$.

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Proof of Auxiliary II We consider that

$$\sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} a_n \left(\frac{z}{z+1}\right)^n \sim \sum_{n=0}^{\infty} \exp\left(\gamma_1\, n^{\frac{1}{2}}\right) \left(\frac{z}{z+1}\right)^n$$
$$= \sum_{n=0}^{\infty} \exp\left(\gamma_1\, n^{\frac{1}{2}}\right) \exp\left[n \ln\left(\frac{z}{z+1}\right)\right]$$
$$= \sum_{n=0}^{\infty} \exp\left[\gamma_1\, n^{\frac{1}{2}} + \ln\left(\frac{z}{z+1}\right) n\right]$$
$$= \sum_{n=0}^{\infty} \exp\left[h(z,n)\right] = \sum_{n=0}^{\infty} u\,(z,n)$$

with

$$u\,(z,n) = \exp\left[h\,(z,n)\right] \overset{\text{Auxiliary I}}{\sim} \exp\left[\gamma_1\, n^{\frac{1}{2}} - \frac{n}{z}\right]$$

as $n,\ z \longrightarrow \infty$.

On the basis of the Euler–MacLaurin sum formula (see, e.g., deBruijn, 1961; Abramowitz and Stegun, 1970), we calculate

$$v[x\,(z)] = \int_{n=0}^{\infty} u(z,n)\, \mathrm{d}n = \int_{n=0}^{\infty} \exp\left[h(z,n)\right]\, \mathrm{d}n = \int_{n=0}^{\infty} \exp\left[\gamma_1\, n^{\frac{1}{2}} - \frac{n}{z}\right] \mathrm{d}n. \quad (1.2.98)$$

It is important to understand that for $\gamma_1 > 0$ or for $\Re(\gamma_1) > 0$, the function $u(z,n) = u_z(n)$ always has a hump that has two properties as $n \longrightarrow \infty,\ z = \text{const.}$:

- The apex of the hump tends to infinity as $n \longrightarrow \infty$.
- The hump becomes smaller and taller, viz. a peak, a delta function in the limit $n \longrightarrow \infty$.

These two properties are the conditions for applying the Lagrange integration method, which is done in the following.

According to this method, the function $u(z,n) = u_z(n)$ is approximated by a second-order polynomial in the limit $n \longrightarrow \infty,\ z = \text{const.}$ (cf., e.g., deBruijn, 1961):

$$u_z(n) \sim g_1(n) = a_1\ [n - n_0]^2 + b_1 \quad \text{as } n \rightarrow \infty \qquad (1.2.99)$$

with

$$\begin{aligned} a_1 &= a_1(n_0,\ \gamma_1,\ z), \\ b_1 &= b_1(n_0,\ \gamma_1,\ z), \\ n_0 &= n_0(\gamma_1,\ z), \end{aligned} \qquad (1.2.100)$$

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where n_0 is the location of the hump on the n-axis. In order to calculate an approximation of the integral function in (1.2.98), this function $g_1(n)$ may be integrated from the lower zero n_{11} of $g_1(n)$ to the upper one n_{12} yielding

$$\int_0^\infty g_1(z,n)\,\mathrm{d}n \sim \int_{n_{11}}^{n_{12}} g_1(z,n)\,\mathrm{d}n \sim v\,(z)$$

as $n \longrightarrow \infty$ and for fixed values of the independent variable z.

According to the Lagrange integration method, there are three determining conditions for the three parameters a_1, b_1, n_0:

$$g_1(n = n_0) = u_z(n = n_0),$$

$$\left.\frac{\mathrm{d}g_1(n)}{\mathrm{d}n}\right|_{n=n_0} = \left.\frac{\mathrm{d}u_z(n)}{\mathrm{d}n}\right|_{n=n_0} = 0,$$

$$\left.\frac{\mathrm{d}^2 g_1(n)}{\mathrm{d}n^2}\right|_{n=n_0} = \left.\frac{\mathrm{d}^2 u_z(n)}{\mathrm{d}n^2}\right|_{n=n_0}.$$

In the following, these three conditions are evaluated:

- The first condition yields

$$\exp\left[\gamma_1\, n_0^{\frac{1}{2}} - \frac{n_0}{z}\right] = a_1\,(n_0 - n_0)^2 + b_1 = b_1 .$$

- The second condition yields

$$\frac{\mathrm{d}u}{\mathrm{d}n} = \frac{\gamma_1\, z - 2\sqrt{n}}{2\, z\, \sqrt{n}}\, \exp\left(\gamma_1\, \sqrt{n} - \frac{n}{z}\right),$$

thus

$$\left.\frac{\mathrm{d}u}{\mathrm{d}n}\right|_{n=n_0} = \frac{\gamma_1\, z - 2\sqrt{n_0}}{2\, z\, \sqrt{n_0}}\, \exp\left[\gamma_1\, n_0 - \frac{n_0}{z}\right] = 0,$$

resulting in

$$\gamma_1\, z - 2\sqrt{n_0} = 0$$

or

$$n_0(\gamma_1,\, z) = \left(\frac{\gamma_1\, z}{2}\right)^2 .$$

On the other hand [cf. (1.2.99)]

$$\frac{\mathrm{d}g_1(n)}{\mathrm{d}n} = 2\, a_1\, (n - n_0)\, ,$$

from which follows

$$\left.\frac{\mathrm{d}g_1(n)}{\mathrm{d}n}\right|_{n=n_0} = 2\, a_1\, (n_0 - n_0) = 0,$$

as is shown to be correct.

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- The third condition yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}n^2} = \frac{\gamma_1\, z^2\left(\gamma_1\sqrt{n}-1\right) - 4\,\gamma_1\, n\, z + 4\, n^{\frac{3}{2}}}{4\, z^2\, n^{\frac{3}{2}}}\,\exp\left(\gamma_1\sqrt{n} - \frac{n}{z}\right),$$

thus

$$\left.\frac{\mathrm{d}^2 u}{\mathrm{d}n^2}\right|_{n=n_0} = \frac{\gamma_1\, z^2\left(\gamma_1\sqrt{n_0}-1\right) - 4\,\gamma_1\, n_0\, z + 4\, n_0^{\frac{3}{2}}}{4\, z^2\, n_0^{\frac{3}{2}}}\,\exp\left(\gamma_1\sqrt{n_0} - \frac{n_0}{z}\right)$$

and

$$\frac{\mathrm{d}^2 g_1}{\mathrm{d}n^2} = 2\, a_1,$$

from which follows

$$\begin{aligned} a_1 &= \frac{1}{2}\left(\frac{\gamma_1\, z^2\left(\gamma_1\sqrt{n_0}-1\right) - 4\,\gamma_1\, n_0\, z + 4\, n_0^{\frac{3}{2}}}{4\, z^2\, n_0^{\frac{3}{2}}}\,\exp\left(\gamma_1\sqrt{n_0} - \frac{n_0}{z}\right)\right) \\ &= -\frac{\exp\left(\frac{\gamma_1^2\, z}{4}\right)}{\gamma_1^2\, z^3} \sim \exp\left(\frac{\gamma_1^2\, z}{4}\right) \end{aligned}$$

as $n_0 \longrightarrow \infty$ or $z \longrightarrow \infty$.

Summarising the calculations, it is to be remarked that the function

$$u_z(n) = u(z,n) = \exp\left(\gamma_1\, n^{\frac{1}{2}} - \frac{n}{z}\right)$$

in the limit $n \longrightarrow \infty$ is approximated by the quadratic parabola

$$g_1(n) = a_1\,[n - n_0]^2 + b_1$$

with[4]

$$\begin{aligned} n_0(\gamma_1,\, z) &= \left(\frac{\gamma_1\, z}{2}\right)^2, \\ a_1(n_0,\, \gamma_1,\, z) &= -\frac{b_1}{\gamma_1^2\, z^3} = -\frac{\exp\left(\frac{\gamma_1^2\, z}{4}\right)}{\gamma_1^2\, z^3}, \\ b_1(n_0,\, \gamma_1,\, z) &= \exp\left(\gamma_1\, n_0^{\frac{1}{2}} - \frac{n_0}{z}\right) = \exp\left(\frac{\gamma_1^2\, z}{4}\right). \end{aligned}$$

For later use, we write

$$-\frac{b_1}{a_1} = \gamma_1^2\, z^3.$$

The calculation of the zeros of $g_1(n)$ is done by

$$a_1\,[n_1 - n_0]^2 + b_1 = 0$$

[4] Thus, for $z \longrightarrow \infty$, we have $a_1 \longrightarrow -b_1$.

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or

$$a_1\, n_1^2 - 2\, a_1\, n_0\, n_1 + a_1\, n_0^2 + b_1 = 0.$$

Thus

$$\begin{aligned} n_{11} &= \frac{1}{2\, a_1}\left[2\, a_1\, n_0 - \sqrt{4a_1^2\, n_0^2 - 4\, a_1\,\left(a_1\, n_0^2 + b_1\right)}\right] \\ &= n_0 - \sqrt{-\frac{b_1}{a_1}} = \left(\frac{\gamma_1\, z}{2}\right)^2 - \gamma_1\,\sqrt{z^3} = n_0 - \gamma_1\,\sqrt{z^3} \end{aligned}$$

and

$$\begin{aligned} n_{12} &= \frac{1}{2\, a_1}\left[2\, a_1\, n_0 + \sqrt{4a_1^2\, n_0^2 - 4\, a_1\,\left(a_1\, n_0^2 + b_1\right)}\right] \\ &= n_0 + \sqrt{-\frac{b_1}{a_1}} = \left(\frac{\gamma_1\, z}{2}\right)^2 + \gamma_1\,\sqrt{z^3} = n_0 + \gamma_1\,\sqrt{z^3}. \end{aligned}$$

The final result is summarised by stating

$$\begin{aligned} v[x(z)] &= \sum_{n=0}^{\infty} a_n\, x^n \\ &\sim \sum_{n=0}^{\infty} \exp\left(\gamma_1\, n^{\frac{1}{2}}\right)\, x^n \text{ as } n \to \infty \\ &\sim \int_{n_{11}}^{n_{12}} \left[a_1\,(n - n_0)^2 + b_1\right]\, \mathrm{d}n \\ &= \frac{a_1}{3}\,(n - n_0)^3 + b_1\, n\Big|_{n_{11}}^{n_{12}} \\ &= \frac{2}{3}\, b_1\,\sqrt{-\frac{b_1}{a_1}} \\ &= \frac{4}{3}\,\gamma_1\,\sqrt{z^3}\,\exp\left(\frac{\gamma_1^2\, z}{4}\right) \sim \exp\left(\frac{\gamma_1^2\, z}{4}\right) \end{aligned}$$

or

$$v[x(z)] \sim \exp\left[\left(\frac{\gamma_1}{2}\right)^2\, z\right] \tag{1.2.101}$$

as $z \to \infty$, which is to be shown. □

In order to prove the proposition, we need Auxiliary III below.

Auxiliary III. *For the coefficient*

$$\left(\frac{\gamma_1}{2}\right)^2 = \frac{\gamma_1^2}{4}$$

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of the exponent of the dominant term of the series (1.2.101) *for* $v[x(z)]$ *of the single confluent case of the Gauss differential equation, it holds that*

$$\frac{\gamma_1^2}{4} = \alpha_{11} - \alpha_{12},$$

with α_{11}, α_{12} *being the characteristic exponents of second kind of the first order of the irregular singularity at infinity:*

$$\alpha_{11} = \frac{1}{2}\left(-G_0 + \sqrt{G_0^2 - 4\,D_0}\right),$$

$$\alpha_{12} = \frac{1}{2}\left(-G_0 - \sqrt{G_0^2 - 4\,D_0}\right).$$

Proof of Auxiliary III From the Birkhoff set (1.2.92), (1.2.93) it is given that

$$\frac{\gamma_1^2}{4} = -\sum_{i=-1}^{+1} \alpha_i .$$

The Jaffé expansion (1.2.68) yields the relations (1.2.70):

$$\begin{aligned} \alpha_1 &= \Gamma_0 + 1,\\ \alpha_0 &= \Gamma_1 + 2,\\ \alpha_{-1} &= \Gamma_2 - 3, \end{aligned}$$

resulting in

$$-\sum_{i=-1}^{+1} \alpha_i = -\sum_{i=0}^{2} \Gamma_i .$$

Furthermore

$$\begin{aligned} \Gamma_2 &= -4 + g_2 - g_1 + g_0 + 2\,g_{-1},\\ \Gamma_1 &= 6 + g_1 - 2\,g_0 - 3\,g_{-1},\\ \Gamma_0 &= -2 + g_0 + g_{-1}, \end{aligned}$$

yielding

$$-\sum_{i=0}^{2} \Gamma_i = \bar{g}_{-2+1} = \tilde{g}_0 = G_0 + 2\,\alpha_{12} = \sqrt{G_0^2 - 4\,D_0}. \tag{1.2.102}$$

So

$$\frac{\gamma_1^2}{4} = -\sum_{i=-1}^{+1} \alpha_i = -\sum_{i=0}^{2} \Gamma_i = -g_2 = -(G_0 + 2\,\alpha_{11}) = \sqrt{G_0^2 - 4\,D_0}. \tag{1.2.103}$$

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On the other hand [cf. (1.2.48)]

$$\alpha_{1\infty 1} = \frac{1}{2}\left(-G_0 + \sqrt{G_0^2 - 4\,D_0}\right) = -\frac{1}{2}\,G_0 + \frac{1}{2}\,\sqrt{G_0^2 - 4\,D_0},$$
$$\alpha_{1\infty 2} = \frac{1}{2}\left(-G_0 - \sqrt{G_0^2 - 4\,D_0}\right) = -\frac{1}{2}\,G_0 - \frac{1}{2}\,\sqrt{G_0^2 - 4\,D_0},$$

from which it is immediately seen that

$$\alpha_{1\infty 1} - \alpha_{1\infty 2} = \sqrt{G_0^2 - 4\,D_0},$$

which is the same as (1.2.102). Thus

$$\left(\frac{\gamma_1}{2}\right)^2 = \alpha_{1\infty 1} - \alpha_{1\infty 2},$$

as has to be shown. □

Proof of the Proposition Summarising the results of Auxiliaries I, II and III above, it becomes obvious that if the parameters $\underline{c}$ are supposed to be *no* eigenvalues, then

$$\begin{aligned}
y(z) \quad &= \quad \exp\left(\alpha_{12}\,z\right)\,z^{\alpha_{0r2}}\sum_{n=0}^{\infty} a_n\left(\frac{z}{z+1}\right)^n \\
&\overset{\text{as } n\to\infty}{\sim} \quad \underbrace{\exp\left(\alpha_{12}\,z\right)}_{\text{asymptotic factor}}\;\sum_{n=0}^{\infty}\exp\left(\gamma_1\,n^{\frac{1}{2}}\right)\left(\frac{z}{z+1}\right)^n \\
&\overset{\text{Auxiliary I}}{\sim} \exp\left(\alpha_{12}\,z\right)\sum_{n=0}^{\infty}\exp\left(\gamma_1\,n^{\frac{1}{2}} - \frac{n}{z}\right) \\
&\overset{\text{Auxiliary II}}{\sim} \exp\left(\alpha_{12}\,z\right)\;\exp\left[\left(\frac{1}{2}\,\gamma_1\right)^2\,z\right] \\
&\overset{\text{Auxiliary III}}{\sim} \exp\left(\alpha_{12}\,z\right)\,\exp\left[\left(\alpha_{11} - \alpha_{12}\right)\,z\right] \\
&= \quad \exp\left(\alpha_{12}\,z\right)\,\exp\left(\alpha_{11}\,z\right)\,\exp\left(-\alpha_{12}\,z\right) \\
&= \quad \exp\left(\alpha_{11}\,z\right),
\end{aligned}$$

which proves the proposition. □

Boundary Eigenvalue Behaviour

In the following we admit that the parameters $\underline{c}$ are supposed to be eigenvalues, viz. we admit values such that the particular solution $y(z)$ of the differential equation (1.2.54), (1.2.55) meets the boundary conditions simultaneously at the regular singularity at the origin (already fulfilled by means of the ansatz (1.2.56), (1.2.58), (1.2.60), (1.2.61), (1.2.68)) as well as at the irregular singularity, located at infinity (to be fulfilled by means of a specific asymptotic behaviour of the particular solution $\{a_n,\, n = 0, 1, 2, 3, \ldots\}$ as $n \to \infty$ of the difference equation (1.2.69), possible only for special

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values $\underline{c}$ (eigenvalues) of the underlying differential equation (1.2.54), (1.2.55). In the eigenvalue case, asymptotics may be settled by means of an estimation.

Proposition 1.1. *Suppose that*

$$u_z(n) = \exp\left(-\alpha\, n^{\frac{1}{2}} - \frac{n}{z}\right),\ \alpha,\ z > 0, \textit{real-valued}$$

holds. Then the integral

$$\int_{n=0}^{\infty} u_z(n) = \int_{n=0}^{\infty} \exp\left(-\alpha\, n^{\frac{1}{2}} - \frac{n}{z}\right)\, \mathrm{d}n$$

is finite, defining a function

$$U(z)$$

that tends to a finite value as $z \longrightarrow \infty$.[5]

Proof of the Proposition Estimating

$$u(z,n) < \exp\left(-\alpha\, n^{\frac{1}{2}} - \frac{n^{\frac{1}{2}}}{z}\right) \text{ for all } z > 0, \text{real-valued}$$

yields an integral

$$\int_{1}^{\infty} u(z,n)\, \mathrm{d}n < \int_{1}^{\infty} \exp\left[-n^{\frac{1}{2}}\left(\alpha + \frac{1}{z}\right)\right] \mathrm{d}n \text{ for all } z > 0, \text{real-valued}, \quad (1.2.104)$$

which may be solved exactly. By means of the substitution

$$N = \sqrt{n}$$

and

$$\frac{\mathrm{d}N}{\mathrm{d}n} = \frac{1}{2\sqrt{n}} = \frac{1}{2\,N},$$

the integral (1.2.104) becomes

$$\begin{aligned}\int_{1}^{\infty} \exp\left[-n^{\frac{1}{2}}\left(\alpha + \frac{1}{z}\right)\right] \mathrm{d}n &= 2\int_{1}^{\infty} N \exp\left[-N\left(\alpha + \frac{1}{z}\right)\right] \mathrm{d}N \\ &= 2\,\frac{\alpha + \frac{1}{z} + 1}{\left(\alpha + \frac{1}{z}\right)^2} \text{ for all } z > 0, \text{real-valued}.\end{aligned}$$

For $z \longrightarrow \infty$, this yields

$$\lim_{z\to\infty} 2\,\frac{\alpha + \frac{1}{z} + 1}{(\alpha + \frac{1}{z})^2} = 2\,\frac{\alpha + 1}{\alpha^2} + \mathcal{O}\left(\frac{1}{z}\right),$$

[5] It is possible to prove the proposition by means of the monotone convergence theorem; the result is $\int_{n=0}^{\infty} \exp\left(-\alpha\, n^{\frac{1}{2}} - \frac{n}{z}\right) \mathrm{d}n = 2\left(\frac{z}{\alpha}\right)^2$.

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thus

$$v[x(z)] = \text{const.} + O\left(\frac{1}{z}\right)$$

as $z \longrightarrow \infty$ if $\underline{c}$ adopts an eigenvalue, which has to be shown. □

As a summary of the foregoing considerations we state the following:

- If $\underline{c}$ is no eigenvalue, then the function $v(x)$ in (1.2.94) behaves like

$$v(x) = \sum_{n=0}^{\infty} a_n\, x^n \sim \exp\left[(\alpha_{11} - \alpha_{12})\; z\right]$$

  as $z \longrightarrow \infty$, whereby $x = \frac{z}{z+1}$ [cf. (1.2.58)].
- If $\underline{c}$ adopts an eigenvalue, then the function $v(x)$ in (1.2.94) behaves like

$$v(x) = \sum_{n=0}^{\infty} a_n\, x^n \sim \text{const.} + O\left(\frac{1}{z}\right)$$

  as $z \longrightarrow \infty$ or $x \longrightarrow 1$. This means that the series in the recessive Thomé solution (1.2.18) at the infinite singularity of the underlying differential equation

$$y_{T2}(z) = \exp\left(\alpha_{1\infty 2}\; z\right)\; z^{\alpha_{0\infty 2}} \left(1 + \sum_{n=1}^{\infty} C_{n2}\, z^{-n}\right) \text{ as } z \longrightarrow \infty,$$

  which are generally asymptotic and divergent, become convergent series.

  Besides the proof above, it is obvious that a series $\sum_{n=0}^{\infty} a_n$, where the a_n are exponentially decreasing, is finite.

Variable Asymptotics

Equation (1.2.54) is linear and of second order. Therefore, so is (1.2.69). Because of this linearity, the general solution of the difference equation (1.2.69) is given by

$$a_n^{(g)} = L_1\, a_{n1} + L_2\, a_{n2}, \quad n = 0, 1, 2, \ldots,$$

whereby a_{n1} and a_{n2} are any two linearly independent particular solutions of (1.2.69) for which it holds that

$$L_1\, a_{n1} + L_2\, a_{n2} = 0 \tag{1.2.105}$$

only for $L_1 = L_2 = 0$ and where $L_1 = L_1(\underline{c})$ and $L_2 = L_2(\underline{c})$ are any complex- or real-valued constants in n that may be dependent, however, on the parameters (denoted $\underline{c}$) of the differential equation (1.2.54), (1.2.55) (but not on n).

From the asymptotic behaviours (1.2.88) it is seen that $a_n^{(1)}$ and $a_n^{(2)}$ form a fundamental system of solutions of the difference equation (1.2.69). Then, as a result of the foregoing considerations, it has been shown that the eigenvalue condition of the

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singular boundary eigenvalue problem of the single confluent case (1.2.54), (1.2.55) of the Gauss differential equation (1.2.2), (1.2.3) is

$$L_1(\underline{c}) = 0 \tag{1.2.106}$$

in (1.2.105) and the eigensolutions are given by (1.2.56), (1.2.58), (1.2.60), (1.2.61), (1.2.68), (1.2.69).

Thus, the singular boundary eigenvalue problem of the single confluent case (1.2.54), (1.2.55) of the Gauss differential equation (1.2.2), (1.2.3) is solved exactly. The Jaffé approach to the singular boundary eigenvalue problem, established above, seems a lot more complicated than the traditional approach, given before. However, the advantage of this approach is its generalisability. From a mathematical point of view, there is no limit on the order of the difference equation coming out of a Jaffé ansatz for the central two-point connection problem to be dealt with along the lines above.

Eigenvalue Condition

According to (1.2.106), the eigenvalue condition of the singular boundary eigenvalue problem is

$$L_1(\underline{c}) = 0.$$

The practical calculation is easy. For arbitrary $\underline{c}$ we need to calculate a value a_{N_0} for a sufficiently large index $n = N_0$. This is possible in a recursive way by means of the linear second-order difference equation (1.2.69), used as a three-term recurrence relation. Hereby, 'sufficiently large' is dependent on the required accuracy of the eigenvalues and in this context it means that the exponentially increasing fundamental solution a_{n1} for $n \to \infty$ is fully developed. Then, we equate $a_{N_0}(\underline{c})$ to zero (or, alternatively, a change of sign) in dependence on $\underline{c}$. This eventually determines the value of the eigenvalue parameter(s) that solve(s) the singular boundary eigenvalue problem.

1.2.5 Example: Laguerre Polynomials

It was pointed out at the beginning that there are special functions whose treatment has been known for a long time; these are called classical special functions and the method of their calculation is presented again here, for reasons of being able to test the new method displayed above. In this way it is possible to draw a comparison with problems whose results are known. One such classical (i.e., well-known) problem is the quantum mechanical calculation of the energy levels (i.e., eigenvalues) of spherically symmetric atoms. Such atoms are described by a so-called Coulomb potential of the stationary three-dimensional Schrödinger equation. This equation can be separated into its spatial dimensions, resulting in the single confluent case (1.2.13) of the Gauss differential equation (1.2.1) for the radial component.

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First this classical solution will be calculated and then the new method, presented above, will be carried out and its results compared with those of the classical calculation. It will turn out that the classical solution is somewhat simpler than the new method; however, while the classical solution focuses on specific properties of the problem at hand and can, therefore, no longer be applied to problems that deviate even slightly from it, the new method is applicable to much more general problems, in principle to all singular boundary eigenvalue problems of differential equations (1.2.2), (1.2.1). It is precisely this limited applicability of the classical method, called the *Sommerfeld polynomial method* – based on focusing on the specific properties of each of the classical problems, that makes a separate treatment necessary, the characteristic property of which is taking care of the general aspects.

Classical Approach

The quantum mechanical treatment of the Schrödinger equation, the potential of which is the Coulomb potential (a spherical one, with first-order pole at its centre; cf. Figure 1.3), results in an algebraic boundary eigenvalue condition. This is shown in the following.

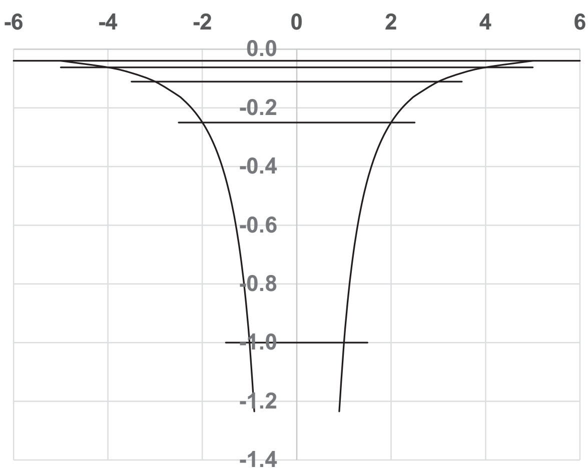

Figure 1.3 Quantum Coulomb problem.

The differential equation of the radial part is the single confluent case of the Gauss equation

$$\frac{d^2 y}{dz^2} + P(z)\,\frac{dy}{dz} + Q(z)\,y = 0,\ z \in \mathbb{R}_0^+, \tag{1.2.107}$$

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the coefficients of which are

$$\begin{aligned} P(z) &= \frac{A_0}{z} + G_0 = \frac{2}{z}, \\ Q(z) &= \frac{B_0}{z^2} + \frac{C}{z} + D_0 = \frac{2}{z} + D_0, \end{aligned} \tag{1.2.108}$$

whereby the coefficients are given by

$$\begin{aligned} G_0 &= 0, \\ A_0 &= 2, \\ B_0 &= 0, \\ C &= 2 \end{aligned}$$

and D_0 is the energy parameter. The local solutions at the two singularities are determined by the characteristic exponents. These are calculated in the following, starting with the singularity at the origin $z = 0$. The particular local solutions are Frobenius solutions

$$y(z) = z^{\alpha_0} \sum_{n=0}^{\infty} a_n \, z^n,$$

whereby the two characteristic exponents are given by

$$\begin{aligned} \alpha_{01} &= \frac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\, B_0}}{2} = 0, \\ \alpha_{02} &= \frac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\, B_0}}{2} = -1. \end{aligned}$$

The local solutions at infinity are the Thomé solutions

$$y(z) = \exp(\alpha_1 \, z) \; z^{\alpha_\infty} \sum_{n=0}^{\infty} C_n \, z^{-n}$$

with

$$\begin{aligned} \alpha_{11} &= \frac{-G_0 + \sqrt{G_0^2 - 4\, D_0}}{2} = \sqrt{-D_0}, \\ \alpha_{12} &= \frac{-G_0 - \sqrt{G_0^2 - 4\, D_0}}{2} = -\sqrt{-D_0} \end{aligned}$$

and

$$\begin{aligned} \alpha_{\infty 1} &= -\frac{A_0 \, \alpha_{11} + C}{G_0 + 2\, \alpha_{11}} = -1 - \frac{1}{\sqrt{-D_0}}, \\ \alpha_{\infty 2} &= -\frac{A_0 \, \alpha_{11} + C}{G_0 + 2\, \alpha_{11}} = -1 + \frac{1}{\sqrt{-D_0}}. \end{aligned}$$

The singular boundary eigenvalue problem of the physical problem is formulated as follows. Look for those values of D_0 that yield solutions of equation (1.2.107)

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that behave holomorphically at the origin $z = 0$ and, simultaneously, behave like $y(z) \sim \exp\left(-\sqrt{-D_0}\, z\right)$ as $z \to \infty$, radially, on the positive real axis.

The classical ansatz in order to solve this singular boundary eigenvalue problem is (cf. Schubert and Weber, 1980, p. 160)

$$y(z) = \exp(\alpha_{12}\, z)\; z^{\alpha_{01}} \sum_{n=0}^{\infty} a_n\, z^n = \exp\left(-\sqrt{-D_0}\, z\right) \sum_{n=0}^{\infty} a_n\, z^n, \tag{1.2.109}$$

resulting in a linear first-order difference equation

$$(n+1)\,(n+2)\, a_{n+1} - \left[n + 1 - \frac{1}{\sqrt{-D_0}}\right] a_n = 0, \quad n = 0, 1, 2, 3, \ldots, \tag{1.2.110}$$

that may be converted into a two-term recurrence relation by fixing $a_0 = 1$, obtaining

$$a_{n+1} = f(n, D_0)\, a_n = \frac{n + 1 - \frac{1}{\sqrt{-D_0}}}{(n+1)\,(n+2)}\, a_n, \quad n = 0, 1, 2, 3, \ldots .$$

The eigenvalue condition of the problem according to the Sommerfeld polynomial method is

$$f(n, D_0) = 0 \rightsquigarrow D_{0n} = -\frac{1}{(n+1)^2}, \quad n = 0, 1, 2, 3, \ldots,$$

which is actually just the condition for the difference equation (1.2.110) to admit trivial solutions. All the non-trivial particular solutions of the difference equation lead to an exponentially increasing behaviour of the solution $y(z)$ of the differential equation at infinity, as may be seen from the characteristic exponent α_{11}. In this case

$$\sum_{n=0}^{\infty} a_n\, z^n \sim \exp\left(2\sqrt{-D_0}\, z\right) \quad \text{as } z \to \infty$$

in (1.2.109).

The result are the eigenvalues

$$\begin{aligned} D_{00} &= -1, \\ D_{01} &= -\frac{1}{4}, \\ D_{02} &= -\frac{1}{9}, \\ &\ldots \\ D_{0n} &= -\frac{1}{n^2}, \\ &\ldots \end{aligned}$$

As may be seen, the quantum mechanical treatment of the Coulomb potential yields energy levels that cumulate at $D_0 = 0$ with

$$D_{0n} = -\frac{1}{n^2}, \quad n = 1, 2, 3, \ldots . \tag{1.2.111}$$

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The corresponding eigensolutions of the underlying equation (1.2.107), (1.2.108) are given by

$$y_n(z) = \exp\left(-\sqrt{-D_0}\, z\right)\, L_n^{(1)}(z) = \exp\left(-\frac{z}{n+1}\right)\, L_n^{(1)}\left(\frac{2\,z}{n+1}\right),\ n = 0,1,2,3,\ldots,$$

where the $L_n^{(1)}(z)$ are polynomials. These polynomials are called *Laguerre polynomials* (see Figure 1.4), bearing the name of **Edmund Nicolas Laguerre** (1834–1886), a French mathematician of the nineteenth century, who discovered them in 1879. The explicit form of the Laguerre polynomials is

$$\begin{aligned}
L_0^{(1)}(z) &= 1,\\
L_1^{(1)}(z) &= 1 - z,\\
L_2^{(1)}(z) &= 1 - 2\,z + \frac{1}{2}\,z^2,\\
L_3^{(1)}(z) &= 1 - 3\,z + \frac{3}{2}\,z^2 - \frac{1}{6}\,z^3,\\
L_4^{(1)}(z) &= 1 - 4\,z + 3\,z^2 - \frac{2}{3}\,z^3 + \frac{1}{24}\,z^4,\\
&\ldots
\end{aligned}$$

or

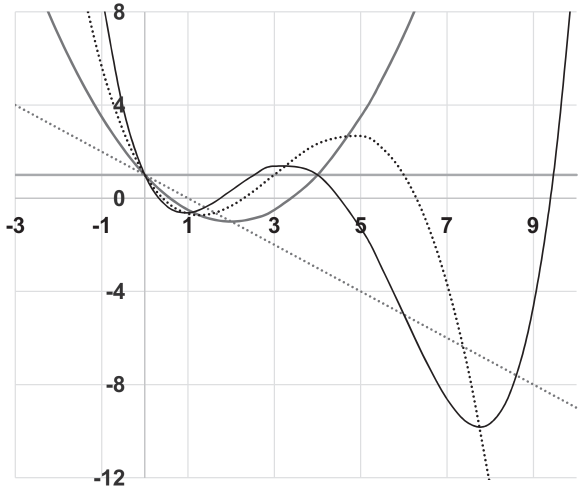

Figure 1.4 Laguerre polynomials.

$$L_n^{(1)}(z) = \sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{k!}\, z^k,\ n = 0,1,2,3,\ldots.$$

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As a final result we give the eigensolutions belonging to the eigenvalues, the five lowest of which are given by (see Figure 1.5)

$$
\begin{aligned}
y_0(z) &= \exp\left(-\sqrt{-D_{00}}\, z\right) L_0^{(1)}(z) = \exp(-z), \\
y_1(z) &= \exp\left(-\sqrt{-D_{01}}\, z\right) L_1^{(1)}(z) = \exp\left(-\frac{1}{2}\, z\right)(1-z), \\
y_2(z) &= \exp\left(-\sqrt{-D_{02}}\, z\right) L_2^{(1)}(z) = \exp\left(-\frac{1}{3}\, z\right)\left(1 - 2\,z + \frac{1}{2}\, z^2\right), \\
y_3(z) &= \exp\left(-\sqrt{-D_{03}}\, z\right) L_3^{(1)}(z) = \exp\left(-\frac{1}{4}\, z\right)\left(1 - 3\,z + \frac{3}{2}\, z^2 - \frac{1}{6}\, z^3\right), \\
y_4(z) &= \exp\left(-\sqrt{-D_{04}}\, z\right) L_4^{(1)}(z) = \exp\left(-\frac{1}{5}\, z\right)\left(1 - 4\,z + 3\,z^2 - \frac{2}{3}\, z^3 + \frac{1}{24}\, z^4\right), \\
&\ldots
\end{aligned}
$$

As one may see, the exponential term dominates the expression of the eigensolutions above.

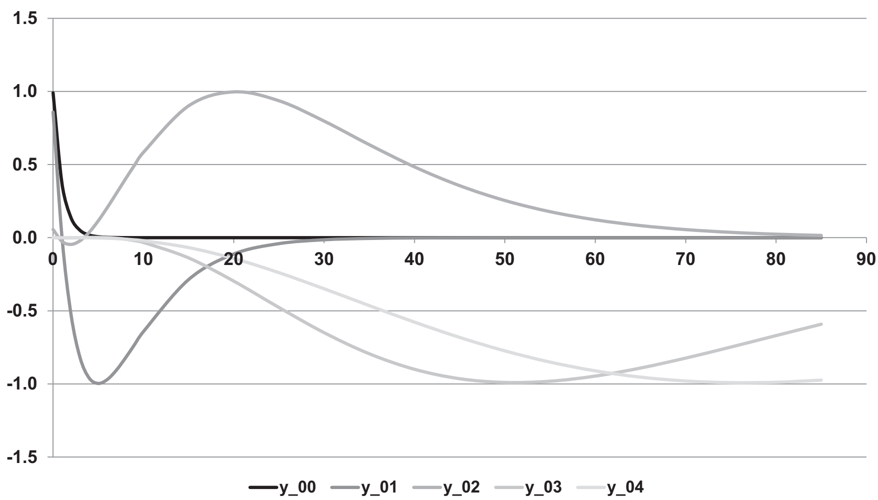

Figure 1.5 Radial eigensolutions of the quantum Coulomb problem.

Jaffé Method

The starting point is the single confluent case of the Gauss differential equation

$$
\frac{d^2 y}{dz^2} + P(z)\,\frac{dy}{dz} + Q(z)\,y = 0
$$

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with

$$P(z) = \frac{A_0}{z} + G_0 = \frac{2}{z},$$
$$Q(z) = \frac{B_0}{z^2} + \frac{C}{z} + D_0 = \frac{2}{z} + D_0. \tag{1.2.112}$$

Thus

$$G_0 = 0,$$
$$A_0 = 2,$$
$$B_0 = 0,$$
$$C = 2.$$

The local (i.e., Frobenius) solutions at the singularity of the differential equation at the origin $z = 0$ have the form

$$y(z) = z^{\alpha_0} \sum_{n=0}^{\infty} a_n \, z^n,$$

the two characteristic exponents of which are

$$l\alpha_{01} = \frac{1 - A_0 + \sqrt{(1 - A_0)^2 - 4\,B_0}}{2} = 0,$$
$$\alpha_{02} = \frac{1 - A_0 - \sqrt{(1 - A_0)^2 - 4\,B_0}}{2} = -1. \tag{1.2.113}$$

The local (i.e., Thomé) solutions at the singularity of the differential equation at infinity have the form

$$y(z) = \exp\left(\alpha_1 \, z\right) \, z^{\alpha_{0\infty}} \sum_{n=0}^{\infty} C_n \, z^{-n},$$

the characteristic exponents of which are

$$\alpha_{11} = \frac{-G_0 + \sqrt{G_0^2 - 4\,D_0}}{2} = \sqrt{-D_0},$$
$$\alpha_{12} = \frac{-G_0 - \sqrt{G_0^2 - 4\,D_0}}{2} = -\sqrt{-D_0} \tag{1.2.114}$$

and

$$\alpha_{0\infty 1} = -\frac{A_0\,\alpha_{11} + C}{G_0 + 2\,\alpha_{11}} = -1 - \frac{1}{\sqrt{-D_0}},$$
$$\alpha_{0\infty 2} = -\frac{A_0\,\alpha_{12} + C}{G_0 + 2\,\alpha_{12}} = -1 + \frac{1}{\sqrt{-D_0}}. \tag{1.2.115}$$

For physical reasons, the boundary eigenvalue problem consists of connecting the Frobenius solution at the origin, related to α_{01}, to the Thomé solution, related to α_{12}.

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This determines the Jaffé ansatz that consists of two parts, the first of which is given by (Jaffé ansatz I) [cf. (1.2.56)]

$$y(z) = \exp(\alpha_{12}\, z)\; z^{\alpha_{01}}\, w(z) = \exp\left(-\sqrt{-D_0}\, z\right)\, w(z), \tag{1.2.116}$$

resulting in a differential equation, having the coefficients

$$\begin{aligned}\tilde{P}(z) &= \tilde{g}_0 + \frac{\tilde{g}_{-10}}{z},\\ \tilde{Q}(z) &= \frac{\tilde{d}_{-10}}{z},\end{aligned}$$

with

$$\begin{aligned}\tilde{g}_0 &= G_0 + 2\,\alpha_{12} = -2\sqrt{-D_0},\\ \tilde{g}_{-10} &= A_0 + 2\,\alpha_{01} = 2,\\ \tilde{d}_{-10} &= 2\,\alpha_{12}\,\alpha_{01} + A_0\,\alpha_{12} + G_0\,\alpha_{01} + C = -2\sqrt{-D_0} + 2.\end{aligned}$$

This determines the exponent β [cf. (1.2.63)] to be

$$\rightsquigarrow \beta = \frac{\tilde{d}_{-10}}{\tilde{g}_0} = 1 - \frac{1}{\sqrt{-D_0}}.$$

A Jaffé transformation (1.2.58)

$$x = \frac{z}{z+1} \tag{1.2.117}$$

changes the coefficients (1.2.112) of the differential equation (1.2.5) according to

$$\begin{aligned}\tilde{P}(x) &= \tilde{g}_0 - \tilde{g}_{-10} + \frac{\tilde{g}_{-10}}{x},\\ \tilde{Q}(x) &= -\tilde{d}_{-10} + \frac{\tilde{d}_{-10}}{x}.\end{aligned}$$

The Jaffé transformation (1.2.117) results in a differential equation, the coefficients of which are given by

$$\begin{aligned}\tilde{\tilde{P}}(x) &= \frac{\tilde{\tilde{g}}_{-2+1}}{(x-1)^2} + \frac{\tilde{\tilde{g}}_{-1+1}}{x-1} + \frac{\tilde{\tilde{g}}_{-10}}{x},\\ \tilde{\tilde{Q}}(x) &= \frac{\tilde{\tilde{d}}_{-10}}{x} + \frac{\tilde{\tilde{d}}_{-3+1}}{(x-1)^3} + \frac{\tilde{\tilde{d}}_{-2+1}}{(x-1)^2} + \frac{\tilde{\tilde{d}}_{-1+1}}{x-1},\end{aligned}$$

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with

$$\begin{aligned}
\tilde{\tilde{g}}_{-2+1} &= \tilde{g}_0 = -2\sqrt{-D_0},\\
\tilde{\tilde{g}}_{-1+1} &= 2 - \tilde{g}_{-10} = 0,\\
\tilde{\tilde{g}}_{-10} &= \tilde{g}_{-10} = 2,\\
\tilde{\tilde{d}}_{-3+1} &= -\tilde{d}_{-10} = 2\sqrt{-D_0} - 2,\\
\tilde{\tilde{d}}_{-2+1} &= \tilde{d}_{-10} = -2\sqrt{-D_0} + 2,\\
\tilde{\tilde{d}}_{-1+1} &= -\tilde{d}_{-10} = 2\sqrt{-D_0} - 2,\\
\tilde{\tilde{d}}_{-10} &= \tilde{d}_{-10} = -2\sqrt{-D_0} + 2.
\end{aligned}$$

The second part of the Jaffé ansatz (Jaffé ansatz II; cf. (1.2.60), (1.2.61)) is

$$w[x(z)] = z^{\alpha_{0\infty 2} - \alpha_{01}}\, v(x) = z^{\alpha_{0\infty 2}}\, v(x) \tag{1.2.118}$$

and results in a differential equation (1.2.107), the coefficients of which are given by

$$\begin{aligned}
\bar{P}(x) &= \frac{\bar{g}_{-2+1}}{(x-1)^2} + \frac{\bar{g}_{-1+1}}{x-1} + \frac{\bar{g}_{-10}}{x},\\
\bar{Q}(x) &= \frac{\bar{d}_{-3+1}}{(x-1)^3} + \frac{\bar{d}_{-2+1}}{(x-1)^2} + \frac{\bar{d}_{-1-1}}{x+1} + \frac{\bar{d}_{-10}}{x},
\end{aligned}$$

with

$$\begin{aligned}
\bar{g}_{-2+1} &= \tilde{g}_0 = -2\sqrt{-D_0},\\
\bar{g}_{-1+1} &= 2 - \tilde{g}_{-10} = 0,\\
\bar{g}_{-10} &= \tilde{g}_{-10} = 2,\\
\bar{d}_{-3+1} &= \tilde{g}_0\,\beta - \tilde{d}_{-10} = 0,\\
\bar{d}_{-2+1} &= \tilde{d}_{-10} + (2 - \tilde{g}_{-10})\,\beta + \beta\,(\beta - 1) = 2 - 2\sqrt{-D_0} - \frac{1}{\sqrt{-D_0}} + \frac{1}{\sqrt{-D_0}^{\,2}},\\
\bar{d}_{-1+1} &= \tilde{g}_{-10}\,\beta - \tilde{d}_{-10} = 2\sqrt{-D_0} - \frac{2}{\sqrt{-D_0}},\\
\bar{d}_{-10} &= -(\tilde{g}_{-10}\,\beta - \tilde{d}_{-10}) = -\bar{d}_{-1+1} = \frac{2}{\sqrt{-D_0}} - 2\sqrt{-D_0}.
\end{aligned}$$

A Taylor expansion

$$v(x) = \sum_{n=0}^{\infty} a_n\, x^n \tag{1.2.119}$$

of the particular solution (that is convergent in the unit disc of the complex number plane) eventually results in the linear second-order difference equation for the coefficients a_n:

$$\begin{aligned}
&(n+1)\,[n + \Gamma_0]\; a_{n+1} + [-2\,(n-1)\,n + n\,\Gamma_1 + \Delta_0]\; a_n\\
&\qquad + [(n-2)\,(n-1) + (n-1)\,\Gamma_2 + \Delta_1]\; a_{n-1} = 0, \quad n = 1,2,3,4,\ldots,
\end{aligned} \tag{1.2.120}$$

and an initial condition

$$\Gamma_0\, a_1 + \Delta_0\, a_0 = 0,$$

the coefficients of which are given by

$$\begin{aligned}
\Gamma_2 &= \bar{g}_{-1+1} + \bar{g}_{-10} = 2,\\
\Gamma_1 &= -(\bar{g}_{-2+1} + \bar{g}_{-1+1} + 2\,\bar{g}_{-10}) = 2\sqrt{-D_0} - 4,\\
\Gamma_0 &= \bar{g}_{-10} = 2,\\
\Delta_1 &= \bar{d}_{-2+1} + \bar{d}_{-1+1} = 2 + \frac{1}{\sqrt{-D_0}^2} - \frac{3}{\sqrt{-D_0}},\\
\Delta_0 &= -\bar{d}_{-1+1} = -2\sqrt{-D_0} + \frac{2}{\sqrt{-D_0}}.
\end{aligned} \tag{1.2.121}$$

The difference equation (1.2.120) may be put into a three-term recurrence relation

$$\begin{aligned}
& a_0 \text{ arbitrary},\\
[3pt]&\Gamma_0\, a_1 + \Delta_0\, a_0 = 0,\\
&\left(1 + \frac{\alpha_1}{n} + \frac{\beta_1}{n^2}\right) a_{n+1} + \left(-2 + \frac{\alpha_0}{n} + \frac{\beta_0}{n^2}\right) a_n\\
&\quad + \left(1 + \frac{\alpha_{-1}}{n} + \frac{\beta_{-1}}{n^2}\right) a_{n-1} = 0, \quad n \geq 1,
\end{aligned} \tag{1.2.122}$$

where the coefficients of the difference equation are

$$\begin{aligned}
\alpha_1 &= \Gamma_0 + 1, \quad \beta_1 = \Gamma_0,\\
\alpha_0 &= \Gamma_1 + 2, \quad \beta_0 = \Delta_0,\\
\alpha_{-1} &= \Gamma_2 - 3, \quad \beta_{-1} = \Delta_1 - \Gamma_2 + 2,
\end{aligned} \tag{1.2.123}$$

which finalises the Jaffé method.

As can be seen from the Jaffé method, the classical approach, presented above, is only possible because $\alpha_{0\infty 1}$ in (1.2.113) is an integer multiple of α_0 [with $D_0 = D_{0n}$ from (1.2.111)] in (1.2.114): $\alpha_{\infty 1} = k\,\alpha_0$ with $k \in \mathbb{Z}$. The Jaffé method, and this is its strength, can be applied even if this condition is no longer fulfilled.

Crucial for the applicability of the classical method is that $\alpha_{0\infty 2}$ in (1.2.116) [for the eigenvalues (1.2.111)] is integer. This is used implicitly in the classical method. In the Jaffé method, however, this fact becomes obvious. Therefore, it can also be understood as a condition, an eigenvalue condition:

$$\alpha_{0\infty 2} = n, \quad n = 0, 1, 2, 3, \ldots.$$

This results in two conditions [cf. (1.2.115)]:

(1)

$$-1 + \frac{1}{\sqrt{-D_{0n}}} = n, \quad n = 0, 1, 2, 3, \ldots,$$

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resulting in

$$D_{0n} = -\frac{1}{(n+1)^2}, \quad n = 0, 1, 2, 3, \ldots.$$

(2)

$$-1 - \frac{1}{\sqrt{-D_{0n}}} = n, \quad n = 0, 1, 2, 3, \ldots,$$

resulting in

$$D_{0n} = -\frac{1}{(n+1)^2}, \quad n = 0, 1, 2, 3, \ldots.$$

The crucial thing about the difference equation (1.2.121) is that all of its partial solutions $\{a_n\}$ oscillate when $n \longrightarrow \infty$. This will be considered in more detail below. The Birkhoff set (cf. §1.2.4) of the difference equation (1.2.120) is given by

$$\begin{aligned} a_{n1}(n) &= \exp\left(\gamma_1 n^{\frac{1}{2}}\right) n^{r_1} \left[1 + \frac{C_{11}}{n^{\frac{1}{2}}} + \frac{C_{12}}{n^{\frac{2}{2}}} + \cdots\right], \\ a_{n2}(n) &= \exp\left(\gamma_2 n^{\frac{1}{2}}\right) n^{r_2} \left[1 + \frac{C_{21}}{n^{\frac{1}{2}}} + \frac{C_{22}}{n^{\frac{2}{2}}} + \cdots\right] \end{aligned} \tag{1.2.124}$$

as $n \longrightarrow \infty$, with

$$\begin{aligned} \gamma_1 &= 2\sqrt{-\sum_{i=-1}^{i=+1} \alpha_i} \neq 0, \\ \gamma_2 &= -\gamma_1, \\ r_1 &= r_2 = \frac{1}{4}\,(2\,\alpha_1 - 2\,\alpha_{-1} - 1) = \frac{1}{4}\,(2\,\Gamma_0 + 2 - 2\,\Gamma_2 - 6 - 1) = -\frac{5}{4}. \end{aligned} \tag{1.2.125}$$

The relationship with the coefficients of the differential equation was given in the central proof of §1.2.4:

$$\begin{aligned} \frac{\gamma_1^2}{4} &= -\sum_{i=-1}^{+1} \alpha_i = -\sum_{i=0}^{2} \Gamma_i \\ &= -\bar{g}_{-2+1} = -(G_0 + 2\,\alpha_{11}) \\ &= -\sqrt{G_0^2 - 4\,D_0} = -4\sqrt{-D_0}, \end{aligned}$$

$$\gamma_1 = 4\sqrt{-\sqrt{-D_0}} = 4\sqrt[4]{|D_0|}\,\imath = \frac{4}{\sqrt[4]{n+1}}\,\imath, \quad n = 0, 1, 2, 3, \ldots.$$

This is seen in Figure 1.6.

However, this means that with

$$\gamma_1 = -\gamma_2 = \gamma$$

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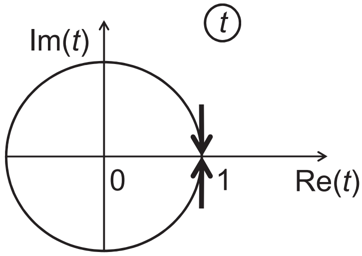

Figure 1.6 Solutions of the first-order characteristic equation.

and because of Euler's formula

$$\exp\left(\imath\,\gamma\,n^{\frac{1}{2}}\right) = \cos\left(\gamma\,n^{\frac{1}{2}}\right) + \imath\,\sin\left(\gamma\,n^{\frac{1}{2}}\right),$$
$$\exp\left(-\imath\,\gamma\,n^{\frac{1}{2}}\right) = \cos\left(\gamma\,n^{\frac{1}{2}}\right) - \imath\,\sin\left(\gamma\,n^{\frac{1}{2}}\right)$$

and thus

$$\begin{aligned}&\exp\left(\imath\,\gamma\,n^{\frac{1}{2}}\right) + \exp\left(-\imath\,\gamma\,n^{\frac{1}{2}}\right)\\ &= \cos\left(\gamma\,n^{\frac{1}{2}}\right) + \imath\,\sin\left(\gamma\,n^{\frac{1}{2}}\right) + \cos\left(\gamma\,n^{\frac{1}{2}}\right) - \imath\,\sin\left(\gamma\,n^{\frac{1}{2}}\right) = 2\,\cos\left(\gamma\,n^{\frac{1}{2}}\right),\end{aligned}$$

an oscillating behaviour.

Since the difference equation has only real-valued coefficients, it also has real-valued solutions. Therefore, the imaginary parts in the partial solutions of the difference equation must cancel each other out. This, in turn, means that for the general solution

$$a_n^{(g)} = L_1\,a_n^{(1)} + L_2\,a_n^{(2)},$$

the two constants L_1 and L_2 must be equal: $L_1 = L_2 = \frac{1}{2}\,L$, meaning

$$\begin{aligned}a_n^{(g)} &\sim L_1\,\left[\exp\left(\imath\,\gamma\,n^{\frac{1}{2}}\right) + \exp\left(-\imath\,\gamma\,n^{\frac{1}{2}}\right)\right]\,n^r\\ &= L_1\,\left[\cos\left(\gamma\,n^{\frac{1}{2}}\right) + \imath\,\sin\left(\gamma\,n^{\frac{1}{2}}\right) + \cos\left(\gamma\,n^{\frac{1}{2}}\right) - \imath\,\sin\left(\gamma\,n^{\frac{1}{2}}\right)\right]\,n^r\\ &= L\,\cos\left(\gamma\,n^{\frac{1}{2}}\right)\,n^r \text{as } n \to \infty\end{aligned}$$

and this is an oscillating behaviour. Because $r < 0$, this is asymptotically decreasing.

As a result, we keep in mind that in this example, in the series (1.2.119), the boundary eigenvalue condition is not hidden in the Taylor series of the solution set, but in the exponent $\alpha_{0\infty 2}$ in (1.2.115). Also, the series in (1.2.119) converges on the entire interval $0 \leq x \leq 1$ over which the singular boundary eigenvalue problem converges, because for $x = 1$ the following holds:

$$v(1) = \sum_{n=0}^{\infty} a_n \sim \sum_{n=0}^{\infty} \cos\left(\gamma\,n\right)\,n^r < \infty,$$

since $r < 0$, as it is an alternating series.

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1.3 Difference Equations of Poincaré–Perron Type

For the sake of simplicity, the most simple but still non-trivial difference equation is of second order and thus has the form

$$p_1(n)\,a_{n+1} + p_0(n)\,a_n + p_2(n)\,a_{n-1} = 0, \quad n = 1,2,3,\dots, \tag{1.3.1}$$

and the initial condition may have the form

$$p_1(0)\,a_1 + p_2(0)\,a_0 = 0. \tag{1.3.2}$$

If, in this case, a_0 is fixed,[6] then (1.3.2) yields the start of a recursive calculation of the a_n for as many terms as are needed by means of the three-term recurrence relation (1.3.1).

Since the treatment of linear difference equations by Henri Poincaré and Oskar Perron (Poincaré, 1885, 1886; Perron, 1909, 1910, 1911), it has been customary not to treat the solutions as sequences of numbers

$$a_0,\ a_1,\ a_2,\ a_3,\ \dots,\ a_n,\ \dots$$

but as the ratio of two consecutive terms

$$t_n = \frac{a_{n+1}}{a_n}. \tag{1.3.3}$$

For our purposes here, the asymptotic behaviour of this ratio for large values of the index n,

$$\lim_{n\to\infty} t_n,$$

is of particular interest, since this behaviour determines the solution of the differential equation in the neighbourhood of the singularity of the differential equation at z_0 [cf. the ansatz (1.2.119)], which generally represents the behaviour of the function at the still undetermined end of the relevant interval.

This asymptotic behaviour of two consecutive terms of the differential equation is determined by an algebraic equation whose order is as large as that of the underlying differential equation. This equation is called a *characteristic equation*; it is the most important criterion for its possible particular solutions.

If a linear, ordinary, homogeneous difference equation has only coefficients that converge to finite values for large indices n, then it is called a *difference equation of Poincaré–Perron type*. For the difference equation (1.3.1), this means

$$\lim_{n\to\infty} p_i(n) = \text{const.} < \infty,\ i = 0,1,2.$$

If the characteristic equation of a linear, ordinary, homogeneous difference equation has only simple roots, such a difference equation is called *regular*, otherwise *irregular*.

Linear ordinary difference equations of the first order

$$p_1(n)\,a_n + p_2(n)\,a_{n-1} = 0, \quad n = 1,2,3,\dots, \tag{1.3.4}$$

[6] This determination gives a normalisation of the function (1.2.119).

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are to be regarded as trivial in the sense of the above explanations. The simplest non-trivial example is the second-order difference equation in which a new term is the sum of the two foregoing ones:

$$a_{n+1} = a_n + a_{n-1}, \quad n = 1, 2, 3, \ldots, \tag{1.3.5}$$

which may be written in the form

$$a_{n+1} - a_n - a_{n-1} = 0, \quad n = 1, 2, 3, \ldots. \tag{1.3.6}$$

Here, all the coefficients $p_0(n)$, $p_1(n)$, $p_2(n)$ are independent of n, thus are constants: $p_0(n) = -p_1(n) = -p_2(n) = 1$. This linear, second-order difference equation is the most instructive one with respect to the principle that is to be shown in this book. Therefore, it is considered in some detail below.

1.3.1 Fibonacci Difference Equation

Fibonacci Sequence

It is easily seen that the initial condition of fixing two consecutive terms, a_0 and a_1, say, transforms the linear, homogeneous, second-order difference equation (1.3.5) or (1.3.6) into a linear, three-term recurrence relation, allowing the successive calculation of as many terms a_n as are needed.

Starting with $a_0 = 0$ and $a_1 = 1$ in (1.3.5) the recursively calculated sequence of numbers

$$a_0 = 0,\ a_1 = 1,\ a_2 = 1,\ a_3 = 2,\ a_4 = 3,\ a_5 =, 5,\ a_6 = 8,\ a_7 = 13,\ a_8 = 21,\ a_9 = 34, \ldots \tag{1.3.7}$$

is the well-known *Fibonacci sequence*. Therefore, I refer to (1.3.6) as the *Fibonacci difference equation*.

The characteristic equation of the difference equation (1.3.6) is given by

$$t^2 - t - 1 = 0, \tag{1.3.8}$$

the two solutions of which are

$$t_1 = \frac{1 + \sqrt{5}}{2} \approx 1.61803, \tag{1.3.9}$$

$$t_2 = \frac{1 - \sqrt{5}}{2} = -\frac{1}{t_1} = 1 - t_1 \approx -0.61803. \tag{1.3.10}$$

The fact that equation (1.3.6) is of second order generates two solutions t_1 and t_2 of its characteristic equation (1.3.8). This is of extraordinary importance, since it shows that there are two fundamental particular solutions of the difference equation (1.3.6) and the general solution $a_n^{(g)}$ of the linear difference equation (1.3.6) has – because of its linearity – the form

$$a_n^{(g)} = C_1\, a_n^{(1)} + C_2\, a_n^{(2)}, \tag{1.3.11}$$

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where $a_n^{(1)}$ and $a_n^{(2)}$ are two linearly independent particular solutions of the difference equation (1.3.6), meaning that one solution is not a multiple of the other. Moreover, C_1 and C_2 do not depend on n, $a_n^{(1)}$ is related to (1.3.9) and $a_n^{(2)}$ is related to (1.3.10), meaning that the ratio of two subsequent terms of the particular solution $a_n^{(1)}$ of (1.3.6) tends to t_1 as $n \longrightarrow \infty$ and that the ratio of two subequent terms of $a_n^{(2)}$ tends to t_2 as $n \longrightarrow \infty$.

It has been known for more than 300 years that

$$a_n^{(1)} = t_1^n \tag{1.3.12}$$

and

$$a_n^{(2)} = t_2^n . \tag{1.3.13}$$

This fact is expressed in the famous *de Moivre–Binet* formula, which, moreover, gives the constants C_1 and C_2 of (1.3.11):

$$a_n = \frac{1}{\sqrt{5}}\, t_1^n - \frac{1}{\sqrt{5}}\, t_2^n . \tag{1.3.14}$$

The French mathematician **Jacques Philippe Marie Binet** (1786–1856) published this formula in 1843, although it was already well known to the French mathematician **Abraham de Moivre** (1667–1754) and to the Swiss mathematicians **Daniel Bernoulli** (1700–1782) and **Leonhard Euler** (1707–1783). The de Moivre–Binet formula gives the exact term a_n in dependence on n for the Fibonacci sequence, i.e., for the initial condition $a_0 = 0$ and $a_1 = 1$. The formula is remarkable since it displays each term of an infinite sequence of integer numbers by a sum of two irrational ones.

It is clearly seen from the de Moivre–Binet formula (1.3.14) that the Fibonacci sequence (1.3.7) consists of two series, one of them increasing exponentially as $n \longrightarrow \infty$, the other decreasing exponentially in this limit since

$$|t_1| > 1 \text{ and } |t_2| < 1 :$$

$$\begin{aligned} a_n &= \frac{1}{\sqrt{5}} \left[\exp\left(n \,\ln t_1\right) - (-1)^n \,\exp\left(-n \,\ln t_1\right)\right] \\ &= \begin{cases} \frac{2}{\sqrt{5}} \,\sinh\left(n \,\ln t_1\right) \text{ for } n \text{ even } (n = 0, 2, 4, 6, \ldots), \\ \\ \frac{2}{\sqrt{5}} \,\cosh\left(n \,\ln t_1\right) \text{ for } n \text{ odd } (n = 1, 3, 5, 7, \ldots). \end{cases} \end{aligned}$$

This composition of the Fibonacci sequence is shown in Figure 1.7.

Thus

$$C_1 = \frac{1}{\sqrt{5}}, \quad a_n^{(1)} = \exp\left(n \,\ln t_1\right), \tag{1.3.15}$$

$$C_2 = -\frac{(-1)^n}{\sqrt{5}}, \quad a_n^{(2)} = \exp\left(-n \,\ln t_1\right). \tag{1.3.16}$$

It is quite understandable that the partial solution $a_n^{(2)}$ in (1.3.16) is often not recognised since the terms become exponentially small on increasing index n:

$$a_n^{(2)} = \exp\left(-n \,\ln t_1\right). \tag{1.3.17}$$

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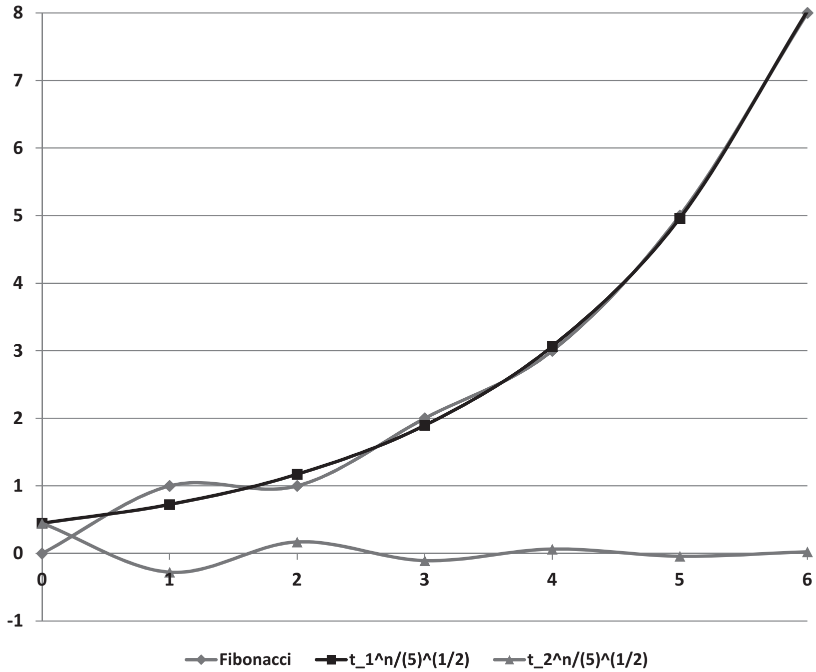

Figure 1.7 Composition of the Fibonacci sequence.

This is called the *recessive particular solution* of the difference equation (1.3.6), while the exponentially increasing particular solution

$$a_n^{(1)} = \exp(n \ln t_1)$$

is called the *dominant particular solution* of the difference equation (1.3.6). So, one can say that the recessive solution is masked by the dominant one. The masking of recessive particular solutions by dominant ones is a quite deep-lying phenomenon of linear equations with order higher than one.

The constants C_1 and C_2 of the general solution (1.3.11) of the difference equation (1.3.6) do not depend on n. These two quantities are exclusively dependent on the initial condition $\{a_0, a_1\}$ in a one-to-one relation, meaning that for each pair $\{a_0, a_1\}$ of initial conditions there is one and just one pair of constants $\{C_1, C_2\}$ and vice versa.

As the formula of de Moivre and Binet (1.3.14) shows, for $a_0 = 0$, $a_1 = 1$ the two constants C_1 and C_2 are equal to one another, having value $1/\sqrt{5}$:

$$a_0 = 0,\ a_1 = 1 \rightsquigarrow C_1 = -C_2 = \frac{1}{\sqrt{5}} \approx 0.44721.$$

There remain two interesting questions on this problem that are to be solved in the following:

(1) For which pair $\{a_0, a_1\}$ of initial conditions does it hold that $C_1 = 0$, $C_2 \neq 0$?
(2) For which pair $\{a_0, a_1\}$ of initial conditions does it hold that $C_1 \neq 0$, $C_2 = 0$?

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Case (1) is the only situation with respect to the initial condition where the recessive solution appears numerically, since the dominant one vanishes. Thus, one may say that in this, and only this, case, the recessive particular solution of the difference equation (1.3.6) is unmasked. Because of this being a deep-lying structure of linear equations, I call it the *mathematical principle of unmasking recessive solutions*.

In case (2), the solution of the difference equation (1.3.6) is a pure exponentially increasing series as $n \to \infty$. It is the fastest growing sequence of numbers as a solution of the difference equation (1.3.6).

Questions (1) and (2), posed above, were answered within a general theory only in the twentieth century when the Danish mathematician and astronomer Niels Erik Nörlund solved the problem in his famous book (Nörlund, 1924) on difference equations. However, as is shown in the following section, the questions in the case of (1.3.6) may be answered by elementary means.

Adjoint Fibonacci Sequence

In order to answer questions (1) and (2) posed at the end of the previous section, the most important equation is (1.3.14), since it displays the structure of the general solution of linear, homogeneous, second-order equations (1.3.6):

$$a_n^{(g)} = C_1\, t_1^n + C_2\, t_2^n,\ n = 0, 1, 2, 3, \ldots, \tag{1.3.18}$$

where t_1 and t_2 are the two solutions (1.3.9) and (1.3.10) of the characteristic equation (1.3.8) of the difference equation (1.3.6). Writing explicitly the equations for a_0 and a_1 yields

$$\begin{aligned} n = 0: &\quad a_0 = C_1 + C_2, \\ n = 1: &\quad a_1 = C_1\, t_1 + C_2\, t_2. \end{aligned}$$

Considering a_0 and a_1 as being given, this is a system of two linear equations of two unknown quantities C_1 and C_2 that may be solved by standard methods, yielding

$$\begin{aligned} C_1(a_0, a_1) &= -\frac{a_1 - a_0\, t_2}{t_2 - t_1} = -\frac{a_1}{t_2 - t_1} + \frac{a_0\, t_2}{t_2 - t_1}, \\ C_2(a_0, a_1) &= \frac{a_1 - a_0\, t_1}{t_2 - t_1} = \frac{a_1}{t_2 - t_1} - \frac{a_0\, t_1}{t_2 - t_1}, \end{aligned}$$

thus linear equations in a_0 and a_1. For the sake of obtaining a one-dimensional problem, we put $a_1 = 1$ in the following (without losing generality), resulting in the solution of the posed problem:

$$\begin{aligned} (1)\ C_1 &= 0 \overset{a_1=1}{\rightsquigarrow} a_0 = \frac{1}{t_2} = -t_1 \approx -1.61803 \\ &\rightsquigarrow C_2 = \frac{1}{t_2} = -t_1 \approx -1.61803, \end{aligned} \tag{1.3.19}$$

$$\begin{aligned} (2)\ C_2 &= 0 \overset{a_1=1}{\rightsquigarrow} a_0 = \frac{1}{t_1} = -t_2 \approx 0.61803 \\ &\rightsquigarrow C_1 = -\frac{1}{t_1} = t_2 \approx -0.61803. \end{aligned} \tag{1.3.20}$$

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In Figure 1.8 it is shown that the two constants C_1 and C_2 are dependent on a_0, while we take $a_1 = 1$. The most important pairs of values are given in the following table:

$a_0 = C_1 + C_2$	C_1	C_2
-2.00	$-\frac{1+2t_2}{t_2-t_1} \approx -0.1056$	$\frac{1+2t_1}{t_2-t_1} \approx -1.8944$
$-t_1 \approx -1.61803$	$0,00$	$-t_1$
-1.00	$-\frac{1+t_2}{t_2-t_1} \approx 0.1708$	$\frac{1+t_1}{t_2-t_1} \approx -1.1708$
$t_2 \approx -0.61803$	$-\frac{1-t_2 t_1}{t_2-t_1} \approx 0.2764$	$\frac{1-t_2 t_1}{t_2-t_1} \approx -0.8944$
0.00	$\frac{1}{\sqrt{5}} \approx 0.4472$	$= \frac{1}{\sqrt{5}} \approx -0.4472$
$-t_2 \approx 0.61803$	$-t_2$	0.00
1.00	$-\frac{1-t_2}{t_2-t_1} \approx 0.7236$	$\frac{1-t_1}{t_2-t_1} \approx 0.2764$
$t_1 \approx 1.61803$	$-\frac{1-t_1 t_2}{t_2-t_1} \approx 0.8944$	$\frac{1-t_1^2}{t_2-t_1} \approx 0.7236$
2.00	1.00	1.00

Equation (1.3.19) answers the question as to the unmasking of the recessive solution (1.3.13) of (1.3.6). Taking the initial condition $a_0 = \frac{1}{t_2} \approx -1.61803$ and $a_1 = 1$ unmasks the recessive solution, the first few terms of which are given by

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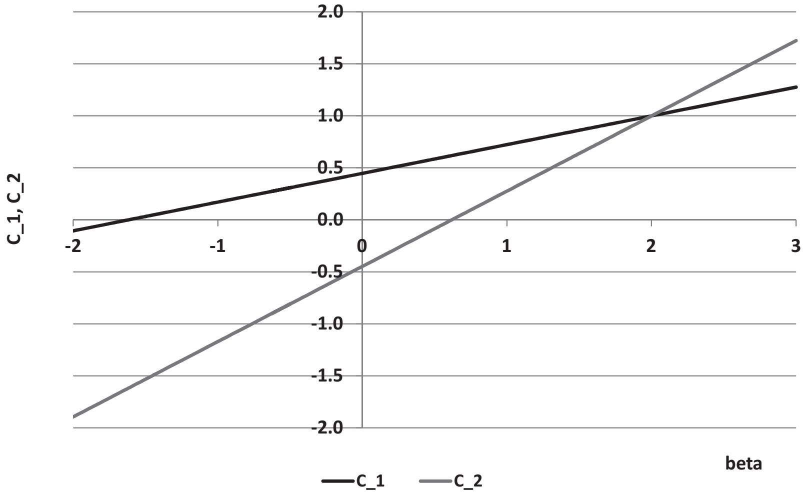

Figure 1.8 Coefficients of the general solution of the difference equation (1.3.6).

Number	Value
a_0	$\mathbf{-t_1 \approx -1.61803}$
a_1	$+\,\mathbf{1.00000}$
a_2	$t_2 \approx -0.61803$
a_3	$\approx +0.38197$
a_4	≈ -0.23607
a_5	$\approx +0.14590$
a_6	≈ -0.09017
a_7	$\approx +0.05573$
a_8	≈ -0.03444
a_9	$\approx +0.02129$
a_{10}	≈ -0.01316
a_{11}	$\approx +0.00813$
...	...

I refer to this sequence of numbers as the *adjoint Fibonacci sequence* for reasons which should be clear from the above execution. This behaviour is shown in Figure 1.9.

As can be seen, the adjoint Fibonacci sequence is an alternating one, as this is expected because of $t_2 < 0$. Its explicit calculation is given by

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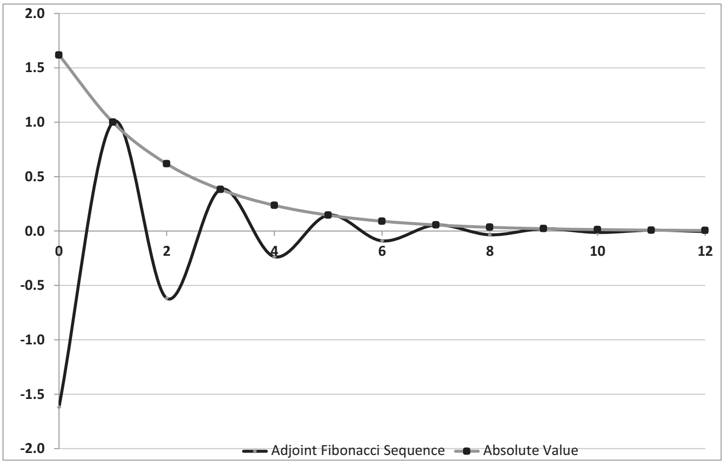

Figure 1.9 The adjoint Fibonacci sequence.

$$a_n^{(2)} = C_2\, t_2^n = -t_1\, t_2^n = t_2^{n-1},\ n = 0, 1, 2, 3, \ldots.$$

Equation (1.3.20) – in contrast to equation (1.3.19) – shows the purely exponentially increasing particular solution of the difference equation (1.3.6), where the exponentially decreasing particular solution is vanishing because $C_2 = 0$. This behaviour is shown in Figure 1.10. Its explicit calculation is given by

$$a_n^{(2)} = C_1\, t_1^n = -t_2\, t_1^n = t_1^{n-1},\ n = 0, 1, 2, 3, \ldots.$$

Final Remarks

It should be mentioned that the quite simple solution of problems (1) and (2) given above is possible only since the fundamental solutions (1.3.12) and (1.3.13) of the difference equation (1.3.1) are explicit, as shown by the formula of de Moivre and Binet. However, in the general case, i.e., if the linear second-order difference equation (1.3.1) has the form

$$a_{n+1} + p_n\, a_n + q_n\, a_{n-1} = 0, \quad n = 1, 2, 3, 4, \ldots, \tag{1.3.21}$$

there is an elaborate theory, based on the fact that this difference equation may be written formally as an infinite continued fraction; this is to be shown in the following. A linear second-order difference equation

$$a_{n+1} + p_n\, a_n + q_n\, a_{n-1} = 0, \quad n = 1, 2, 3, \ldots, \tag{1.3.22}$$

may be considered as a three-term recurrence relation if two subsequent numbers, a_0 and a_1, say, are fixed. This then determines a sequence a_n, $n = 1, 2, 3, \ldots$, of numbers. Dividing (1.3.22) by a_{n-1} yields

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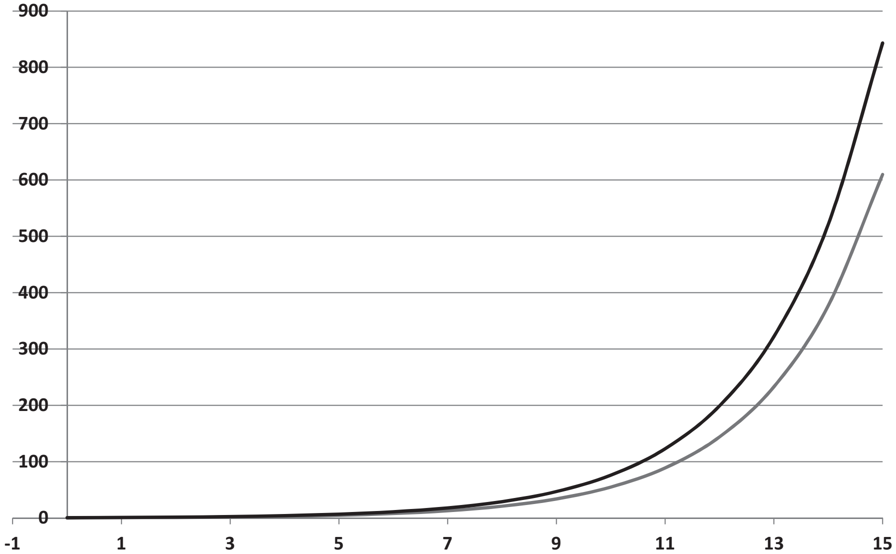

Figure 1.10 Purely exponentially increasing solution.

$$\frac{a_{n+1}}{a_{n-1}}\frac{a_n}{a_n}+p_n\frac{a_n}{a_{n-1}}+q_n=0$$

or

$$\frac{a_n}{a_{n-1}}\left[\frac{a_{n+1}}{a_n}+p_n\right]+q_n=0. \tag{1.3.23}$$

This is an equation for the ratio

$$\frac{a_n}{a_{n-1}}$$

of any two subsequent numbers a_{n+1} and a_n of this difference equation (1.3.21) that may be transformed into an infinite continued fraction. For this, equation (1.3.23) is written as

$$\begin{aligned}\frac{a_n}{a_{n-1}}&=-\frac{q_n}{p_n+\dfrac{a_{n+1}}{a_n}}\\&=-\cfrac{q_n}{p_n-\cfrac{q_{n+1}}{p_{n+1}-\cfrac{q_{n+2}}{p_{n+2}-\cfrac{q_{n+3}}{\cdots}}}}\end{aligned}$$

an infinite continued fraction. For $n=1$ this results in

$$\frac{a_1}{a_0}=-\cfrac{q_1}{p_1-\cfrac{q_2}{p_2-\cfrac{q_3}{p_3-\cfrac{q_4}{\cdots}}}} \tag{1.3.24}$$

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This theory was developed by Nörlund (1924); see in particular page 438.

The question of unmasking the recessive solutions is answered by the theory of Nörlund, such that equation (1.3.24) may be written as

$$\frac{a_1^{(2)}}{a_0^{(2)}} = -\cfrac{q_1}{p_1 - \cfrac{q_2}{p_2 - \cfrac{q_3}{p_3 - \cfrac{q_4}{\dots}}}}.$$

The first theory of transcendental boundary eigenvalue conditions of singular boundary eigenvalue problems of linear, second-order differential equations was built on the basis of this (Gelfand and Schilow, 1964; Gelfand and Wilenkin, 1964).

The theory of Nörlund, in a sense, completed the mathematical questions posed by Fibonacci's presentation of the sequence that nowadays bears his name.

I do not want to finish this section without having pointed to the relevance of these results to what is called the *golden section*, sometimes called the golden ratio. Although already known in antiquity, it became the most important ratio in Renaissance times. It is the dividing of a length into two unequal parts such that the ratio of the whole distance to the larger part x is the same as the ratio of the larger part x to the smaller one $1 - x$:

$$\frac{1}{x} = \frac{x}{1-x}.$$

As is easily seen, this equation is similar to the characteristic equation (1.3.8) of the difference equation (1.3.5). The Fibonacci sequence is a particular solution of

$$x^2 + x - 1 = 0,$$

given by

$$x_1 = -\frac{1+\sqrt{5}}{2} = -t_1 \approx -1.61803,$$
$$x_2 = -\frac{1-\sqrt{5}}{2} = -t_2 = \frac{1}{t_1} = 1 - t_1 \approx 0.61803.$$

The example of the Fibonacci difference equation and its solutions has been dealt with in such detail since it shows clearly the mathematical principle of unmasking the recessive solution that is at the centre of this book. In particular, it may be seen by elementary means how we can carry out this unmasking, technically. Since the recessive solutions are normally eigensolutions of singular boundary eigenvalue problems, this is the mathematical mechanism underlying the solution method that is presented in this book. In the general case, the problem is a technical one. How can we get the coefficient of the dominant particular solution of the difference or differential equation to vanish in dependence on the parameters involved.

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1.3.2 Regular Difference Equations

Consider the first-order linear difference equation

$$p_1(n+1)\,a_{n+1} - p_2(n)\,a_n = 0, \quad n = 0,1,2,3,\ldots. \tag{1.3.25}$$

If one determines the value of a_0 here, then a linear two-term recurrence relation results from (1.3.25):

$$a_{n+1} - t_n\,a_n = 0, \quad n = 0,1,2,3,\ldots, \tag{1.3.26}$$

with

$$t_n = \frac{p_2(n)}{p_1(n+1)}, \quad n = 0,1,2,3,\ldots, \tag{1.3.27}$$

with the help of which one can calculate as many values of a_n in a recursive way as needed. If one writes (1.3.25) in the form

$$a_{n+1} = t_n\,a_n, \quad n = 0,1,2,3,\ldots, \tag{1.3.28}$$

then it may be seen that the solution of (1.3.25) is given by

$$a_{n+1} = a_0\,\Pi_{i=1}^{n} t_i, \quad n = 0,1,2,3,\ldots.$$

If the function t_n is a constant in n, i.e., $t_n = t$, then the result is

$$a_{n+1} = a_0\,t^n = a_0\,\exp\left[n\,\ln(t)\right], \quad n = 0,1,2,3,\ldots. \tag{1.3.29}$$

This was the case for the Fibonacci difference equation (1.3.6). For $t > 1$ this means an exponential growth of the solution a_n as $n \to \infty$ and for $t < 1$ an exponential decay of the solution a_n towards zero as $n \to \infty$.

For $t = 1$ the solution is given by a constant:

$$a_n = a_0, \quad n = 1,2,3,\ldots. \tag{1.3.30}$$

It is, in a certain sense, the transition from exponentially increasing to exponentially decreasing behaviour.

If t_n is not constant but tends to unity, $t \to 1$, then the solution a_n of the difference equation (1.3.28) is given by a power behaviour. For

$$t_n = 1 + \frac{r}{n}, \quad n = 1,2,3,\ldots, \tag{1.3.31}$$

we have

$$a_n = a_0\,\Pi_{i=1}^{n}\left(1 + \frac{r}{n}\right) = a_0\,n^r, \quad n = 1,2,3,\ldots. \tag{1.3.32}$$

This formula is also valid if $r < 0$ holds.

The characteristic behaviour displayed above remains the same for difference equations of higher order. However, one must now take into account that the function spaces

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of the solutions basically have vector-space structure, i.e., the general solution of the difference equation of mth order is the linear hull of its particular solutions:

$$a_n^{(g)} = \sum_{i=1}^{m} L_i \, a_n^{(i)}. \tag{1.3.33}$$

This will be focussed on in the following.

It was important above that t_n tends to a finite, non-vanishing number as $n \to \infty$. Therefore, we start with a second-order linear difference equation having the form

$$p_0(n)\, a_{n+1} + p_1(n)\, a_n + p_2(n)\, a_{n-1} = 0, \quad n = 1, 2, 3, \ldots, \tag{1.3.34}$$

with

$$\begin{aligned} \lim\nolimits_{n\to\infty} p_0(n) &= p_0 = \text{const.},\\ \lim\nolimits_{n\to\infty} p_1(n) &= p_1 = \text{const.},\\ \lim\nolimits_{n\to\infty} p_2(n) &= p_2 = \text{const.} \end{aligned} \tag{1.3.35}$$

Dividing (1.3.34) by a_n, defining

$$t_n = \frac{a_{n+1}}{a_n}, \quad n = 0, 1, 2, 3, \ldots, \tag{1.3.36}$$

leads to

$$p_0(n)\, t_{n+1} + p_1(n) + \frac{p_2(n)}{t_{n-1}} = 0, \quad n = 1, 2, 3, \ldots. \tag{1.3.37}$$

Under the conditions (1.3.35), the limit $t = \lim_{n\to\infty} t_n$ exists, resulting in what is called the *characteristic equation*

$$p_0\, t^2 + p_1\, t + p_2 = 0, \quad n = 1, 2, 3, \ldots, \tag{1.3.38}$$

a second-order algebraic equation for t, indicating the possible asymptotic behaviours of the ratio of two consecutive term a_{n+1}, a_n of the particular solutions of the difference equation (1.3.34).

If the characteristic equation of a linear, ordinary, homogeneous difference equation has only simple roots, such a difference equation is called *regular*, otherwise it is called *irregular*.

1.3.3 Irregular Difference Equations

The characteristic equation of the Fibonacci difference equation (1.3.6) does have the two different values

$$t_1 = \frac{1 + \sqrt{5}}{2}$$

and

$$t_2 = \frac{1 - \sqrt{5}}{2}.$$

As can be seen, the quantity t on its own is decisive for the asymptotic behaviour of the partial solutions of the difference equation (1.3.6) as $n \to \infty$; and the behaviour

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for $n \to \infty$ is, after all, the natural behaviour of the difference equation, because it is used as a recursion and thus the index n grows in a natural way beyond all limits.

The characteristic of the particular solutions presented in §1.3.2 changes fundamentally, as soon as the characteristic equation of type (1.3.34) has multiple roots. This, of course, is possible only for difference equations whose order is larger than one. Let us therefore consider a second-order difference equation of the form

$$\left(1 + \frac{\alpha_1}{n}\right) a_{n+1} + \left(-2 + \frac{\alpha_0}{n}\right) a_n + \left(1 + \frac{\alpha_{-1}}{n}\right) a_{n-1} = 0, \quad n = 1, 2, 3, \dots .$$

Its characteristic equation

$$(t-1)^2 = 0 \tag{1.3.39}$$

has the twofold solution

$$t_1 = t_2 = 1. \tag{1.3.40}$$

The considerations in §1.3.2 can no longer help here. What matters, and this is decisive, is *how* the twofold solution t_1 in (1.3.40) of the characteristic equation (1.3.39) is approximated by the two particular solutions of the difference equation. Information about this is therefore not provided by the characteristic equation (1.3.39), but by a characteristic equation extended by a term:

$$\left(1 + \frac{\alpha_1}{n}\right) t^2(n) + \left(-2 + \frac{\alpha_0}{n}\right) t(n) + \left(1 + \frac{\alpha_{-1}}{n}\right) = 0. \tag{1.3.41}$$

Its asymptotic solutions are

$$t_1 = 1 + \sqrt{\frac{-\sum_{i=-1}^{+1} \alpha_i}{n}} + \mathcal{O}\left(\frac{1}{n}\right),$$
$$t_2 = 1 - \sqrt{\frac{-\sum_{i=-1}^{+1} \alpha_i}{n}} + \mathcal{O}\left(\frac{1}{n}\right)$$

as $n \to \infty$. This shows that the one particular solution approaches the unit circle in the complex t-plane from inside and the other from outside.

A linear difference equation, the coefficients of which tend to finite values as the index tends to infinity and the characteristic equation of which has exclusively simple finite roots, is called a *regular difference equation of Poincaré–Perron type*.

A linear difference equation, the coefficients of which tend to finite values as the index tends to infinity and the characteristic equation of which has multiple finite roots, is called an *irregular difference equation of Poincaré–Perron type*.

A second-order characteristic equation (1.3.41) of an irregular difference equation is named after the American mathematician **Clarence Raymond Adams** (1898–1965), who applied it for the first time in Adams (1928).

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1.4 Underlying Principle and Basic Method

The method for solving singular boundary eigenvalue problems of linear, ordinary, homogeneous differential equations of second order

$$P_0(z)\,\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + P_1(z)\,\frac{\mathrm{d}y}{\mathrm{d}z} + P_2(z)\,y = 0, \quad z \in \mathbb{C}, \tag{1.4.1}$$

$P_i(z)$, $i = 0, 1, 2$, being polynomials in z, presented in this book relies essentially on approaches of generalised power series of the form

$$y(z) = f(z) \sum_{n=0}^{\infty} a_n\,(z - z_0)^n, \tag{1.4.2}$$

where $f(z)$ – being called an *asymptotic factor* – is either a power or an exponential function, at least an explicit function. These series (1.4.2) converge on dotted circular domains of the complex number plane around the point of expansion z_0. If one enters the differential equations with such power series, then the linear, ordinary, homogeneous differential equation of second order (1.4.1) is transformed into linear, ordinary, homogeneous difference equations for the coefficients a_n. These always have a start-up calculation in the form of an initial condition which turns the difference equation into a linear recursion, sometimes called a recurrence relation, with the help of which one can recursively calculate as many coefficients a_n of the power series as needed.

The special thing about this transformation is that the difference equation resulting from the differential equation is again linear, ordinary and homogeneous, but it does not necessarily have to be of second order. The order of the difference equation is not determined by the order of the differential equation behind it, but by the number, position and nature of its singularities. Therefore, it is useful to mention some properties of linear, ordinary, homogeneous difference equations, whereby their order should not be limited from the outset.

1.4.1 The Mathematical Principle of Unmasking Recessive Solutions

Consider the Frobenius solutions (1.2.8)

$$y_{F1}(z) = z^{\alpha_{01}} \sum_{n=0}^{\infty} a_{n1}\, z^n,$$

$$y_{F2}(z) = z^{\alpha_{02}} \sum_{n=0}^{\infty} a_{n2}\, z^n$$

about the singularity $z = 0$ of the differential equation (1.2.2), (1.2.1) as well as the Thomé solutions (1.2.18)

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$$y_{T1}(z) = \exp(\alpha_{1\infty 1}\, z)\; z^{\alpha_{0\infty 1}} \left(1 + \sum_{n=1}^{\infty} C_{n1}\, z^{-n}\right) \text{ as } z \to \infty,$$

$$y_{T2}(z) = \exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{0\infty 2}} \left(1 + \sum_{n=1}^{\infty} C_{n2}\, z^{-n}\right) \text{ as } z \to \infty$$

at the irregular singularity at infinity. Both pairs of solutions may serve as a fundamental system in order to represent the general solution $y^{(g)}(z)$ of the differential equation (1.2.2), (1.2.1):

$$y^{(g)}(z) = c_1\, y_{F1}(z) + c_2\, y_{F2}(z)$$

and

$$y^{(g)}(z) = C_1\, y_{T1}(z) + C_2\, y_{T2}(z)$$

where c_1, c_2 as well as C_1, C_2 are arbitrary constants in z, viz. the totality of complex numbers. Thus, it is clear that the particular solution of the differential equation (1.2.2), (1.2.1) that is written by the Jaffé ansatz presented in §1.2.4 [cf. (1.2.56), (1.2.60), (1.2.61)]

$$y(z) = \exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{02}}\, (z+1)^{\alpha_{0\infty 2} - \alpha_{02}} \sum_{n=0}^{\infty} a_n \left(\frac{z}{z+1}\right)^n$$

is representable by means of either of these fundamental systems $y_{F1}(z)$, $y_{F2}(z)$ and $y_{T1}(z)$, $y_{T2}(z)$ by choosing specific, yet generally unknown, values $c_1 = c_{10}$, $c_2 = c_{20}$ or $C_1 = C_{10}$, $C_2 = C_{20}$:

$$y(z) = c_{10}\, y_{F1}(z) + c_{20}\, y_{F2}(z)$$

and

$$y(z) = C_{10}\, y_{T1}(z) + C_{20}\, y_{T2}(z).$$

Thus, it is always possible to write

$$\begin{aligned} y(z) &= \exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{02}}\, (z+1)^{\alpha_{0\infty 2} - \alpha_{02}} \sum_{n=0}^{\infty} a_n\, x^n \\ &= C_{10}\, \exp(\alpha_{1\infty 1}\, z)\; z^{\alpha_{01}} \left(1 + \sum_{n=0}^{\infty} C_{n1}\, z^{-n}\right) \\ &\quad + C_{20}\, \exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{02}} \left(1 + \sum_{n=0}^{\infty} C_{n2}\, z^{-n}\right) \end{aligned}$$

as $z \to \infty$, whereby the in general unknown constants C_{10}, C_{20} are dependent on the parameters of the differential equation (1.2.2), (1.2.1):

$$C_{10} = C_{10}(\underline{c}),\; C_{20} = C_{20}(\underline{c})$$

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whereby $\underline{c}$ is the totality of parameters of the differential equation (1.2.2), (1.2.1). If the parameters of the differential equation $\underline{c}$ are arbitrary, then in general

$$C_{10} = C_{10}(\underline{c}) \neq 0.$$

Thus, because $\alpha_{1\infty 1} > 0$ and $\alpha_{1\infty 2} < 0$:

$$\exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{02}}\; (z+1)^{\alpha_{0\infty 2}-\alpha_{02}} \sum_{n=0}^{\infty} a_n\, x(z)^n$$
$$\sim C_1\; \exp(\alpha_{1\infty 1}\, z)\; z^{\alpha_{01}} \left(1 + \sum_{n=0}^{\infty} C_{n1}\, z^{-n}\right)$$

as $z \longrightarrow \infty$, from which results

$$\sum_{n=0}^{\infty} a_n\, x^n \sim C_1\; \exp\left[(\alpha_{1\infty 1} - \alpha_{1\infty 2})\; z\right]\; z^{\alpha_{01}-\alpha_{02}}\; (z+1)^{-\alpha_{02}}$$
$$\times \left(1 + \sum_{n=0}^{\infty} C_{n1}\, z^{-n}\right)$$
$$\sim \exp\left[(\alpha_{1\infty 1} - \alpha_{1\infty 2})\; z\right]$$

as $z \longrightarrow \infty$.

If the parameters of the differential equation $\underline{c}$ take on eigenvalues, then

$$C_{10} = C_{10}(\underline{c}) = 0.$$

Thus

$$\exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{02}}\; (z+1)^{\alpha_{0\infty 2}-\alpha_{02}} \sum_{n=0}^{\infty} a_n\, x^n$$
$$\sim C_2\; \exp(\alpha_{1\infty 2}\, z)\; z^{\alpha_{0\infty 2}} \left(1 + \sum_{n=0}^{\infty} C_{n2}\, z^{-n}\right)$$

as $z \longrightarrow \infty$, from which follows

$$\sum_{n=0}^{\infty} a_n\, x^n \sim C_2 \left(1 + \sum_{n=0}^{\infty} C_{n2}\, z^{-n}\right) \sim \text{const.}$$

as $z \longrightarrow \infty$, whereby

$$(z+1)^{-\alpha_{0\infty 2}} = z^{-\alpha_{0\infty 2}} \left(1 + \frac{1}{z}\right)^{-\alpha_{0\infty 2}}$$

as $z \longrightarrow \infty$.

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The asymptotic considerations above [cf. (1.2.79)] tell us that there are two different asymptotic behaviours of the coefficients a_n as $n \longrightarrow \infty$:

$$a_{n1} = \exp\left(\gamma_1 \sqrt{n}\right) n^r \left(1 + \sum_{i=1}^{\infty} C_{1i}\, n^{-i/2}\right),$$
$$a_{n2} = \exp\left(\gamma_2 \sqrt{n}\right) n^r \left(1 + \sum_{i=1}^{\infty} C_{2i}\, n^{-i/2}\right).$$

Since the difference equation (1.2.69) is linear, the general solution of which is a two-dimensional vector space given by

$$a_n^{(g)} = L_1\, a_{n1} + L_2\, a_{n2} \tag{1.4.3}$$

where L_1 and L_2 in general is the totality of complex constants in n, but dependent on the parameters $\underline{c}$ of the difference equation (1.2.69) and thus of the differential equation (1.2.2), (1.2.1):

$$L_1 = L_1(\underline{c});\ L_2 = L_2(\underline{c}).$$

Thus, if

$$C_1(\underline{c}) \neq 0 \text{ then } L_1(\underline{c}) \neq 0 \text{ and } a_n = a_{n,1},$$

and if

$$C_2(\underline{c}) = 0 \text{ then } L_2(\underline{c}) = 0 \text{ and } a_n = a_{n,2}. \tag{1.4.4}$$

Hereby, the latter behaviour (1.4.4) solves the singular boundary eigenvalue problem of the single confluent case of the Gauss differential equation (1.2.2), (1.2.1), since the term on the right-hand side of (1.4.3) is dominated by the exponentially decreasing asymptotic behaviour of the Jaffé ansatz (1.2.56), (1.2.60), (1.2.61) as $z \longrightarrow \infty$. Therefore, the eigenvalue condition of the singular boundary eigenvalue problem

$$L_1(\underline{c}) = 0$$

in (1.4.3), which determines the eigenvalues.

The mechanism that makes the eigensolutions of the singular boundary eigenvalue problem – generally asymptotically dominated, viz. recessive – appear numerically as soon as the eigenvalue parameter adopts an eigenvalue, I call the mathematical principle of unmasking recessive solutions. It seems to be the crucial and rather general mechanism in solving boundary eigenvalue problems of linear differential equations.

1.4.2 The Method of Realising the Mathematical Principle

The mathematical principle of unmasking recessive solutions (of both difference and differential equations), as outlined in §1.4.1, is a very general one, which is, however, nothing more than a consequence of the structure of the function spaces that the particular solutions in question originate from: the linear structure of vector spaces.

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This fundamental structure of abstract spaces is one thing; something quite different is its realisation in the concrete case. This requires a mathematical method. This method is now almost one hundred years old and comes from the mathematical physicists of quantum mechanics: Wolfgang Pauli, Friedrich Hund and above all George Cecil Jaffé. As explained in §1.2, it consists of several steps, aimed at finding the particular solution that solves the singular boundary eigenvalue problem of the differential equation in question, into a problem for a difference equation, which is first solved and, above all, whose solution can be interpreted in terms of the original boundary eigenvalue problem. The goal is a computationally solvable boundary eigenvalue condition.

Starting from the formulation of the problem by a differential equation, this solution proposed by Jaffé consists of four steps:

- First of all, an approach is needed that includes the asymptotic factors of the local solutions when radially approximating the two relevant singularities of the differential equation in question.
- If the relevant interval is not the unit interval, a Moebius transformation is performed that maps the relevant interval of the singular boundary eigenvalue problem to the unit interval.
- The connection must be established by introducing an additional power term.
- The differential equation resulting from these three steps has a particular solution that may be written in the form of a power series whose radius of convergence is the unit circle of the complex number plane. Its coefficients obey a linear ordinary difference equation of Poincaré–Perron type, which can be solved recursively. With this recursive solution, the boundary eigenvalue problem is formally solved, exactly.

The interpretation of the particular solution of the difference equation then succeeds on the basis of this approach in three further steps:

- Within the framework of an index asymptotics, it can be shown which behaviour the partial solutions exhibit for the difference equation mentioned above with the coefficients of the power series.
- Within the framework of a variable asymptotics, the index asymptotics of the coefficients of the series is related to the asymptotic behaviour of the particular solutions of the difference equation that solves the singular boundary eigenvalue problem.
- With this, a boundary eigenvalue condition can finally be formulated.

In this way, a solution of the problem is achieved, which also includes an understanding of the numerical procedure. The special feature – and this is the novelty – is the general validity of the method. It can be applied in principle to all singular boundary eigenvalue problems where linear, ordinary, homogeneous differential equations (1.2.2), (1.2.1) have polynomial-like coefficients, with the order of the polynomials not restricted, principally.

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The analytical path from the differential equation to the difference equation is characterised by analytical and algebraic computations, which require a certain amount of calculatory effort. Already the new procedure for the classical case of the Coulomb potential of the Schrödinger equation requires – as explained – considerably more computational effort than the classical method. It is therefore advisable to use algebraic computer programs here. Otherwise, the generality of the method is limited by the manual computational effort, which can stand in the way here.

Higher special functions differ from the classical special functions in that the eigenvalue conditions are transcendental functions, whereas those of the classical special functions are of algebraic nature. It is immediately obvious that the variety of higher special functions is far greater than that of the classical ones. It is therefore a strategic advantage to specify a procedure for the extraction of the higher special functions and no longer to catalogue them explicitly, as was the case with the classical special functions. The main aim of this book is to indicate the procedure that will enable the reader to find solutions to his or her own problems. Thus, it is hoped that in this way new higher special functions will be found, which are hitherto unknown. Apart from algebraic and numerical computation possibilities, the procedure given in this book is a suitable means for this.

The principles and methods presented in this first chapter can be applied unchanged to more general differential equations. Only there is more computational effort necessary. This is done in Chapters 3 and 4 for the equations that are in a certain sense next to the Gaussian-type equations, namely, the differential equations of Heun type.

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