THE LAPLACE TRANSFORM OF GENERALIZED FUNCTIONS

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Introduction. In this paper we develop a theory of Laplace transforms for generalized functions. Some fundamental results in this field were given by Schwartz in **(3)** for the n-dimensional bilateral case from the point of view of topological vector spaces, and in **(4)** in a form amenable to operational use. Our presentation characterizes a one-dimensional theory of Laplace transforms with a half-plane of convergence (indicating an extension of the usual classical transform) and with the property that Laplace transforms are analytic functions satisfying the fundamental convolution–multiplication theorem. Section 1 is devoted to defining the Laplace transform of generalized functions and also to showing how the property of a half-plane of convergence is intrinsic to this definition. In §5 we show that the Laplace transform is an analytic function and then prove that the Laplace transform of the convolution of two distributions is the product of the Laplace transforms of these distributions.

A vital part of any transform theory is the representation problem; in the Laplace transform of distributions, the representation problem was solved by Schwartz (**3**, pp. 202–203) for the bilateral and complex case. We shall quote this result for the usual case in §7 and give the accompanying inversion formula; further, we shall state and prove real representation and inversion theorems in §7. We shall also introduce the concept of uniform convergence for the Laplace transform of distributions and give a result analogous to the classical case in §4. Another notion that can be extended from the classical theory is that of the "growth" of Laplace transforms as the independent variable goes to infinity in certain ways. We shall discuss this concept in §6. As preliminaries to these fundamental results in a theory of Laplace transforms, we give a Lebesgue integral definition of the Laplace transform in §2, and in §3 we show that our definition of §1 is a legitimate generalization of the classical one. In §8 we consider the space of those distributions having a Laplace transform as an algebraic inductive limit, and give an indication of certain related and natural problems concerning some topological vector spaces.

Finally, because of the traditional connection **(5)** between Laplace transforms and Tauberian theorems, we prove in §9 Wiener's Tauberian theorem for distributions.

As is standard **(2)**, we let $\mathscr{D}$ be the complex-valued $C^{\infty}(-\infty, \infty)$ functions with compact support, and $\mathscr{D}_K$ the subspace of $\mathscr{D}$ whose elements have their

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supports in the compact set K. The topology on $\mathscr{D}_K$ is given by the following convergence criterion: f_n converges to 0 if for each p, $f_n^{(p)} \to 0$ uniformly on $(-\infty, \infty)$. A distribution T is a linear functional on $\mathscr{D}$ whose restriction to each $\mathscr{D}_K$ is continuous. The space of distributions is denoted by $\mathscr{D}'$. Also, we let $\mathscr{S}$ be those C^∞ functions f for which given any k and p we have

$$\lim_{|x|\to\infty} \left| x^k \frac{d^{(p)}}{dx^p} f(x) \right| = 0.$$

A fundamental system of neighbourhoods of 0 for a locally convex topology on $\mathscr{S}$ is given by all sets $V(m, k, \epsilon)$ where $f \in V$ if for all $p \leqslant m$ and all $x \in (-\infty, \infty)$,

$$\left| (1 + x^2)^k \frac{d^{(p)}}{dx^p} f(x) \right| < \epsilon.$$

The strong dual of $\mathscr{S}$ is then denoted by $\mathscr{S}'$.

We now let $\mathscr{D}'_+$ be the space of 1-dimensional distributions with support contained in $[0, \infty)$. Also, we denote the field of complex numbers by $\mathbf{C}$, and a typical member of $\mathbf{C}$ by $s = \sigma + i\tau$. When any confusion arises concerning which variable is to be considered when a function belongs to a given space, we write the subscript on the space symbol accordingly. Lastly, we recall Schwartz's fundamental result (**2**, p. 95), which states that $T \in \mathscr{S}'$ if and only if $T = f^{(m)}$, where f is continuous and of the form

$$f(t) = g(t)(1 + t^2)^{k/2}$$

with g bounded and for some k. Such an f will be called *slowly increasing*.

1. Definition of the Laplace transform of generalized functions.

DEFINITION 1. *Let $\mathscr{S}_+$ be the vector space of all complex-valued infinitely differentiable functions f defined on $(-\infty, \infty)$, for which, given any non-negative integers k and m, we have*

$$\lim_{t\to\infty} |t^k D^{(m)} f(t)| = 0,$$

where, as usual, $D^{(m)} = d^{(m)}/dt^m$.

Let k and m be non-negative integers and let ϵ be any positive real number; further, define $V(m, k, \epsilon)$ to be the set of those elements $f \in \mathscr{S}_+$ for which, given any $t \in [0, \infty)$ and any non-negative integer $p \leqslant m$, we have

$$|(1 + t^2)^k D^{(p)} f(t)| < \epsilon.$$

It is easy to check that a base at 0 for a locally convex topology on $\mathscr{S}_+$ is given by all sets of the form $V(m, k, \epsilon)$. As such, $\mathscr{S}_+$ is a topological vector space and the space of continuous linear functionals on $\mathscr{S}_+$ is designated by $\mathscr{S}'_+$.

THEOREM 1. *Let $T \in \mathscr{D}'_+$. Then $T \in \mathscr{S}'_+$ if and only if $T \in \mathscr{S}'$.*

Proof. One way is obvious. Let $T \in \mathscr{S}'$ and $\phi \in \mathscr{S}_+$, and define a complex-valued infinitely differentiable function h on $(-\infty, \infty)$ with the properties: (a) for some $\epsilon > 0$ and for all $t \in (-\epsilon, \infty)$, $h(t) = 1$, (b) for some $M > \epsilon$ and for all $t < -M$, $h(t) = 0$. Then $h\phi \in \mathscr{S}$ and $\langle T, h\phi\rangle$ has meaning. We now define

$$(1) \qquad \langle T, \phi\rangle = \langle T, h\phi\rangle.$$

It is trivial to show that (1) is independent of the particular h, satisfying (a) and (b), chosen.

By (1), T is clearly linear on $\mathscr{S}_+$.

Further, it is straightforward to show that if $\phi_\gamma \to 0$ in $\mathscr{S}_+$, then

$$\langle T, \phi_\gamma\rangle \to 0.$$

In fact, this is done by representing T as $f^{(m)}$, where f is a slowly increasing continuous function with support contained in $[0, \infty)$; and then noting that

$$(2) \qquad \langle T, \phi\rangle = \lim_{R\to\infty} \langle f^{(m)}(t), h(t)\phi_\gamma(t)\zeta_R(t)\rangle,$$

where, for $R > 0$, $\zeta_R(t)$ is an infinitely differentiable function on $(-\infty, \infty)$ that satisfies

$$\zeta_R(t) = \begin{cases} 1 & \text{for } t \in (-\infty, R], \\ 0 & \text{for } t \in [R + r, \infty),\ r > 0 \text{ being fixed.} \end{cases}$$

We then compute the right-hand side of (2) and find that

$$\langle T, \phi_\gamma\rangle = (-1)^m \int_0^\infty f(t)\phi_\gamma^{(m)}(t)\,dt.$$

Then, because f is slowly increasing and $\phi_\gamma \in \mathscr{S}_+$, it is easy to show that $\langle T, \phi_\gamma\rangle \to 0$.

It is clear that

$$\exp[-(s - \sigma_0)t] \in \mathscr{S}_{+t} \quad \text{for all } \sigma > \sigma_0 > -\infty.$$

THEOREM 2. *Let $T \in \mathscr{D}'_+$ and assume that* $\exp(-\sigma_0 t)T_t \in \mathscr{S}'_{+t}$. *Then,*

(a) $\exp(-\sigma t)T_t \in \mathscr{S}'_{+t}$ *for all* $\sigma > \sigma_0$,

and, further,

(b) *if* $\sigma_1, \sigma_2 \in [\sigma_0, \sigma)$, *then*

$$\langle \exp(-\sigma_1 t)T, \exp(-[\sigma - \sigma_1)t\rangle = \langle \exp(-\sigma_2 t)T, \exp[-(\sigma - \sigma_2)t]\rangle.$$

Proof. (a) Let $\phi \in \mathscr{S}_+$ and $\exp(-\sigma_0 t)T_t = f^{(m)}(t)$, where f is slowly increasing, continuous, and with support contained in $[0, \infty)$. Then, because

$$\exp(-\sigma t)T = \exp[-(\sigma - \sigma_0)t]\exp(-\sigma_0 t)T$$

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and $\bar{\mathscr{D}} = \mathscr{S}_+$, we can define

$$\langle \exp(-\sigma t)T, \phi(t)\rangle = (-1)^m \int_0^\infty f(t) \frac{d^{(m)}}{dt^m} \{\exp[-(\sigma - \sigma_0)t]\phi(t)\}\, dt.$$

Clearly then $e^{-\sigma t}T$ is linear on $\mathscr{S}_+$, and, in a manner similar to that of Theorem 1, $e^{-\sigma t}T$ is shown to be continuous on $\mathscr{S}_+$.

(b) It is obviously sufficient to show that

$$\text{(3)} \quad \langle \exp(-\sigma_0 t)T, \exp[-(\sigma - \sigma_0)t]\rangle = \langle \exp(-\sigma_1 t)T, \exp[-(\sigma - \sigma_1)t]\rangle.$$

Now,

$$\text{(4)} \quad \langle \exp(-\sigma_0 t)T, \exp[-(\sigma - \sigma_0)t]\rangle = (-1)^m \int_0^\infty f(t) \frac{d^{(m)}}{dt^m} \exp[-(\sigma - \sigma_0)t]\, dt,$$

and because $\bar{\mathscr{D}} = \mathscr{S}_+$ and $\exp[-(\sigma - \sigma_1)t]$ is multiplied by the rapidly decreasing (on the right) function $\exp[-(\sigma_1 - \sigma_0)t]$, it follows that

$$\text{(5)} \quad \langle \exp(-\sigma_1 t)T, \exp[-(\sigma - \sigma_1)t]\rangle$$
$$= (-1)^m \int_0^\infty f(t) \frac{d^{(m)}}{dt^m} \{\exp[-(\sigma_1 - \sigma_0)t]\exp[-(\sigma - \sigma_1)t]\}\, dt.$$

The right-hand sides of (4) and (5) are identical, and, hence, (3) holds.

DEFINITION 2. *Let $T \in \mathscr{D}'_+$ and assume there is $\sigma_0 \in (-\infty, \infty)$ such that $\exp(-\sigma_0 t)T \in \mathscr{S}'_+$. Further, let* $\sigma_c = \inf\{\sigma \in (-\infty, \infty) | \exp(-\sigma t)T \in \mathscr{S}'_+\}$, *and define $A \subseteq \mathbf{C}$ to be*

$$A = \{s \in \mathbf{C} | \sigma > \sigma_c\}.$$

The Laplace transform of T is a map $\mathfrak{L}(T)$

$$A \to \mathbf{C}$$

where

$$\mathfrak{L}(T)(s) = \langle \exp(-\sigma_1 t)T_t, \exp[-(s - \sigma_1)t]\rangle$$

and $\sigma_c \leqslant \sigma_1 < \sigma$ (clearly, the equality is allowed only when

$$\exp(-\sigma_c t)T \in \mathscr{S}'_+).$$

Also, σ_c is the abscissa of convergence of the Laplace transform of T.

2. Integral definition of the Laplace transform.

DEFINITION 3. *Let B be the set of maps β such that β maps $(-\infty, \infty)$ into* $[0, 1]$, *and where*

(a) $\beta \in C^\infty$,

(b) $\beta(t) = \begin{cases} 1 & \textit{for } t \leqslant 0, \\ 0 & \textit{for } t \geqslant \delta, \textit{ where } \delta > 0 \textit{ is fixed.} \end{cases}$

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Further, for $R \geqslant 0$, we define

$$\alpha_{\beta R}(t) = \beta(t - R).$$

The following example shows that B is not empty.

Example. For $t \in [R, R + \delta]$, consider

$$\theta_R(t) = \exp\left(-\frac{1}{(R+\delta) - t} + \frac{1}{R - t}\right),$$

where $R \geqslant 0$. We define $\alpha_R(t)$ as follows:

$$\alpha_R(t) = \begin{cases} 1 & \text{for } t \in [-\infty, R], \\ \dfrac{-1}{\delta\theta_R(t_0)} \displaystyle\int_{R+\delta}^{t} \theta_R(x)\,dx & \text{for } t \in [R, R+\delta], \\ 0 & \text{for } t \in [R+\delta, \infty), \end{cases}$$

where t_0 is such that

$$\int_R^{R+\delta} \theta_R(x)\,dx = \delta\theta_R(t_0)$$

by the mean-value theorem. Note that since $t_0 \in (R, R + \delta)$, then $\theta_R(t_0) \neq 0$. Clearly, α_R satisfies the conditions of Definition 3.

Note that given any $\beta \in B$ and any integer $k \geqslant 0$, there exists a constant $N(k) > 0$ such that for all t

$$|\beta^{(k)}(t)| < N(k).$$

THEOREM 3. *Let $T \in \mathscr{D}'_+$ and let $\sigma_0 \in (-\infty, \infty)$ be such that for all $\sigma > \sigma_0$, $\exp(-\sigma t)T \in \mathscr{S}'_+$. Then, for any $\sigma_1 \in (\sigma_0, \sigma)$ there exists*

$$\lim_{R\to\infty} \langle \exp(-\sigma_1 t)T, \exp[-(s - \sigma_1)t]\alpha_{\beta R}(t)\rangle = \mathfrak{L}(T)(s).$$

Proof. Since the intersection of the support of T and the support of $\alpha_{\beta R}$ is compact, we know that

$$\langle \exp(-\sigma_1 t)T, \exp[-(s - \sigma_1)t]\alpha_{\beta R}(t)\rangle$$

exists. Thus, since it is readily seen that

$$\exp[-(s - \sigma_1)t]\alpha_{\beta R}(t) \xrightarrow{R} \exp[-(s - \sigma_1)t] \text{ in } \mathscr{S}_+,$$

the result follows immediately.

THEOREM 4. *If $\exp(-\sigma t)T \in \mathscr{S}'_+$ for all $\sigma > \sigma_0$ and if $\sigma_1 \in (\sigma_0, \sigma)$, then*

$$\mathfrak{L}(T)(s) = (s - \sigma_1)^m \int_0^\infty f(t)\exp[-(s - \sigma_1)t]\,dt,$$

where $\exp(-\sigma_1 t)T = f^{(m)}(t)$ and f is continuous and slowly increasing.

Proof. A simple calculation shows that

$$\langle \exp(-\sigma_1 t)T, \exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\rangle = (s-\sigma_1)^m \int_0^R f(t)\exp[-(s-\sigma_1)t]\,dt$$

$$+ (-1)^m \sum_{k=0}^{m} (-1)^k \binom{m}{k} (s-\sigma_1)^k \int_R^{R+\delta} f(t)\exp[-(s-\sigma_1)t]\alpha_{\beta R}^{(m-k)}(t)\,dt.$$

Because of Theorem 3, it is sufficient to show that for $k = 0, 1, \ldots, m$,

$$\lim_{R\to\infty} \int_R^{R+\delta} f(t)\exp[-(s-\sigma_1)t]\alpha_{\beta R}^{(m-k)}(t)\,dt = 0,$$

but this is clear since f is slowly increasing and $\alpha_{\beta R}^{(m-k)}$ is bounded.

3. Relation between classical and generalized function definition of Laplace transform.

THEOREM 5. *Let f be locally integrable on $(-\infty, \infty)$ and with support contained in $[0, \infty)$. If*

$$\int_0^\infty f(t)\exp(-s_0 t)\,dt$$

exists, then for all $\sigma > \sigma_1 > \sigma_0$,

$$\mathfrak{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

Proof. We have that $e^{-\sigma_1 t}f(t) \in \mathscr{S}'_+$ since

$$\lim_{t\to\infty} \int_{-\infty}^{t} \exp(-\sigma_1 x)f(x)\,dx \equiv \lim_{t\to\infty} g(t)$$

exists, implying that $g(t)$ is bounded and continuous, and $g'(t) = \exp(-\sigma_1 t)f(t)$. Classically, we know that for all $\sigma > \sigma_0$,

$$\int_0^\infty f(t)e^{-st}\,dt$$

exists. We need only show that for $\sigma > \sigma_1 > \sigma_0$, there exists

$$\lim_{R\to\infty} \langle \exp(-\sigma_1 t)f(t), \exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\rangle = \int_0^\infty f(t)e^{-st}\,dt.$$

Now,

$$\langle \exp(-\sigma_1 t)f(t), \exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\rangle$$
$$= \int_0^R e^{-st}f(t)\,dt + \int_R^{R+\delta} f(t)e^{-st}\alpha_{\beta R}(t)\,dt,$$

and thus it is only necessary to prove that

$$\lim_{R\to\infty} \int_R^{R+\delta} f(t)e^{-st}\alpha_{\beta R}(t)\,dt = 0. \tag{6}$$

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In fact, when integrating by parts, we have

$$\left|\int_R^{R+\delta} f(t)e^{-st}\alpha_{\beta R}(t)\,dt\right| = \left|\int_R^{R+\delta} g'(t)\exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\,dt\right|$$
$$\leqslant K\exp[-(\sigma-\sigma_1)R] + (|s-\sigma_1|KM(\beta,0)+KM(\beta,1))$$
$$\times \int_R^{R+\delta}\exp[-(\sigma-\sigma_0)t]\,dt,$$

where K is a bound on g, $M(\beta, 0)$ bounds $\alpha_{\beta R}$, and $M(\beta, 1)$ bounds $\alpha'_{\beta R}$. Since

$$\lim_{R\to\infty}\int_R^{R+\delta}\exp[(\sigma-\sigma_1)t]\,dt = 0,$$

(6) follows immediately.

THEOREM 6. *Let f be locally integrable on $(-\infty, \infty)$ and with support contained in $[0, \infty)$. If $\mathfrak{L}(f)(s)$ exists, then*

$$\mathfrak{L}(f)(s) = \int_0^\infty f(t)e^{-st}\,dt. \tag{7}$$

Proof. Since $\mathfrak{L}(f)(s)$ exists, there is $\sigma_1 < \sigma$ such that

$$\mathfrak{L}(f)(s) = \langle \exp(-\sigma_1 t)f(t), \exp[-(s-\sigma_1)t]\rangle$$
$$= \lim_{R\to\infty}\left(\int_0^R f(t)e^{-st}\,dt + \int_R^{R+\delta}\{\exp(-\sigma_1 t)f(t)\}\exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\,dt\right)$$

where $\exp(-\sigma_1 t)f(t) \in \mathscr{S}'_+$.

Thus, $\exp(-\sigma_1 t)f(t) = g^{(k)}(t)$, where g is continuous and slowly increasing, and hence

$$\lim_{R\to\infty}\int_R^{R+\delta} g^{(k)}(t)\{\exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\}\,dt = 0,$$

as is easily seen by integrating by parts k times. Therefore, (7) follows.

4. Uniform convergence.

DEFINITION 4. *The Stoltz region at s_0 with half-angle $\gamma < \pi/2$ is the set*

$$\mathfrak{St}_{s_0\gamma} = \{s\in\mathbf{C}|\ |\arg(s-s_0)| < \gamma\}.$$

DEFINITION 5. *Assume that $\mathfrak{L}(T)(s_0)$ exists and let D be a subset of $\mathbf{C}$ such that $\sigma > \sigma_0$ if $s \in \mathbf{C}$; also, let $\sigma_1 < \sigma_0$ and $\exp(-\sigma_1 t)T \in \mathscr{S}'_+$. $\mathfrak{L}(T)(s)$ converges uniformly in D as $R \to \infty$ if and only if for any $\beta \in B$ and any $\epsilon > 0$, there is an $R_0 > 0$ such that for all $R > R_0$ and for all $s \in D$, we have*

$$|\mathfrak{L}((T)(s) - \langle\exp(-\sigma_1 t)T, \exp[-(s-\sigma_1)t]\alpha_{\beta R}(t)\rangle| < \epsilon.$$

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We now have the following result, which is identical with the classical statement; Landau's original proof remains valid except for that part where a number of extra integral terms must be considered.

THEOREM 7. *Given* $\mathfrak{S}t_{s_0\gamma}$ *and assuming that* $\mathfrak{L}(T)(s_0)$ *exists, then* $\mathfrak{L}(T)(s)$ *converges uniformly in* $\mathfrak{S}t_{s_0\gamma}$.

Proof. Let $s \in \mathfrak{S}t_{s_0\gamma}$. Also, for $\sigma_1 < \sigma_0 < \sigma$, $\exp(-\sigma_1 t)T = f^{(m)}(t) \in \mathscr{S}'_+$, where f is continuous, slowly increasing, and with support contained in $[0, \infty)$.

Given $\epsilon > 0$, we must find R' such that for all $R > R'$ and for all $s \in \mathfrak{S}t_{s_0\gamma}$,

$$|\mathfrak{L}(T)(s) - \langle f^{(m)}(t), \exp[-(s - \sigma_1)t]\alpha_{\beta R}(t)\rangle| < \epsilon. \tag{8}$$

Note first that for any $k = 0, 1, \ldots, m$, there exists $S_k > 0$ such that for any $t > S_k$ and for any $s \in \mathfrak{S}t_{s_0\gamma}$,

$$|s - \sigma_1|^k \exp[-(\sigma - \sigma_0)t] < \exp[-\tfrac{1}{2}(\sigma - \sigma_0)t].$$

Also, since $\mathfrak{L}(T)(s_0)$ exists, there is an $R_0 > 0$ such that

$$\left| \int_{R_0}^{R} f(t)\exp[-(s_0 - \sigma_1)t]\, dt \right| < \frac{\epsilon \cos \gamma}{2M} \qquad \text{for all } R > R_0,$$

where

$$M = 2 + 4\sum_{k=0}^{m} \binom{m}{k} \max(M(\beta, m - k), M(\beta, m - k + 1))$$

and $M(\beta, m - k)$ is the uniform bound on $\alpha_{\beta R}^{(m-k)}(t)$.

These last two inequalities, along with a derivation following Landau's (for the classical case) (**5**, p. 56), imply that the left-hand side of (8) is less than

$$\frac{2\epsilon \cos \gamma \sec \theta}{M} + 4\sum_{k=0}^{m} \binom{m}{k} \max(M(\beta, m - k), M(m - k + 1, \beta)) \frac{\epsilon \cos \gamma \sec \theta}{M},$$

where

$$\sin \theta = \frac{\tau - \tau_0}{-|s - s_0|}.$$

Since the secant is an increasing function on $[0, \pi/2)$, we have $\sec \theta < \sec \gamma$ and hence (8) follows.

5. Analyticity and the convolution–multiplication theorem. The analyticity of the Laplace transform was shown by Schwartz (**3**, pp. 202–203) in his representation result. Using the result of §4, we arrive at analyticity in the classical fashion by employing Weierstrass' Double Series Theorem.

THEOREM 8. *Let* $F(s) = \mathfrak{L}(T)(s)$. *Then* $F(s)$ *is analytic for* $\sigma > \sigma_c$, *and*

$$\mathfrak{L}((-t)^p T)(s) = F^{(p)}(s) \qquad \textit{for all } p \geqslant 0.$$

Proof. With the usual terminology,

$$F(s) = \sum_{n=0}^{\infty} (s - \sigma_1)^m \int_n^{n+1} f(t)\exp[-(s - \sigma_1)t]\,dt \equiv \sum_{n=0}^{\infty} g_n(s).$$

Then $g_n(s)$ is entire (**5**, p. 57). With $\sigma > \sigma_c$, consider any neighbourhood Γ of s such that for all $z = x + iy \in \Gamma$ we have $x > \sigma_0 > \sigma_c$ for any such $\sigma_0 \in (-\infty, \infty)$. Then $\mathfrak{S}t_{\sigma_0\gamma}$ exists so that $\Gamma \subseteq \mathfrak{S}t_{\sigma_0\gamma}$.

Thus, by Theorem 7,

$$\sum_{n=0}^{\infty} g_n(z)$$

converges uniformly in Γ. Hence, by the Double Series Theorem, $F(s)$ is analytic for $\sigma > \sigma_c$. Now, for $p = 1$,

$$\mathfrak{L}(-tT)(s) = -\langle t\exp(-\sigma_1 t)T, \exp[-(s - \sigma_1)t]\rangle,$$

and

$$F'(s) = \left\langle \exp(-\sigma_1 t)T, \exp(\sigma_1 t)\frac{d}{ds}e^{-st}\right\rangle;$$

thus the result holds for $p = 1$ and is proved for all p by a routine calculation using induction.

Because of Theorem 8 it is clear that a number of formulas can be derived connecting the Laplace transforms of distributions with related slowly increasing functions. In fact, if $\mathfrak{L}(T)(s)$ exists and for $b \in (\sigma_c, \sigma)$, $e^{-bt}T = f^{(m)}(t)$ with f continuous and slowly increasing, we have

$$\frac{m}{s-b}\mathfrak{L}(T)(s) + \mathfrak{L}(tT)(s) = (s - b)^m \int_0^{\infty} tf(t)\exp[-(s - b)t]\,dt.$$

THEOREM 9. *Assume that $\mathfrak{L}(T)(s)$ exists with abscissa of convergence $\sigma = \sigma_{c1}$, and that $\mathfrak{L}(S)(s)$ exists with abscissa of convergence $\sigma = \sigma_{c2}$. Then*

$$\mathfrak{L}(T * S)(s) = \mathfrak{L}(T)(s)\mathfrak{L}(S)(s) \qquad \textit{for all } \sigma > \max(\sigma_{c1}, \sigma_{c2}).$$

Convolution, here, is defined as usual in the theory of distributions, and the result itself is trivial if we use the tensor product definition of convolution. The theorem could also be proved by giving integral representations of the Laplace transforms (in terms of slowly increasing functions). This latter approach, however, becomes a long-winded exercise in manipulating Fubini's theorem.

COROLLARY 9.1. *If $\mathfrak{L}(T)(s)$ exists, then for all $\sigma > \sigma_c$ and for all $m \geqslant 0$,*

$$\mathfrak{L}(T^{(m)})(s) = s^m\mathfrak{L}(T)(s).$$

Proof. $\mathfrak{L}(T^{(m)})(s) = \mathfrak{L}(T * \delta^{(m)})(s) = \mathfrak{L}(\delta^{(m)})(s)\mathfrak{L}(T)(s)$. Clearly,

$$\mathfrak{L}(\delta^{(m)})(s) = s^m,$$

since

$$\mathfrak{L}(\delta^{(m)})(s) = \lim_{R\to\infty} \langle \delta^{(m)}, e^{-st}\alpha_{\beta R}(t)\rangle$$
$$= \lim_{R\to\infty} (-1)^m\left(\sum_{k=0}^{m} \binom{m}{k}(-s)^k\alpha_{\beta R}^{(m-k)}(0)\right) = s^m.$$

6. The growth of Laplace transforms. Classically, we know that if g is locally integrable with support contained in $[0, \infty)$, and if $\mathfrak{L}(g)(s)$ exists for all $\sigma > \sigma_c$, then for $\sigma' > \sigma_c$,

$$\mathfrak{L}(g)(s) = o(|\tau|), \qquad |\tau| \to \infty,$$

uniformly on $[\sigma', \infty)$. Unfortunately, we are not so lucky in the case of generalized functions; in fact, we have the following result.

THEOREM 10. *Assume $\mathfrak{L}(T)(s)$ exists with abscissa of convergence $\sigma = \sigma_c$. Then for any $\sigma > \sigma_c$ there is an $n \geqslant 0$ such that*

$$F(\sigma + i\tau) \equiv \mathfrak{L}(T)(s) = o(|\tau|^n), \; |\tau| \to \infty.$$

Proof. Given $\sigma > \sigma_c$, we choose $\sigma_1 = (\sigma_c + \sigma)/2$ and let $\exp(-\sigma_1 t)T = f^{(m)}(t)$, where, as usual, f is continuous and slowly increasing. Then,

$$G(s) \equiv \frac{F(s)}{(s - \sigma_1)^m} = \int_0^\infty \{f(t)\exp(\sigma_1 t)\}e^{-st}\,dt,$$

and we readily prove the result (**5**, pp. 92–93), noting that the Laplace transform of $f(t)\exp(\sigma_1 t)$ is $G(s)$.

The following example shows that we cannot strengthen Theorem 10 in the direction of the classical result.

Example. If $T = \delta'$, then for all $\sigma \in (-\infty, \infty)$, $\mathfrak{L}(T)(s) = s$. Given any integer $n > 1$,

$$\lim_{|\tau|\to\infty} \frac{|\sigma + i\tau|}{|\tau|^n} = 0.$$

It is not true, however, that given $\epsilon > 0$ there is a τ_0 such that for all σ in an interval $[\sigma', \infty)$ and for all τ satisfying $|\tau| > |\tau_0|$, we have

$$\left|\frac{\sigma + i\tau}{\tau^n}\right| < \epsilon_0.$$

In fact, it is obvious that the choice of τ_0 depends on the σ chosen.

We now define the real Laplace transform in the expected way: let $T \in \mathscr{D}'_+$ and assume that there is an $x_0 \in (-\infty, \infty)$ such that

$$\exp(-x_0 t)T \in \mathscr{S}'_+;$$

further, let

$$x_c = \inf\{x \in (-\infty, \infty) | e^{-xt}T_t \in \mathscr{S}'_+\}$$

and define

$$\mathfrak{L}(T)(x) = \langle \exp(-x_1 t)T_t, \exp[-(x - x_1)t] \rangle,$$

where $x_c \leqslant x_1 < x$ (as in Definition 2, the equality is allowed only if $\exp(-x_c t)T \in \mathcal{S}'_+$).

The natural partial representation result corresponding to Theorem 10 is

THEOREM 10′. *Assume* $\mathfrak{L}(T)(x) = F(x)$ *exists for all* $x > x_c$. *Then an* $n \geqslant 0$ *exists such that*

$$F(x) = o(x^n), \qquad x \to \infty.$$

The proof of this result is again classical and straightforward although somewhat different from that of Theorem 10. We shall not prove it here, since it will be given in a stronger and more general way in our real representation theorem. We note now that Theorem 10′ is a corollary of this representation theorem when n is taken large enough.

7. Representation and inversion results. The fundamental complex representation theorem proved by Schwartz (**3**, pp. 202–203; **4,** pp. 249–252) is the following.

THEOREM 11. *An analytic function* $F(s)$ *is the Laplace transform of a generalized function* $T \in \mathcal{D}'_+$ *if and only if there are a non-negative integer* n *and constants* $K > 0$ *and* c *such that*

$$|F(s)| \leqslant K|s|^n \qquad \textit{for all } \sigma > c.$$

We recall that the difficult part of the theorem is to construct a distribution T such that $\mathfrak{L}(T)(s) = F(s)$. In order to prove this part, we first consider the particular case of $F(s)$ bounded in the half-plane $\sigma > c > 0$ and such that

$$F(s) = O(1/|s|^2), \qquad |s| \to \infty.$$

For this situation we construct a continuous function $f(t)$, with support contained in $[0, \infty)$, satisfying

$$|f(t)| \leqslant Ce^{ct},$$

for some $C > 0$, and then show $\mathfrak{L}(f) = F$. To extend the result for the statement of Theorem 11, we let $F(s)$ be any analytic function for $\sigma > c > 0$ and for which

$$F(s) = O(|s|^m), \qquad |s| \to \infty.$$

Then

$$\frac{F(s)}{s^{m+2}} = O\left(\frac{1}{|s|^2}\right), \qquad |s| \to \infty;$$

hence, there exists a continuous function $g(t)$ for which $|g(t)| \leqslant Me^{ct} (M > 0)$ and such that

$$\frac{F(s)}{s^{m+2}} = \int_0^\infty g(t)e^{-st}\,dt \qquad \text{for all } \sigma > c.$$

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The result is proved when we let $T = g^{(m+2)}(t)$.

We have the following useful corollaries to Theorem 11.

COROLLARY 11.1. *If* $\mathfrak{L}(T)(s) = F(s)$ *exists, then there is a* $K > 0$ *and a* c *such that*

$$|F(s)| \leqslant K|s|^m, \qquad \text{for all } \sigma > c$$

where $e^{-ct}T = f^{(m)}(t)$, *with* f *continuous and slowly increasing.*

COROLLARY 11.2. *If the conditions of Theorem* 11 *are satisfied for* F, *analytic, then there is a continuous function* f *such that*

$$\mathfrak{L}(f^{(n+2)}(t))(s) = F(s).$$

We note that although $n + 2$ plays the role of an upper bound (for order of derivation) in Corollary 11.2, it is possible to consider a lower order of derivation. In fact, if

$$Y(t) = \begin{cases} t, & t \geqslant 0, \\ 0, & t < 0, \end{cases}$$

then we can choose $n = 0$ in Corollary 11.2, but we can take $f = Y$ such that

$$\mathfrak{L}(f)(s) = 1/s^2.$$

We shall now state the complex inversion result. The proof is essentially Bromwich's as given by Widder (**5**, p. 69).

THEOREM 12. *Assume* $\mathfrak{L}(T)(s) = F(s)$ *exists with abscissa of convergence* $\sigma = \sigma_c$. *Further, let* n *be such that for some right half-plane* $\sigma > c > 0$, *where* $c > \sigma_c$, *we have that there is a* $K > 0$ *for which*

$$|F(s)| \leqslant K|s|^n. \tag{9}$$

Then for any $b > c$ *and* $a > b$,

$$T_t = e^{bt}\frac{d^{(n+2)}}{dt^{n+2}}(e^{-bt}g'(t)), \tag{10}$$

where

$$g(t) = \lim_{R\to\infty}\frac{1}{2\pi i}\int_{a-iR}^{a+iR}\frac{F(s)e^{st}ds}{s(s-b)^{n+2}}.$$

Proof. By Corollary 11.1, (9) is clear and there is a continuous function f, with support contained in $[0, \infty)$, such that for some $M > 0$,

$$|f(t)| \leqslant Me^{ct}$$

and

$$\mathfrak{L}(T)(s) = \mathfrak{L}(f^{(n+2)}(t))(s).$$

Now, $e^{-bt}T \in \mathscr{S}'_+$ implies that there is a continuous and slowly increasing function h for which

$$h^{(n+2)}(t) = e^{-bt}T,$$

and such that for all $\sigma > b$

$$\frac{F(s)}{(s-b)^{n+2}} = \int_0^\infty h(t)\exp[-(s-b)t]\,dt.$$

We now apply the above-mentioned proof of Bromwich to deduce (10).

COROLLARY 12.1 (*uniqueness*). If $\mathfrak{L}(T)(s) \equiv 0$, *then* $T \equiv 0$.

We shall now prove our real representation theorem, and begin by defining the following differential operator.

DEFINITION 6. *Let* $G(x) \in C^\infty$. *Then for all* $t > 0$ *and for any non-negative integer* k, *we define*

$$\tilde{l}_{k,t}(G(x)) = (-1)^k \frac{ke^k}{t^{k+1}} G^{(k)}\left(\frac{k}{t}\right) + (-1)^k k\left(\frac{e}{t}\right)^k G^{(k-1)}\left(\frac{k}{t}\right).$$

LEMMA 13.1. *Let* ϕ *be a real-valued and locally integrable function for which*

$$F(x) = \int_0^\infty e^{-xt}\phi(t)\,dt$$

is defined and converges for some $x \in (-\infty, \infty)$; *for all* $t > 0$ *we assume that* $\phi(t+)$ *and* $\phi(t-)$ *exist. Then*

$$\lim_{k\to\infty} \tilde{l}_{k,t}(F(x)) = \phi(t+) - \phi(t-).$$

Proof. Given $t > 0$ and fixed. If k is so large that k/t lies in the region of convergence of $F(x)$, then

$$\begin{aligned}\tilde{l}_{k,t}(F(x)) &= (-1)^k \frac{ke^k}{t^{k+1}} \int_0^\infty (-1)^k u^k e^{-ku/t}\phi(u)\,du \\ &\quad + (-1)^k k\left(\frac{e}{t}\right)^k \int_0^\infty (-1)^{k-1} u^{k-1} e^{-ku/t}\phi(u)\,du \\ &= \left(\frac{e}{t}\right)^k \int_0^\infty e^{-ku/t} ku^{k-1}\phi(u)\frac{u}{t}\,du - \left(\frac{e}{t}\right)^k \int_0^\infty e^{-ku/t} ku^{k-1}\phi(u)\,du \\ &= -\left(\frac{e}{t}\right)^k \int_0^\infty e^{-ku/t} ku^{k-1}\left(1 - \frac{u}{t}\right)\phi(u)\,du.\end{aligned}$$

Making the change of variable $u = ty$, we obtain

$$\begin{aligned}\tilde{l}_{k,t}(F(x)) &= -ke^k \int_0^\infty e^{-ky}\phi(ty)y^{k-1}(1-y)\,dy \\ &= -ke^k \int_0^\infty e^{kh(y)}\frac{\phi(ty)}{y}(1-y)\,dy,\end{aligned}$$

where $h(y) = \log y - y$ and satisfies the conditions of Theorem 8a and Theorem 8b in (**5**, pp. 296–298).

We then follow the proof of Widder's Theorem 9 (**5**, pp. 298–299), verbatim.

In proving our representation theorem we shall also make use of Bernstein's representation result (**5**, pp. 160–162), which states that f is completely monotonic on (a, ∞) if and only if

$$f(x) = \int_0^\infty e^{-xt}\, d\alpha(t),$$

where $\alpha(t)$ is non-decreasing and the integral converges for $x \in (a, \infty)$.

Finally, before giving our theorem, we give the following convenient definition.

DEFINITION 7. *Let $S \in \mathscr{S}'_+$. An order of derivation of S is an integer m for which there is a continuous, slowly increasing function f, with support contained in $[0, \infty)$, such that $f^{(m)} = S$. We denote an order of derivation of S by*

$$\bar{O}(S) = m.$$

THEOREM 13. (a) *If $\mathfrak{L}(T)(x) = F(x)$ exists with abscissa of convergence $x = x_c$, and if $\bar{O}(Te^{-bt}) = m$ for $b > x_c$, then for any $x_0 > x_c$ there is a $K > 0$ and an $n \geqslant 0$ such that for all $x > x_0$*

$$\left|\frac{d^{(k)}}{dx^k}\left(\frac{F(x)}{(x - x_0)^m}\right)\right| \leqslant \frac{K(k+n)!}{(x - x_0)^{k+n+1}}, \qquad k = 0, 1, 2, \ldots. \tag{11}$$

$$\textit{For all } t \in (0, \infty), \quad \lim_{k\to\infty} \tilde{l}_{k,t}\left(\frac{F(x)}{(x - x_0)^m}\right) = 0. \tag{12}$$

(b) *Given $F(x)$. If there is an x' such that $F(x) \in C^\infty$ for all $x > x'$, and if for all $x_0 > x'$ there is a $K > 0$ and integers $m, n \geqslant 0$ such that* (11) *and* (12) *hold, then $T \in \mathscr{D}'_+$ exists so that $\mathfrak{L}(T)(x) = F(x)$ and $\bar{O}(Te^{-bt}) = m$ for all $b > x'$.*

Our following proof is for real-valued functions $F(x)$, and the extension to complex-valued F is obvious.

Proof. (a) Given $x > x_c$ and let $x_0 \in (x_c, x)$. Then $\exp(-x_0 t)T = f^{(m)}(t)$ for some $m \geqslant 0$ and some f continuous and slowly increasing; in fact,

$$G(x) \equiv \frac{F(x)}{(x - x_0)^m} = \int_0^\infty \exp(-xt) f(t) \exp(x_0 t)dt.$$

Thus, there is an $n \geqslant 0$ and a $K > 0$ such that

$$|f(t)\exp(x_0 t)| \leqslant K \exp(x_0 t)t^n \qquad \text{for all } t > 0.$$

Now,

$$G^{(k)}(x) = (-1)^k \int_0^\infty t^k \exp(-xt) f(t) \exp(x_0 t)dt,$$

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and hence, for all k,

$$|G^{(k)}(x)| \leqslant K \int_0^\infty t^{k+n} \exp[-(x-x_0)t]\,dt = \frac{K(k+n)!}{(x-x_0)^{k+n+1}}.$$

Since $f(t)\exp(x_0 t)$ is continuous, (12) is immediate from Lemma 13.1.

(b) If (11) is satisfied, then for $G(x) = F(x)/(x-x_0)^m$ we have

$$(-1)^k\left[G^{(k)}(x) + (-1)^k \frac{K(n+k)!}{(x-x_0)^{n+k+1}}\right] \geqslant 0$$

and

$$(-1)^k\left[-G^{(k)}(x) + (-1)^k \frac{K(n+k)!}{(x-x_0)^{n+k+1}}\right] \geqslant 0.$$

Thus, the functions $[n!\,K(x-x_0)^{-(n+1)} + G(x)]$ and

$$[n!\,K(x-x_0)^{-(n+1)} - G(x)]$$

are completely monotonic on (x_0, ∞). Hence, by Bernstein's theorem, there is a non-decreasing function $\beta(t)$ such that

$$G(x) = \frac{n!\,K}{(x-x_0)^{n+1}} + \int_0^\infty e^{-xt}\,d\beta(t),$$

with the integral converging for $x > x_0$. But,

$$\frac{n!\,K}{(x-x_0)^{n+1}} = \int_0^\infty e^{-\alpha t}\,d\gamma(t), \qquad \text{for } x > x_0,$$

where

$$\gamma(t) = \int_0^t u^n \exp(x_0 u)\,du.$$

Thus,

$$G(x) = \int_0^\infty e^{-xt}\,d\alpha(t),$$

where

$$\alpha(t) = \beta(t) - K\gamma(t).$$

Clearly, α is of bounded variation, and it can be shown (**6**, pp. 873–875) that a function $\phi(t)$ exists for which

$$\alpha(t) = \int_0^t \phi(x)\,dx$$

and

(13) $$|\phi(t)| \leqslant K\gamma(t).$$

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Now, by the definition of $\gamma(t)$, we have for all $t > 0$ and for some $M > 0$

$$|\phi(t)| \leqslant K \exp(x_0 t) \left(\frac{n!}{|x_0|^{n+1}} + \sum_{k=1}^{n} \frac{n!\, t^k}{k!\, |x_0|^{n-(k-1)}}\right) + \frac{n!}{|x_0|^{n+1}}$$
$$\leqslant M \exp(x_0 t) \left(\frac{n!}{|x_0|^{n+1}} + \sum_{k=1}^{n} \frac{n!\, t^k}{k!\, |x_0|^{n-(k-1)}}\right),$$

where $x_0 > 0$.

If $x_0 < 0$, (13) implies that

$$|\phi(t)| \leqslant K \exp(x_0 t) \left(\left[\sum_{k=1}^{n} (-1)^{n-k} \frac{n!\, t^k}{k!\, x_0^{n-(k-1)}}\right] + (-1)^{n-1} \frac{n!}{x_0^{n+1}}\right) + (-1)^n \frac{n!}{x_0^{n+1}}$$
$$\leqslant K \exp(x_0 t) \left(\left[\sum_{k=1}^{n} (-1)^{n-k} \frac{n!\, t^k}{k!\, x_0^{n-(k-1)}}\right] + (-1)^{n-1} \frac{n!}{x_0^{n+1}}\right)$$

since

$$(-1)^n \frac{n!}{x_0^{n+1}} < 0$$

if $x_0 < 0$. Thus, $\phi(t) = \exp(x_0 t) f(t)$ where $f(t)$ is majored by some polynomial. Since

$$G(x) = \int_0^\infty e^{-xt} \phi(t)\, dt,$$

and because we have (12), it follows that ϕ, and hence f, is continuous.

By the definition of γ, $\phi(0) = 0$ and hence we can assume that the support of f is contained in $[0, \infty)$.

Let $T = f^{(m)}(t) \exp(x_0 t)$.

To avoid problems in calculation we have not taken $x_0 = 0$; this is clearly no restriction.

Our final result in this section is the real inversion formula. We derive this formula, as in the complex case, from the classical setting (**5**, pp. 288–293). It is then trivial to show that the following theorem holds.

THEOREM 14. *Let $T \in \mathscr{D}'_+$ and assume that $\mathfrak{L}(T)(x) = F(x)$ exists with abscissa of convergence $x = x_c$. Then, for $b > x_c$ and for all $t > 0$, we have*

$$T_t = e^{bt} \frac{d^{(m)}}{dt^m} \left(\lim_{k\to\infty} e^{-bt} L_{k,t}\left[\frac{F(x)}{(x-b)^m}\right]\right),$$

where $m = \bar{O}(e^{-bt}T)$ and

$$L_{k,t}\left[\frac{F(x)}{(x-b)^m}\right] = \frac{(-1)^k}{k!} \frac{d^{(k)}}{dt^k} \left(\frac{F(k/t)}{(k/t - b)^m}\right) \left(\frac{k}{t}\right)^{k+1}.$$

8. An inductive limit space. Let $\mathscr{D}'_+$ have the induced topology from $\mathscr{D}'$ and define $\mathscr{L}'_\sigma$ to be the set of those elements $T \in \mathfrak{D}'_+$ for which $T \in e^{\sigma t}\mathscr{S}'$. Also, let

$$\mathscr{L}' = U\{\mathscr{L}'_\sigma | \sigma \in (-\infty, \infty)\}.$$

Clearly if $\sigma_1 < \sigma_2$, then $\mathscr{L}'_{\sigma_1} \subseteq \mathscr{L}'_{\sigma_2}$; and $T \in \mathscr{L}'$ if and only if there is a σ_0 such that $\mathscr{L}(T)(s)$ exists for all $\sigma > \sigma_0$. For the real Laplace transform, the analogous notation for $\mathscr{L}'_\sigma$ and $\mathscr{L}'$ will be $\mathscr{L}'_x$ and $\mathscr{L}'_R$, respectively. Now, we let A_c be the set of analytic functions defined by the complex representation theorem, and A_R be the corresponding set (of C^∞ functions) for the real representation result. We consider the topology of compact convergence on both these sets. It is then natural to investigate the dual spaces and maps of

$$\mathscr{L}' \to A_c$$

and

$$\mathscr{L}'_R \to A_R.$$

This programme has been carried out for the bilateral Laplace transform in the cases of distributions with compact support (by Lions) and analytic functionals (by Malgrange), and will be carried out for the map $\mathfrak{L}$. We note in closing that $\mathscr{L}'$ and $\mathscr{L}'_R$ can be considered as strict inductive limits.

9. Wiener's Tauberian theorem. Using the standard spaces of distributions, $\mathscr{D}'_{L'}$ and $\mathscr{D}_{L^\infty}$ (**1**, p. 55), we shall prove the following.

THEOREM 15. *Let $f \in L^1(-\infty, \infty)$ and assume that*

$$\int_{-\infty}^{\infty} e^{ixt} f(t)\, dt \neq 0 \qquad \text{for all } x.$$

Further, let $T \in \mathscr{D}'_{L'}$ and $a \in \mathscr{D}_{L^\infty}$. If there exists a constant K such that

$$\lim_{x\to\infty} (f * a)(x) = K \int_{-\infty}^{\infty} f(t)\, dt,$$

then

$$\lim_{x\to\infty} (T * a)(x) = K\langle T, 1\rangle.$$

Our proof of Theorem 15 involves the following lemma and Beurling's (**1**, pp. 134–136) proof of the classical Wiener theorem.

LEMMA 15.1. *Let $f \in L'(-\infty, \infty)$, assume that $a(t)$ is n times differentiable and that for any $i = 1, \ldots, n$, there exists a constant $M_i > 0$ such that*

$$|a^{(i)}(t)| < M_i$$

for all $t \in (-\infty, \infty)$. Then if $(f * a)(x) \to 0$ as $x \to \infty$, we have

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(t) a^{(i)}(x - t)\, dt = 0 \qquad \text{for any } i = 1, \ldots, n - 1.$$

Proof. Let $n = 2$; $f * a(x) \to 0$ as $x \to \infty$ implies that, for all $h \neq 0$, there exists $\theta = \theta(x - t, h) \in (0, 1)$ such that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(t)[a'(x - t) + \tfrac{1}{2} h a''(x - t + \theta h)]\, dt = 0.$$

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Arguing by contradiction, it is a simple matter to show that

$$\lim_{x\to\infty} \int_{-\infty}^{\infty} f(t)\alpha'(x-t)\,dt = 0.$$

We proceed in a similar manner for the other values of n.

Proof of Theorem 15. Using Lemma 15.1 and the properties of $\mathscr{D}'_{L'}$, it is easy to show that

$$\lim_{m\to\infty} [(f * a) * T](x + \gamma_m) = 0$$

whenever

$$\lim_{m\to\infty} \gamma_m = \infty.$$

We then rewrite Beurling's proof.

We note that our Tauberian condition, the hypothesis that $a \in \mathscr{D}_{L^\infty}$, is stronger than the corresponding classical condition which only demands that $a \in L^\infty(-\infty, \infty)$. This strengthening of the Tauberian condition is, of course, compensated for by the fact that our theorem is true for the elements of $\mathscr{D}'_{L'}$ instead of $L'(-\infty, \infty)$. Further note that our Tauberian condition cannot be weakened effectively to achieve the same conclusion since it is necessary (in general) that $a \in \mathscr{D}_{L^\infty}$ in order to define $T * a$ for $T \in \mathscr{D}'_{L'}$.

We remark that Lemma 15.1 has recently been strengthened by S. Silverman so that the conclusion is true for $i = 1, \ldots, n$. Also, our Wiener theorem has been proved using the fact that the translates of f are dense in $\mathscr{D}'_{L'}$.

Finally, if $T \in \mathscr{D}'_{L'}$ and

$$T = \sum_{i=1}^{k} f_i^{(n_i)}, \qquad n_i \geqslant 1, f_i \in L',$$

then $\langle T, 1\rangle = 0$. This is clearly a generalization of the property that if f and f' are in $L'(-\infty, \infty)$, then

$$\int f' = 0.$$

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