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CROWNS, FENCES, AND DISMANTLABLE LATTICES

DAVID KELLY AND IVAN RIVAL

A finite lattice L of order n is *dismantlable* [**6**] if there is a chain $L_1 \subset L_2 \subset \ldots \subset L_n = L$ of sublattices of L such that $|L_i| = i$ for every $i = 1, 2, \ldots, n$. In [**1**] it was shown that every finite planar lattice is dismantlable. Furthermore, every lattice L with $|L| \leqq 7$ is dismantlable [**6**]; in fact, every large enough lattice contains a dismantlable sublattice with precisely n elements [**4**]. As well, such lattices are closed under the formation of sublattices and homomorphic images [**6**]. In section 2, we shall extend the definition of *dismantlable* to infinite lattices.

For an integer $n \geqq 3$ a *crown* [**1**] is a partially ordered set

$$\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$$

in which $x_i \leqq y_i$, $y_i \geqq x_{i+1}$, for $i = 1, 2, \ldots, n - 1$, and $x_1 \leqq y_n$ are the only comparability relations (see Figure 1).

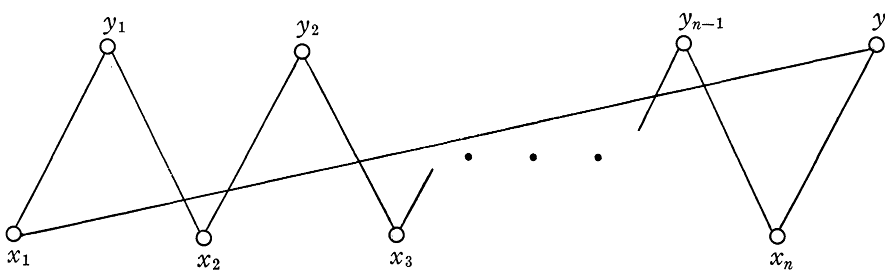

FIGURE 1. A crown of order $2n$

The main result of this paper is a characterization (Theorem 3.1) of dismantlable lattices in terms of crowns. In fact, we show that *every finite lattice is either dismantlable or it contains a crown but not both.* For more familiar classes of lattices we prove (Theorem 3.5) that *a modular lattice of finite length is dismantlable if and only if it has breadth* $\leqq 2$ (or, *equivalently, it contains no crown of order* 6). It now follows (Corollary 3.6) that *a finite distributive lattice is dismantlable if and only if it is planar.*

1. Preliminaries.

An element x in a lattice L is *join-reducible* (*meet-reducible*) *in* L if there exist $y, z \in L$ both distinct from x such that $x = y \vee z (x = y \wedge z)$; x is *join-irreducible* (*meet irreducible*) *in* L if it is not join-

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reducible (meet-reducible) in L; x is *doubly irreducible in* L if it is both join-irreducible and meet-irreducible in L. Let $J(L)$, $M(L)$, and $\mathrm{Irr}(L)$ denote the set of all join-irreducible elements in L, meet-irreducible elements in L, and doubly irreducible elements in L, respectively. The *length* of an n-element chain is $n - 1$ and the length of a partially ordered set P is the least upper bound of the lengths of the chains in P. For $x, y \in P$, x is *incomparable* with $y(x\|y)$ if $x \nleqq y$ and $y \nleqq x$. For all further terminology we refer to [**2**].

Let us observe that a partially ordered set which contains no infinite chains and in which all maximal chains have the same length is itself of finite length. Furthermore, since in a lattice L with no infinite chains, any join equals a finite join, L is complete and for every $x \in L$, $x = \bigvee (a \in J(L) | a \leqq x)$.

LEMMA 1.1. *If x and y are elements in a lattice L with no infinite chains and $x \nleqq y$, then there exists $a \in J(L)$ such that $a \leqq x$ but $a \nleqq y$.*

(Of course, the dual of the preceding lemma holds as well.)

A *fence* F [**1**] is a partially ordered set $\{x_1, x_2, \ldots, x_n, \ldots\}$ in which either

(1) $\quad x_1 \leqq x_2, x_2 \geqq x_3, \ldots, x_{2m-1} \leqq x_{2m}, x_{2m} \geqq x_{2m+1}, \ldots$

or

(1′) $\quad x_1 \geqq x_2, x_2 \leqq x_3, \ldots, x_{2m-1} \geqq x_{2m}, x_{2m} \leqq x_{2m+1}, \ldots$

are the only comparability relations (denoted by $F = (x_1, x_2, \ldots, x_n, \ldots)$). $F = (x_1, x_2, \ldots, x_n, \ldots)$ is a *lower fence* (see Figure 2) if (1) holds and an *upper fence* if (1′) holds. We shall also denote a crown on $\{x_1, y_1, x_2, y_2, \ldots, x_n, y_n\}$ by $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$.

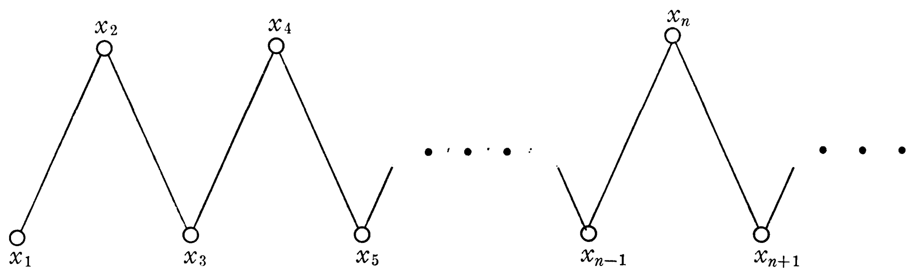

FIGURE 2. A lower fence $(x_1, x_2, \ldots, x_n, \ldots)$

The concept of a fence turns out to be the natural link between that of a crown and that of a dismantlable lattice. (Indeed, observe that the removal of one element, or two comparable elements from a crown leaves a fence.)

We now establish some elementary results about fences in partially ordered sets which contain no crowns.

LEMMA 1.2. *Let P be a partially ordered set containing no crowns and $F =$*

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$\{x_1, x_2, \ldots, x_n\}$, $(n \geqq 3)$, *be a subset of* P *satisfying* (1). *Then* F *is a fence in* P *for even (odd)* n *if and only if*

$$x_1 \ngeq x_3, x_{n-2} \ngeq x_n \ (x_{n-2} \nleq x_n),$$

$$x_i \nleq x_{i+3}, \text{for } i \text{ odd and } i \leqq n-3, \text{and}$$

$$x_i \ngeq x_{i+3}, \text{for } i \text{ even and } i \leqq n-3.$$

Proof. It is enough to consider the case in which n is even. Clearly $x_i \neq x_{i+1}$ for all $i < n$, (for example, if $x_1 = x_2$ then $x_1 \geqq x_3$). If F is not a fence in P, choose the least integer $k \geqq 2$ such that either $x_i \leqq x_{i+k}$, for some odd i, or $x_i \geqq x_{i+k}$, for some even i. By hypothesis, $k > 3$ and, in fact, k must be odd. But then $(x_i, x_{i+1}, \ldots, x_{i+k})$ is a crown, which is a contradiction.

The next two corollaries describe how fences can be constructed by "pasting" together smaller fences.

COROLLARY 1.3. *If* $n \geqq 3$ *and* $(x_1, x_2, \ldots, x_n)$ *is a lower fence in a partially ordered set* P *containing no crowns and* $y \in P$, *then* $(y, x_1, x_2, \ldots, x_n)$ *is a fence in* P *if and only if* $y \geqq x_1$, $y \nleq x_2$ *and* $y \geqq x_3$.

COROLLARY 1.4. *Let* $(x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j)$ *and* $(y_1, y_2, \ldots, y_j, z_1, z_2, \ldots, z_k)$ $(j \geqq 3)$ *be fences in a partially ordered set* P *containing no crowns. Then* $(x_1, x_2, \ldots, x_i, y_1, y_2, \ldots, y_j, z_1, z_2, \ldots, z_k)$ *is a fence in* P.

If $(x_1, x_2, \ldots, x_n)$ is a lower fence in a partially ordered set P and $x_2' \in P$, then $(x_1, x_2', x_3, \ldots, x_n)$ is a lower fence in P whenever $x_1 \leqq x_2'$, $x_3 \leqq x_2'$, and $x_2' \leqq x_2$.

LEMMA 1.5. *Let* $n \geqq 3$ *and* $(x_1, x_2, \ldots, x_n)$ *be a lower fence in a lattice* L *containing no crowns and* $y \in L$. *Then* $(y, x_1, x_1 \vee x_3, x_3, \ldots, x_n)$ *is a fence in* L *if and only if* $y \geqq x_1$ *and* $y \| x_1 \vee x_3$.

2. Dismantlable lattices. For finite lattices, the following result was proven in [**6**, Theorem 2].

PROPOSITION 2.1. *For any lattice* L *the following two conditions are equivalent:*

(i) *every nonempty sublattice* S *of* L *contains an element which is doubly irreducible in* S;

(ii) *there is an ordinal* α *and a family* $(L_\gamma | 0 \leqq \gamma \leqq \alpha)$ *of sublattices of* L *with* $L_0 = L$, $L_\alpha = \emptyset$, *and satisfying the following conditions:*

(a) *if* $\beta < \alpha$ *and* $L_\beta \neq \emptyset$, *there exists* $x \in L_\beta$ *such that* $L_{\beta+1} = L_\beta - \{x\}$; *if* $L_\beta = \emptyset$, *then* $L_{\beta+1} = \emptyset$; *and,*

(b) *for a limit ordinal* $\beta \leqq \alpha$, $L_\beta = \bigcap_{\gamma<\beta} L_\gamma$.

Proof. For (i) implies (ii), it is enough to define a family $(L_\gamma | 0 \leqq \gamma \leqq \alpha)$ by setting x in (ii)(a) to be a doubly irreducible element in L_β.

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Suppose now that (ii) holds. Let S be an arbitrary nonempty sublattice of L and let β be the least ordinal such that $S \nsubseteq L_\beta$. We choose some $x \in S - L_\beta$. If β were a limit ordinal, then $x \notin L_\gamma$ for some $\gamma < \beta$ which, however, would contradict the minimality of β; therefore, $\beta = \delta + 1$ for some δ. Since $S \subseteq L_\delta$ and x is doubly irreducible in L_δ, x is also doubly irreducible in S.

Since Proposition 2.1 is a natural extension of [**6**, Theorem 2] to arbitrary lattices, we define a lattice to be *dismantlable* whenever it satisfies either of the equivalent conditions of Proposition 2.1. Obviously every sublattice of a dismantlable lattice is dismantlable. Furthermore, it is possible to extend the proof in [**6**, Corollary 2] to show that any homomorphic image of an arbitrary dismantlable lattice is dismantlable.

We can now prove a strong version of one direction of our main result (Theorem 3.1).

PROPOSITION 2.2. *Every lattice which contains a crown is not dismantlable.*

Proof. Let L be a lattice containing a crown $(x_1, y_1, x_2, y_2, \ldots, x_n, y_n)$, where $n \geqq 3$. Then L also contains the crown $(x_1', y_1', x_2', y_2', \ldots, x_n', y_n')$ where $y_i' = x_i \vee x_{i+1}$ and $x_i' = y_{i-1}' \wedge y_i'$, for $1 \leqq i \leqq n$, (with $x_{n+1} := x_1$ and $y_0' := y_n'$). Then $y_i' = x_i' \vee x_{i+1}'$, for $1 \leqq i \leqq n$, (with $x_{n+1}' := x_1$), so that every element in the sublattice of L generated by $(x_1', y_1', x_2', y_2', \ldots, x_n', y_n')$is either join-reducible or meet-reducible.

Remark. The argument used in the proof of Proposition 2.2 also shows that a lattice which contains a fence (crown) also contains a fence (crown) of the same order in which for any pair of elements having an upper (lower) bound, the respective bound is actually the join (meet) of the pair in the lattice.

The direct product of a two-element chain and the integers $\mathbf{Z}$ is a (distributive) non-dismantlable lattice containing no crowns. Therefore, only lattices containing no infinite chains will be considered in the sequel.

3. The main result of this paper is

THEOREM 3.1. *A finite lattice is dismantlable if and only if it contains no crowns.*

Actually we shall prove a stronger result:

THEOREM 3.2. *A lattice which contains no infinite chains and no infinite fences is dismantlable if and only if it contains no crowns.*

The necessity is just a special case of Proposition 2.2. Since the property of non-containment of certain partially ordered sets in a lattice is inherited by its sublattices, the proof of Theorem 3.2 finally amounts to proving

THEOREM 3.3. *Every lattice which contains no infinite chains, no infinite fences, and no crowns must contain at least one doubly irreducible element.*

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The non-containment of infinite fences in Theorem 3.3 is essential. The lattice on $\{x_i | i \in \mathbf{Z}\} \cup \{0, 1\}$ with the partial ordering defined by $0 < x_i < 1$ ($i \in \mathbf{Z}$), $x_i < x_{i+1}$ (i even and $i \in \mathbf{Z}$), $x_i > x_{i+1}$ (i odd and $i \in \mathbf{Z}$), contains no crowns, yet it is not dismantlable. In fact, the lattice $L = F_1 \cup F_2 \cup \{c, d, 0, 1\}$, (see Figure 3) where $F_1 = (a_1, a_2, \ldots, a_n, \ldots)$ is an infinite upper fence, $F_2 = (b_1, b_2, \ldots, b_n, \ldots)$ is an infinite lower fence, $F_1 \cap F_2 = \emptyset$, $d > x$ for every $x \in F_1$, $d > a$, $c < x$ for every $x \in F_2$, $c < d$, and 0, 1 are the universal bounds is not dismantlable, yet it contains no crowns; note that $\{x_i | i \in \mathbf{Z}\}$ as ordered above is not isomorphic to any subset of L.

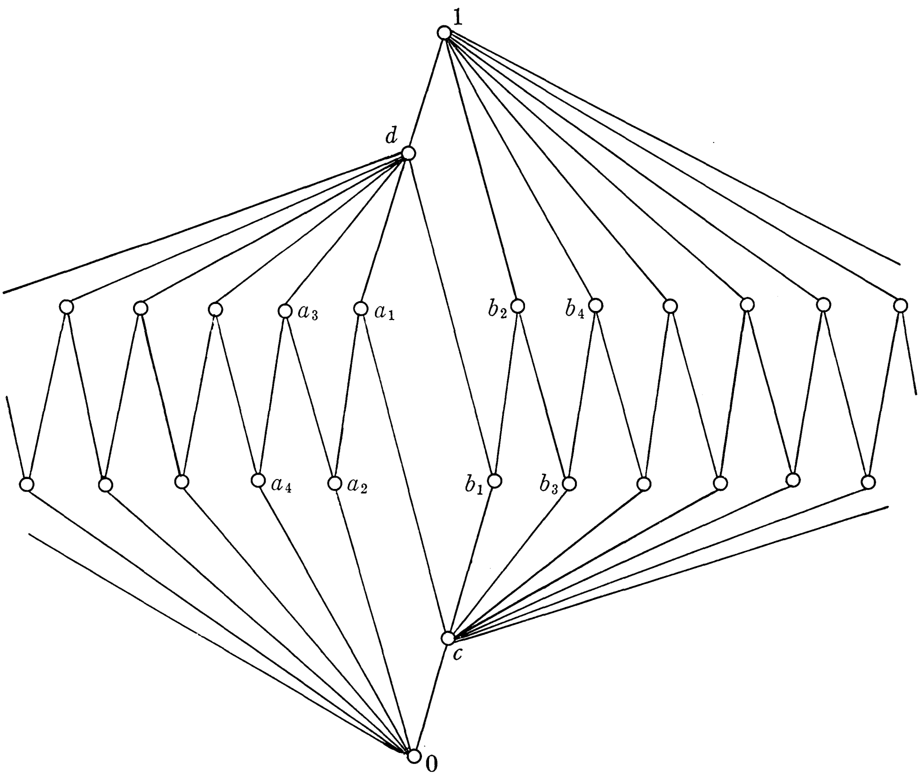

FIGURE 3. A non-dismantlable lattice with no crowns

Furthermore, the non-dismantlable lattice consisting of a crown of order $2n$, ($n \geqq 3$), with universal bounds 0 and 1 adjoined does not contain any crown of order $2m$, for $m \neq n$, so that all crowns must be omitted in Theorems 3.1 to 3.3.

Before proceeding to the proof in Section 4, we prove an analogue of Theorem 3.2 for modular lattices.

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The *breadth* of a lattice L ($b(L)$) is the least positive integer b such that any join

$$\bigvee_{i=1}^{n} x_i, \quad (n > b),$$

is always a join of a subset of b of the x_i's.

LEMMA 3.4. *For a lattice L the following conditions are equivalent:*
(i) $b(L) \leqq 2$;
(ii) *L contains no crown of order* 6;
(iii) *L contains no sublattice isomorphic to the Boolean lattice* $\mathbf{2}^3$.

Proof. To prove that (iii) implies (ii) we need only recall the remark following Proposition 2.2; the rest is routine.

THEOREM 3.5. *A modular lattice L of finite length is dismantlable if and only if $b(L) \leqq 2$.*

Proof. The "only if" part follows from Proposition 2.2 and Lemma 3.4.
Suppose that L is a modular lattice of finite length and L contains no crown of order 6. We show, in fact, that every maximal join-irreducible in L is already doubly irreducible in L. If this were not so then there would exist a maximal join-irreducible a with at least two distinct covers x_1, x_2. By the maximality of a, x_1 and x_2 must be join-reducible. Now let $y_1, y_2 \in L$ both distinct from a such that x_1 covers y_1 and x_2 covers y_2. By the lower semimodularity of L, a covers both $a \wedge y_1$ and $a \wedge y_2$, and if c is the unique element which a covers then we must have $y_1 \wedge y_2 = a \wedge y_1 = a \wedge y_2 = c$. Finally, by upper semimodularity $y_1 \vee y_2$ must cover both y_1 and y_2 so that $(y_1, x_1, a, x_2, y_2, y_1 \vee y_2)$ is a crown of order 6 in L which is impossible. Thus, $a \in \operatorname{Irr}(L)$.

The crown of order 8 with universal bounds 0 and 1 adjoined shows that modularity is essential in Theorem 3.5. Modular lattices of finite length with breadth $\leqq 2$ have been studied in [**5**] where they are called *quasiplanar*. Although not every such lattice is planar (see Figure 4), we have the following

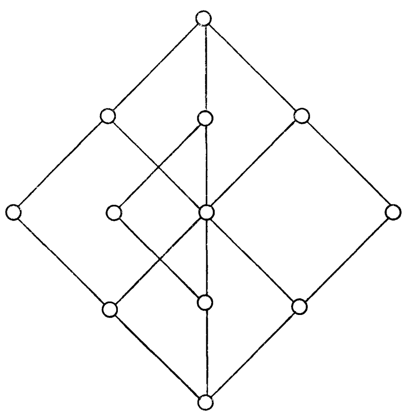

FIGURE 4. A modular non-planar dismantlable lattice

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COROLLARY 3.6. *A finite distributive lattice is dismantlable if and only if it is planar.*

Proof. It is well-known that in a finite distributive lattice L, $b(L)$ equals the largest integer n such that there exists $x \in L$ and x covers k elements. But then it follows from [**3**, Theorem 1.2] that L can be embedded into a direct product of $b(L)$ chains. (The special case of this latter statement for $b(L) \leqq 2$ is also proven in [**7**].)

In Section 5, we shall show that a simple application of the construction used in the proof of Theorem 3.3, yields a proof of

THEOREM 3.7. *A dismantlable lattice which is not a chain and which contains no infinite chains and no infinite fences, contains at least two incomparable doubly irreducible elements.*

For a finite dismantlable lattice L, Theorem 3.7 has a rather simple proof by induction on $|L|$.

Indeed, suppose $b \in \text{Irr}(L)$. (We may without loss of generality assume that $0 < b < 1$.) Then there exist unique elements $a, c \in L$ such that b covers a and c covers b. Without loss of generality the dismantlable sublattice $L' = L - \{b\}$ of L is not a chain so that by the inductive hypothesis it contains two incomparable elements x, y both doubly irreducible in L'. If neither x nor y is in $\{a, c\}$ then $x, y \in \text{Irr}(L)$ and we are done. Otherwise, $x \in \{a, c\}$ and $y \notin \{a, c\}$, say, so that $y \in \text{Irr}(L)$, and since $b \in \text{Irr}(L)$, $b \| y$.

The next result was established for finite lattices in [**6**, Theorem 2.].

COROLLARY 3.8. *A lattice L which contains no infinite chains and no infinite fences is dismantlable if and only if for every chain C in L, there is an ordinal α and a family $(L_\gamma | 0 \leqq \gamma \leqq \alpha)$ of sublattices of L with $L_0 = L$, $L_\alpha = C$, and satisfying the following conditions:*

(a) *if $\beta < \alpha$ and $L_\beta \supset C$, there exists $x \in L_\beta - C$ such that $L_{\beta+1} = L_\beta - \{x\}$; if $L_\beta = C$, then $L_{\beta+1} = C$; and,*

(b) *for a limit ordinal $\beta \leqq \alpha$, $L_\beta = \bigcap_{\gamma<\beta} L_\gamma$.*

Proof. The sufficiency is obvious. Let C be a chain in the dismantlable lattice L; we may assume that C is maximal. We show that for every sublattice S of L such that $S \supset C$ there is an element in $S - C$ which is doubly irreducible in S. But every such S is dismantlable and is not a chain. Thus, by Theorem 3.7, S contains two incomparable doubly irreducible elements so that one of them must be in $S - C$.

Let L be the lattice consisting of the infinite fence $(x_1, x_2, \ldots, x_n, \ldots)$ with universal bounds 0 and 1 adjoined. L is dismantlable, yet it contains only one doubly irreducible element. Thus, the hypothesis regarding infinite fences is necessary in Theorem 3.7. This hypothesis is also essential in Corollary 3.8 since for the chain $C = \{x_1, x_2\}$ there is no suitable family $(L_\gamma)_\gamma$ of sublattices that will do.

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4. Proof of Theorem 3.3. Throughout this section we shall assume that L is a lattice which contains no infinite chains and no infinite fences, that L contains no crowns, and that Irr $(L) = \emptyset$.

Let Q be a convex subset of L and $F = (y_1, y_2, \ldots, y_n)$, $n \geqq 3$, be a fence in Q. We shall call F *maximal* in Q whenever

(2) there is no fence $(x_1, x_2, y_2, u, y_4, \ldots, y_n)$ such that $u = y_2 \wedge y_4$ if F is lower, and, $u = y_2 \vee y_4$ if F is upper, and,

(2′) there is no fence $(y_1, y_2, \ldots, y_{n-3}, v, y_{n-1}, z_1, z_2)$ such that $v = y_{n-1} \vee y_{n-3}$ if F is lower (upper) and n is even (odd), and, $v = y_{n-1} \wedge y_{n-3}$ if F is upper (lower) and n is even (odd).

A fence $F = (y_1, y_2, \ldots, y_n)$, $n \geqq 3$, is *left-maximal* in Q if (2) holds and *right-maximal* if (2′) holds.

LEMMA 4.1. *Every convex subset Q of L which contains a fence of order 3 also contains a left-maximal and a right-maximal fence both of order $\geqq 3$. Indeed, if Q contains a fence of order 5 it contains a maximal fence of order $\geqq 5$.*

Proof. In view of Corollary 1.4 it suffices to show that Q contains a left-maximal fence of order $\geqq 3$ if it contains a fence of order 3. If F is a fence in Q of order 3 and Q contains no left-maximal fence then there exists a sequence $(F_m)_m$ of fences in Q such that $F_1 = F$ and, for $m > 1$, F_m is obtained from F_{m-1} as described in (2). But now if, for every $m \geqq 1$, y_m is the fourth entry in F_{m+1}, then $(y_1, y_2, \ldots, y_m, \ldots)$ would be an infinite fence in Q, which is impossible.

Continuing the proof of Theorem 3.3, let $Q_0 = L$. For $n \geqq 1$, we shall construct a sequence $((F_n, Q_n))_n$ of pairs such that, for $n \geqq 1$, Q_n is a nonempty convex subset of L, F_1 is a left-maximal fence in Q_0 and, for $n \geqq 2$, F_n is a maximal fence in Q_{n-1} (see Figure 5). For notational ease, we shall always label F_n so that $F_n = (e_n, f_n, g_n, h_n, \ldots)$. Furthermore, by virture of the remark following Proposition 2.2, we may assume that $g_n = f_n \vee h_n (g_n = f_n \wedge h_n)$ if F_n is upper (lower).

Once F_n has been chosen, Q_n will be defined by

(3) $Q_n = \{x \in L | x \geqq f_n \text{ and } x \| g_n\}$ if F_n is upper
$Q_n = \{x \in L | x \leqq f_n \text{ and } x \| g_n\}$ if F_n is lower.

It is obvious that Q_n is convex in L. We shall show by induction on $n \geqq 1$, that a maximal fence F_n in Q_{n-1} (left-maximal for $n = 1$) can be chosen so that

(i) *for every* $n \geqq 1$, $Q_n \subseteq Q_{n-1}$,

(ii) $|F_1| \geqq 3$ *and, for* $n > 1$, $|F_n| \geqq 5$, *and*

(iii) *if* F_n *is upper (lower),* $x \in Q_n$, $y \leqq x (y \geqq x)$, *and* $y \nleqq g_n (y \ngeqq g_n)$, *then* $y \in Q_n$.

For $n = 1$, property (i) is obvious. If b is join-reducible in L and $a \vee c = b$, $a \neq c$, then (a, b, c) is a fence of order 3 in $Q_0 = L$. Thus, by Lemma 4.1, Q_0

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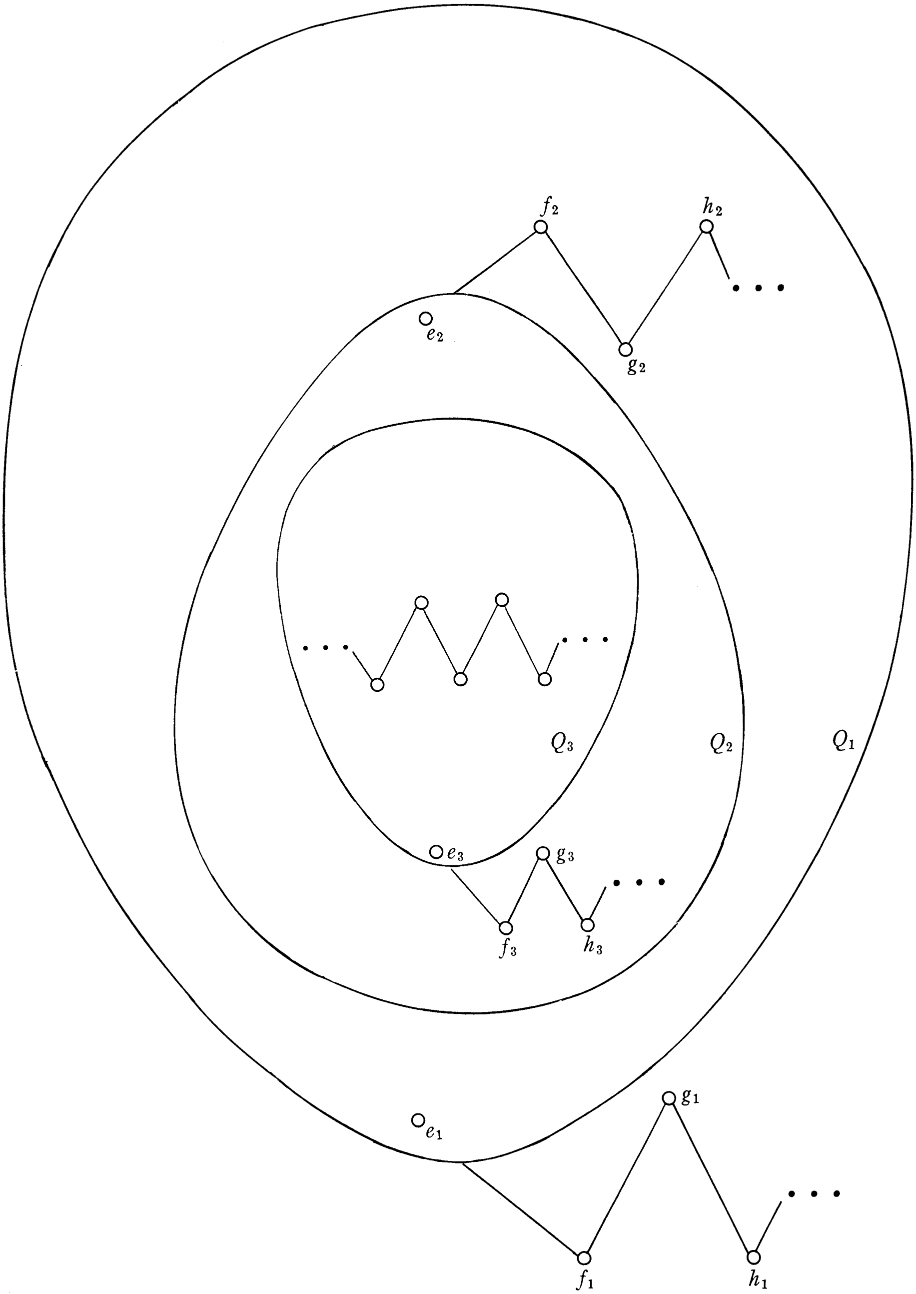

FIGURE 5. The construction scheme

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must contain a left-maximal fence $F_1 = (e_1, f_1, g_1, h_1, \ldots)$ of order $\geqq 3$. Property (iii) holds for $n = 1$, precisely because F_1 is left-maximal in $Q_0 = L$.

Let $n \geqq 2$. We assume that (F_i, Q_i) have been defined for $i = 1, 2, \ldots, n - 1$ and that (i), (ii), and (iii) hold whenever n is replaced by a smaller positive integer.

LEMMA 4.2. *Let $x \in Q_{n-1}$ and $y \notin Q_{n-1}$.*
(a) *If $y \geqq x$, then $y > g_{n-1}$.*
(b) *If $y \leqq x$, then $y < g_{n-1}$.*

Proof. (a) If F_{n-1} is upper, y must be comparable with g_{n-1} and, in fact, $y > g_{n-1}$. If F_{n-1} is lower, $y > g_{n-1}$ by (iii) for $n - 1$.

By Lemma 3.4, every join-reducible (meet-reducible) element x in L has a *join* (*meet*) *representation* $x = b_1 \vee b_2$ $(x = c_1 \wedge c_2)$, where $b_1, b_2 \in J(L)$ $(c_1, c_2 \in M(L))$. A join representation $x = b_1 \vee b_2$ of a join-reducible element x in L is *maximal* if whenever $x = c_1 \vee c_2$, $c_1, c_2 \in J(L)$, and $c_1 \geqq b_1$, $c_2 \geqq b_2$ then $c_1 = b_1$ and $c_2 = b_2$. (*Mininal* meet representations are defined dually.)

The next lemma together with Lemmas 4.1 and 4.2 shows that Q_{n-1} does contain a maximal fence satisfying (ii).

LEMMA 4.3. *Let Q be a nonempty subset of L and $g \in L - Q$ such that $x \| g$ for all $x \in Q$. Furthermore, suppose that whenever $x \in Q$ and $y \notin Q$ the following conditions are satisfied:*
(a) *if $y \geqq x$, then $y > g$;*
(b) *if $y \leqq x$, then $y < g$.*
Then Q contains a fence of order 5.
(For this lemma the only assumptions needed on L are that L contains no infinite chains, Irr $(L) = \emptyset$, and L contains no crown of order 6.)

Proof. It is easy to check that under the hypotheses of the lemma Q is convex. Furthermore, if $x \in Q$ but $x \notin M(L)$ then there exist $y, z \in M(L)$ such that $x = y \wedge z$. At least one of y, z lies in Q since otherwise $y, z > g$ so that $x = y \wedge z \geqq g$. Thus, $Q \cap M(L) \neq \emptyset$ and dually, $Q \cap J(L) \neq \emptyset$.

For $a, b, c \in L$, we write a/bc for the ordered set $\{a, b, c\}$ whenever $b, c \in J(L)$, $b \neq c$, $a \in M(L)$, and $b, c < a$. These "quotient pairs" are strictly partially ordered by $a_1/b_1c_1 < a_2/b_2c_2$ if and only if $b_1 < b_2, c_2$ or $c_1 < b_2, c_2$. Quotient pairs bc/a are defined dually. Only certain quotient pairs will be of interest here; in fact, let $\triangle$ be the set of all a/bc such that $a, b, c \in Q$, $a = b \vee c$, and if $b < x < a$ or if $c < x < a$ then x is both join- and meet-reducible; and dually, let ∇ be the set of all bc/a such that $a, b, c \in Q$, $a = b \wedge c$, and if $a < x < b$ or if $a < x < c$ then x is both join- and meet-reducible.

Let us assume that Q contains no fence of order 5.

First, we show that either $\triangle$ or ∇ is nonempty. Suppose that $\triangle = \emptyset$. Take b_1 a minimal meet-irreducible in Q and $b_2, b_3 \in J(L)$ a maximal join representation of b_1. If $b_2, b_3 \notin Q$ then $b_2, b_3 < g$ so that $b_1 = b_2 \vee b_3 \leqq g$,

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which is impossible. On the other hand, b_2 and b_3 cannot both be in Q since in that case $b_1/b_2b_3 \in \triangle$, contradicting our assumption. Thus, we may assume that $b_2 \in Q$ but that $b_3 \notin Q$, (i.e., $b_3 < g$).

Now, let us take $b_4, b_5 \in M(L)$ a minimal meet representation of b_2. If one of b_4, b_5 is not in Q, say b_4, then $b_4 > g > b_3$ so that $b_1 = b_2 \vee b_3 \leqq b_4$ which, by the minimality of b_4 would imply that $b_4 = b_1 \in Q$. Therefore, we may assume that $b_4, b_5 \in Q$. If $b_1 \wedge b_4, b_1 \wedge b_5 > b_2$ we get a fence$(b_4, b_1 \wedge b_4, b_1, b_1 \wedge b_5, b_5)$ of order 5 so that without loss of generality we may take $b_2 = b_1 \wedge b_4$.

Let $b_6, b_7 \in J(L)$ be a maximal join representation of b_4. If $b_6, b_7 \geqq b_2$, then $b_4/b_6b_7 \in \triangle$. Thus, we may suppose that $b_6 < b_4$ but that $b_6 \ngeqq b_2$. If $b_6 \leqq b_1$ then $b_6 \leqq b_1 \wedge b_4 = b_2$ which, by the maximality of b_6, gives that $b_6 = b_2$, contradicting $b_6 \ngeqq b_2$. Therefore, $b_6 \nleqq b_1$. If $b_6 \notin Q$ we get a crown $(g, b_6, b_4, b_2, b_1, b_3)$. Thus, $b_6 \in Q$.

Now take $b_8 \in M(L)$ minimal with respect to $b_8 > b_6$ and $b_8 \ngeqq b_2$. Suppose there exists such a b_8 with $b_8 \nleqq b_2 \vee b_6$. If $b_8 \notin Q$ we get a crown $(b_8, b_6, b_2 \vee b_6, b_2, b_1, b_3)$; thus, $b_8 \in Q$. In this case Q contains a fence $(b_8, b_6, b_2 \vee b_6, b_2, b_1)$ of order 5. Thus, we may assume that every $x \in M(L)$ such that $x > b_6$ but $x \ngeqq b_2$ also satisfies $x \leqq b_2 \vee b_6$.

Finally, we take $c_1, c_2 \in M(L)$ a minimal meet representation of b_6. Clearly, one of c_1, c_2 must satisfy $x \ngeqq b_2$, say c_1. Then $c_1 \leqq b_2 \vee b_6$. If $c_2 \nleqq b_2 \vee b_6$ then $c_2 \geqq b_2$; however, in this case, $c_2 \geqq b_2 \vee b_6 \geqq c_1$, which is impossible. Thus, $c_2 \leqq b_2 \vee b_6$ and $c_1c_2/b_6 \in \nabla$. We have then shown that either $\triangle \neq \emptyset$ or $\nabla \neq \emptyset$.

Suppose now that $\triangle \neq \emptyset$ and take c/ab a maximal element in $\triangle$ with respect to the strict partial ordering of $\triangle$. (Note that an infinite chain in $(\triangle, <)$ would give an infinite chain in Q.) Take $a_1, a_2, b_1, b_2 \in M(L)$ such that a_1, a_2 is a minimal meet representation of a and, b_1, b_2 is a minimal meet representation of b. If $a_1, a_2 \notin Q$ then $a = a_1 \wedge a_2 \geqq g$, which is impossible. Thus, we may assume that $a_1 \in Q$; similarly, we may assume that $b_1 \in Q$. If $a_1 \neq c \neq b_1$ we get a fence (a_1, a, c, b, b_1) in Q of order 5 (for example, if $a_1 \geqq b$, then $a_1 \geqq a \vee b = c$, contradicting the minimality of a_1). Without loss of generality we may assume then that $a_1 = c \neq b_1$. If $b_2 \notin Q$ then $a_2 \in Q$ (otherwise we get the crown (g, a_2, a, c, b, b_2)). But then (a_2, a, c, b, b_1) is a fence of order 5 in Q (note that $a_2 \neq b_1$ since otherwise $b_1 \geqq c$). Therefore, $b_2 \in Q$. Suppose that $b_2 \neq c$. If $b_2 \wedge c, b_1 \wedge c > b$ then $(b_2, b_2 \wedge c, c, b_2 \wedge c, b_2)$ is a fence in Q of order 5. Thus, we may assume that $b_1 \wedge c = b$, (which is a minimal meet representation of b). Furthermore, $a_2 \notin Q$ since otherwise (a_2, a, c, b, b_1) is a fence in Q of order 5.

Let $B = \{x \in J(L) | x < b_1 \text{ and } x \nleqq c\}$. If there exists $x \in B$ such that $x \ngeqq b$ then $x \| b$. Take a minimal such x. If $x \notin Q$ we get a crown (a_2, a, c, b, b_1, x); therefore, $x \in Q$, in which case we get a fence (a, c, b, b_1, x) in Q of order 5. Thus, every $x \in B$ satisfies $x \geqq b$. Let $b_3, b_4 \in J(L)$ be a maximal join representation of b_1. Obviously, one of b_3, b_4 lies in B, say b_3, so that $b_3 \geqq b$. If

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$b_4 \not\geqq b$ then $b_4 \leqq c$; thus, $b_4 \leqq c \wedge b_1 = b \leqq b_3$, which is impossible. Therefore, $b_4 \geqq b$ and, in fact, $b_1/b_3b_4 \in \Delta$, and $b_2/b_3b_4 > c/ab$, contradicting the maximality of c/ab. The case $\nabla \neq \emptyset$ is handled dually. The proof of the lemma is now complete.

Thus, we are assured of the existence of a maximal fence in Q_{n-1} of order $\geqq 5$. Let $G = (a_1, a_2, \ldots, a_m)$ be any such fence. Furthermore, we may assume that $a_3 = a_2 \vee a_4(a_2 \wedge a_4)$ if G is upper (lower) and that $a_{m-2} = a_{m-3} \vee a_{m-1}$ $(a_{m-3} \wedge a_{m-1})$ if G is upper and m is odd, or if G is lower and m is even (if G is upper and m is even, or if G is lower and m is odd). There is no loss in generality in assuming that F_{n-1} is upper. It remains now to choose $F_n = (e_n, f_n, g_n, h_n, \ldots)$ in Q_{n-1} and then Q_n as defined by (3) such that (F_n, Q_n) satisfies properties (i), (ii), and (iii). To this end we shall distinguish four cases, and in each case choose $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_1, a_2, \ldots, a_m)$ or $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_m, a_{m-1}, \ldots, a_1)$ so that properties (i) and (iii) hold. Either choice, of course, already satisfies (ii). Furthermore, for either choice, $Q_n \neq \emptyset$ since $a_1 \in Q_n$ or $a_m \in Q_n$.

Case (a). *G is upper and m is odd* (see Figure 6). Set

$$A_1 = \{x | x \geqq a_2 \text{ and } x \| a_3\},\ A_2 = \{x | x \geqq a_{m-1} \text{ and } x \| a_{m-2}\},$$
$$B_1 = \{y | y \not\leqq a_3 \text{ and } y \leqq x \text{ for some } x \in A_1\}, \text{ and}$$
$$B_2 = \{y | y \not\leqq a_{m-2} \text{ and } y \leqq x \text{ for some } x \in A_2\}.$$

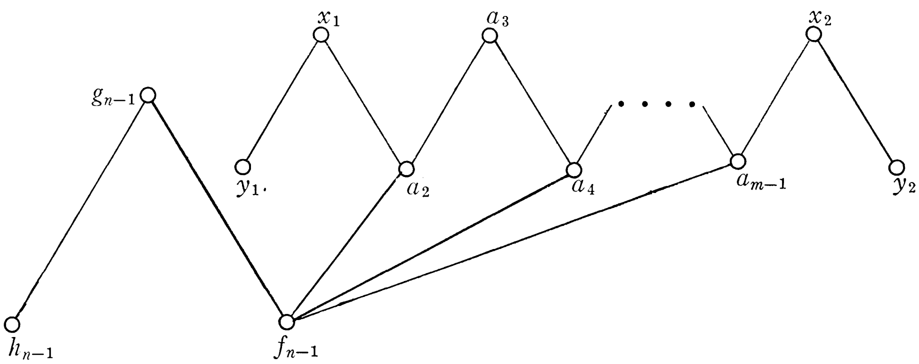

FIGURE 6. Theorem 3.6. Case (a)

Either every $x \in A_1$ satisfies $x \not\geqq g_{n-1}$ or, every $x \in A_2$ satisfies $x \not\geqq g_{n-1}$, (since otherwise, $(g_{n-1}, x_1, a_2, \ldots, a_{m-1}, x_2)$ would be a crown in L, where $x_i \in A_i$ and $x_i \geqq g_{n-1}$, for $i = 1, 2$). Suppose that every x in A_1 or A_2 satisfies $x \not\geqq g_{n-1}$; then, either every $y \in B_1$ satisfies $y \not\leqq g_{n-1}$ or, every $y \in B_2$ satisfies $y \not\leqq g_{n-1}$, (since otherwise $(y_1, x_1, a_2, \ldots, a_{m-1}, x_2, y_2, g_{n-1})$ would be a crown in L, where $y_i \in B_i$, $y_i \leqq x_i$ for some $x_i \in A_i$ and $y_i \leqq g_{n-1}$, for $i = 1, 2$). Without loss of generality we may assume that every y in B_1 satisfies $y \not\leqq g_{n-1}$. But then $A_1 \subseteq Q_{n-1}$ so that by property (iii) for $n - 1$, $B_1 \subseteq Q_{n-1}$ and,

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furthermore, if $y \in B_1$ then $y \geqq a_2$ (since otherwise $(y_1, x_1, a_2, \ldots, a_{m-1}, a_m)$, where $y_1 \in B_1$, $y_1 \leqq x_1$ for some $x_1 \in A_1$ and $y_1 \ngeqq a_2$, would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \ldots, a_m)$). If we now set $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_1, a_2, \ldots, a_m)$ and, therefore, $Q_n = A_1$ we have that the pair (F_n, Q_n) satisfies properties (i), (ii), and (iii).

Therefore, we may, without loss of generality, assume that every $x \in A_2$ satisfies $x \ngeqq g_{n-1}$ but that there exists $x_1 \in A_1$ such that $x_1 \geqq g_{n-1}$. In this case, we set $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_m, a_{m-1}, \ldots, a_1)$ and therefore, $Q_n = A_2$. Obviously (F_n, Q_n) satisfies (i) and (ii). Furthermore, every $y \in B_2$ satisfies $y \nleqq g_{n-1}$ (since otherwise $(x_1, a_2, \ldots, a_{m-1}, x_2, y_2)$ is a crown in L, where $y_2 \in B_2$, $y_2 \leqq x_2$ for some $x_2 \in A_2$ and $y_2 \leqq g_{n-1}$). To show (iii) we must prove that $B_2 \subseteq A_2$. Again applying (iii) for $n - 1$ to $y \in B_2$ we get that $B_2 \subseteq Q_{n-1}$ and if there were $y_2 \in B_2$ such that $y_2 \ngeqq a_{m-1}$ and $y_2 \leqq x_2$ for some $x_2 \in A_2$, then $(a_1, a_2, \ldots, a_{m-1}, x_2, y_2)$ would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \ldots, a_m)$. Thus, (F_n, Q_n) also satisfies property (iii).

Case (*b*). *G is lower and m is odd.* This case is completed by dualizing the argument of case (a).

Case (*c*). *G is upper and m is even* (see Figure 7). Define A_1, B_1 as in Case (a) and set $A_2 = \{x | x \leqq a_{m-1} \text{ and } x \| a_{m-2}\}$ and $B_2 = \{y | y \ngeqq a_{m-2} \text{ and } y \geqq x \text{ for some } x \in A_2\}$. (Note that A_2, B_2 here are just the duals of A_2, B_2, respectively, in Case (a).)

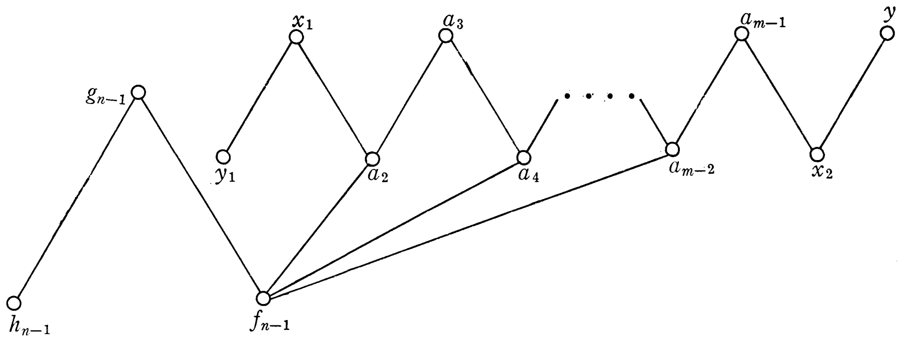

FIGURE 7. Theorem 3.6. Case (c)

Either every $x \in A_1$ satisfies $x \ngeqq g_{n-1}$ or, every $x \in A_2$ satisfies $x \nleqq g_{n-1}$, (since otherwise, $(x_1, a_2, a_3, \ldots, a_{m-1}, x_2)$ is a crown in L, where $x_1 \in A_1$, $x_1 \geqq g_{n-1}$ and $x_2 \in A_2, x_2 \leqq g_{n-1}$). Suppose that every x in A_1 satisfies $x \ngeqq g_{n-1}$ and every x in A_2 satisfies $x \nleqq g_{n-1}$); then either every $y \in B_1$ satisfies $y \nleqq g_{n-1}$ or, every $y \in B_2$ satisfies $y \ngeqq g_{n-1}$ (since otherwise, $(y_1, x_1, a_2, \ldots, a_{m-1}, x_2, y_2)$ is a crown in L, where $y_1 \in B_1$, $y_1 \leqq x_1$ for some $x_1 \in A_1$, $y_1 \leqq g_{n-1}$, and $y_2 \in B_2$, $y_2 \geqq x_2$ for some $x_2 \in A_2$, $y_2 \geqq g_{n-2}$). If every y in B_1 satisfies $y \nleqq g_{n-1}$ then the corresponding argument of Case (a) shows that $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_1, a_2, \ldots, a_m)$ and therefore $Q_n = A_1$ satisfy

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properties (i), (ii), and (iii). If, on the other hand, B_2 satisfies $y \ngeqq g_{n-2}$ then we simply dualize the corresponding argument in Case (a) with the fence $(a_1, a_2, \ldots, a_m)$ replaced by the fence $(a_m, a_{m-1}, \ldots, a_1)$.

Suppose now that every $x \in A_1$ satisfies $x \ngeqq g_{n-1}$ but that there exists $x_2 \in A_2$ such that $x_2 \leqq g_{n-1}$. In this case we set $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_1, a_2, \ldots, a_m)$ and therefore, $Q_n = A_1$. Obviously (F_n, Q_n) satisfies (i) and (ii). Furthermore, every $y \in B_1$ satisfies $y \nleqq g_{n-1}$ (since otherwise $(y_1, x_1, a_2, \ldots, a_{m-1}, x_2, g_{n-1})$ is a crown in L, where $y_1 \in B_1$, $y_1 \leqq x_1$ for some $x_1 \in A_1$, and $y_1 \leqq g_{n-1}$). To show (iii) we must show that $B_1 \subseteq A_1$. Applying (iii) for $n - 1$ to $y \in B_1$ we get that $B_1 \subseteq Q_{n-1}$ and if there were $y_1 \in B_1$ such that $y_1 \ngeqq a_2$, $y_1 \leqq x_1$ for some $x_1 \in A_1$ then $(y_1, x_1, a_2, \ldots, a_m)$ would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \ldots, a_m)$. Thus, also (iii) holds.

If every $x \in A_2$ satisfies $x \nleqq g_{n-1}$ and there exists $x_1 \in A_1$ such that $x_1 \geqq g_{n-1}$ we choose $F_n = (e_n, f_n, g_n, h_n, \ldots) = (a_m, a_{m-1}, \ldots, a_1)$ and therefore, $Q_n = A_2$. Clearly, (F_n, Q_n) satisfy (i) and (ii). Again every $y \in B_2$ satisfies $y \ngeqq g_{n-1}$ (since otherwise $(g_{n-1}, x_1, a_2, \ldots, a_{m-1}, x_2, y_2)$ is a crown in L, where $y_2 \in B_2$, $y_2 \geqq x_2$ for some $x_2 \in A_2$, and $y_2 \geqq g_{n-1}$). Thus, $B_2 \subseteq Q_{n-1}$ and if there were $y_2 \in B_2$, $y_2 \geqq x_2$ for some $x_2 \in A_2$, and $y_2 \nleqq a_{m-1}$ then $(a_1, a_2, \ldots, a_{m-1}, x_2, y_2)$ would be a fence in Q_{n-1} contradicting the maximality of $(a_1, a_2, \ldots, a_m)$. Thus, (F_n, Q_n) also satisfies (iii).

Case (d). G is lower and m is even. This case is completed by replacing the fence G in Case (c) by the fence $(a_m, a_{m-1}, \ldots, a_1)$.

Therefore, we now have a sequence of pairs (F_n, Q_n) for every $n \geqq 1$ satisfying properties (i), (ii), and (iii); furthermore, for all $n \geqq 1$ $Q_n \neq \emptyset$, $Q_n \subset Q_{n-1}$ since $f_{n-1} \in Q_{n-1} - Q_n$, and every $x \in Q_n$ is comparable with f_{n-1}.

But then whenever $1 \leqq n < m$, $f_n \rho_n f_m$, where ρ_n is either "$<$" or "$>$". Without loss of generality "$<$" appears an infinite number of times among the ρ_n's which, in turn, gives an infinite increasing chain in L. This is impossible since L has no infinite chains. We conclude that L must contain a doubly irreducible element.

5. Proof of Theorem 3.7. Let L be a dismantlable lattice which is not a chain and let us assume again that L contains no infinite chains and no infinite fences.

We show first that for some $x \in \text{Irr}\ (L)$ there exists $y \in L$ such that $x \| y$. Otherwise $\text{Irr}(L)$ is a chain $x_1 < x_2 < \ldots < x_k$ in L and if C is any maximal chain in L then $\text{Irr}(L) \subseteq C$. For each $i = 1, 2, \ldots, k$ take $y_i(z_i)$ to be the unique element covered by (covering) x_i (if $x_1 = 0$ take $y_1 = 0$ and, if $x_k = 1$ take $z_k = 1$). If we set $z_0 = 0$ and $y_{k+1} = 1$ then

$$L = \text{Irr}(L) \cup \bigcup ([z_i, y_{i+1}] | 0 \leqq i \leqq k).$$

For some i, $(0 \leqq i \leqq k)$, $S = [z_i, y_{i+1}]$ is not a chain and $S \cap \text{Irr}(L) = \emptyset$. On the other hand, S is a sublattice of a dismantlable lattice and is therefore

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dismantable, i.e., $\text{Irr}(S) \neq \emptyset$. But clearly, $\text{Irr}(S) \subseteq \text{Irr}(L)$ which is a contradiction.

Thus, there exists $x \in \text{Irr}\ (L)$ and $y \in L$ such that $x \| y$. We now have a fence $G = (y, x \wedge y, x)$ in L and by Lemma 4.1 there exists a left-maximal fence $F_1 = (a_1, a_2, \ldots, a_{m-1}, x)$. We may now define Q_1 as in Section 4. If $Q_1 \cap \text{Irr}(L) = \emptyset$ we can proceed to construct the sequence $((F_n, Q_n))_n$, $(n \geqq 2)$, as before which leads to a contradiction. Thus, $Q_1 \cap \text{Irr}(L) \neq \emptyset$. This means there exists $z \in \text{Irr}(L)$ such that $z \| x$.

Added in proof. We have learned that an alternate proof of Theorem 3.1 has been provided by M. Ajtai (see Period. Math. Hungar. *4* (1973), 217–220).

References

1. K. A. Baker, P. C. Fishburn, and F. S. Roberts, *Partial orders of dimension 2, interval orders, and interval graphs*, The Rand Corp. *P-4376* (1970).
2. G. Birkhoff, *Lattice theory*, 3rd. ed. (American Mathematical Society, Providence, R.I., 1967).
3. R. P. Dilworth, *A decomposition theorem for partially ordered sets*, Ann. of Math. *51* (1950), 161–166.
4. G. Havas and M. Ward, *Lattices with sublattices of a given order*, J. Combinatorial Theory *7* (1969), 281–282.
5. C. Herrmann, *Quasiplanare Verbände*, Arch. Math. (Basel) *24* (1973), 217–220.
6. I. Rival, *Lattices with doubly irreducible elements*, Can. Math. Bull. *17* (1974).
7. R. Wille, *Modular lattices of order dimension 2*, Proc. Amer. Math. Soc. *43* (1974), 287–292.

University of Manitoba,
Winnipeg, Manitoba

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