RADIATIVE TRANSFER AND SOLAR OSCILLATIONS*

(Invited Review)

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Abstract. We consider the atmospheric behaviour of solar oscillations in a model including a detailed, semi-empirical atmosphere. Equations of radial and non-radial oscillation, with consistent treatment of the radiation field, are derived. These equations are solved in the radial, grey case; the results show that departure from the Eddington approximation has little effect on the properties of the oscillations. Preliminary results are presented for the non-radial case, indicating substantial deviations from the Eddington approximation when the optical thickness across a horizontal wavelength is of the order of or less than unity.

1. Introduction

A consistent calculation of solar (or indeed stellar) oscillations must take into proper account the interaction between the hydrodynamics of the motion and the radiative energy transfer: the radiation contributes to the heating or cooling of the gas and so affects the dynamics of the oscillation. In addition observations of solar oscillations are made through their effect on the solar spectrum of radiation and so the diagnostics of such oscillations depends on an understanding of how they affect the radiation field.

The dynamical effect of the radiation occurs through the energy equation which relates two of the thermodynamic fluctuations, e.g. of pressure and density, through the fluctuation in the divergence of the flux of energy. The radiative flux has traditionally been treated in the diffusion approximation, although more recently the Eddington approximation (Unno and Spiegel, 1966) has been commonly applied. These approximations both become exact in the limit of large optical depth, and so they are probably adequate for calculations of global properties of the oscillations. This is true in particular for low order modes that reside predominantly in the solar interior. On the other hand

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the solar atmosphere makes a significant contribution to the damping of the solar 5 min oscillations (e.g. Ando and Osaki, 1975, 1977; see also Section 4 below). Furthermore the very high accuracy now attained in the observational determination of the frequencies of the 5 min modes of low degree (Grec *et al.*, 1980; Claverie *et al.*, 1981) suggests that effects of the atmosphere on the computation of oscillation frequencies, although certainly small, may eventually become significant; thus it was shown by Christensen-Dalsgaard and Gough (1980) that for modes close to the upper end of the observed frequency range the atmospheric boundary condition has some influence on the eigen-frequencies. Finally it seems likely that the dynamics of the possible chromospheric mode (Ando and Osaki, 1977; Ulrich and Rhodes, 1977), trapped between the temperature minimum and the transition region, is affected by the treatment of radiative transfer.

The interpretation of observations of solar oscillations in terms of their behaviour in the atmosphere requires theoretical calculations of the response of the features in the solar spectrum to the oscillations. The simplest, and least reliable, approach is to assume that an observed quantity, e.g. a lineshift in a given part of a spectral line, can be ascribed an effective height of formation, determined as an average over a contribution or response function for the observed quantity, such that the observation characterizes the oscillation at this height (e.g. Fossat and Ricort, 1975; Schmieder, 1976, 1978). Thus a lineshift would be interpreted as measuring the oscillation velocity at a given height in the atmosphere. This procedure, however, clearly ignores the variation of the oscillation through the region where the spectral line is formed. To go beyond it one must make a detailed calculation of the perturbations in the atmosphere associated with the oscillation and of the effects of these perturbations on the observed quantities. The latter part of the process has been considered by, e.g., Mein (1971), Canfield (1976), Cram *et al.* (1979), Gouttebroze and Leibacher (1980), and Keil (1980), and clearly involves the interaction between the radiation field, typically in a spectral line, and the oscillation. However radiative effects may also be important in the calculation of the variation of the oscillations with height in the atmosphere.

The diagnostic part of the theory of oscillations in the solar atmosphere has become increasingly important as observations relating to the atmospheric behaviour of the oscillations have become more detailed. These observations include well-resolved two-dimensional power spectra ($k - \omega$ diagrams) in chromospheric spectral lines in addition to the commonly observed photospheric lines (Rhodes *et al.*, 1983) as well as a resolved $k - \omega$ diagram in continuum intensity (Brown and Harrison, 1980), and the whole-disk measurements of intensity oscillations by Woodard and Hudson (1983). In addition there is a large volume of observations with less resolution in space and time, but with simultaneous measurements of line shifts and intensities in several different spectral lines, or in different parts of the same line (e.g. Cram, 1978; Lites and Chipman, 1979; Stebbins *et al.*, 1980). Such observations give potentially detailed information about the variation in oscillation amplitude and phase with the height in the atmosphere, and are thus of great interest as tests of the theory of oscillation. Furthermore, once the theoretical treatment of the oscillations has been sufficiently developed, observations of phase and amplitude variations might possible be used to infer properties of the mean structure of the atmosphere, and thus supplement traditional atmospheric diagnostics.

A special problem is presented by the SCLERA observations of oscillations in the limb darkening function close to the solar limb (e.g. Brown *et al.*, 1978). When interpreted as straight displacement of the solar limb these observations yield amplitudes apparently in contradiction to other observations in the same period range. Calculations by Hill *et al.* (1978) which treated the radiation field in the grey Eddington approximation showed that this discrepancy could be explained by taking into account the variation in the intensity perturbation with the distance from the limb; but this required modifications to the outer mechanical boundary condition that so far have received little additional support. Further, more detailed, oscillation calculations are needed to throw light on this problem.

Apart from calculations using the grey Eddington approximation or the even simpler Newton's law of cooling (e.g. Noyes and Leighton, 1963) relatively little work has been done on the atmospheric behaviour of oscillations in the Sun or other stars. Davis (1971) described a technique for including the radiation field in non-linear computations of radial pulsations of Cepheids. Similar calculations have been made by Karp (1975). Kalkofen and Ulmschneider (1977) presented a method for including radiative transfer in the calculation of radially propagating, non-linear waves in the solar atmosphere. Apparently the only attempt at a better treatment of radiative transfer in linear theory of solar oscillations was made by Schmieder (1977) who considered radially propagating waves in the photosphere. Her calculations essentially used the Eddington approximation, but she went beyond the grey case by considering the radiation field at a number (4 or 6) of different wavelengths. It might also be pointed out that the problem of convective stability, taking into account a consistent, grey radiation field, was studied by Legait (1982a, b), and that detailed dynamical models of the solar granulation with non-grey radiation have been developed by, e.g., Levy (1974) and Nordlund (1982).

There is little doubt that non-linear effects are important for the behaviour of waves in the solar atmosphere, especially in the chromosphere where the combined velocity amplitude approaches the sound speed. On the other hand linear calculations are computationally far less costly than the corresponding non-linear calculations, and so permit a more detailed exploration of the dependence of the oscillations on the parameters of the problem. This, together with the relative simplicity of linear oscillations, make such calculations well suited for attempts to understand the basic physics of the oscillations and perhaps to delimit the regions of validity of commonly used approximations. Furthermore the wave amplitudes are probably sufficiently small in the lower parts of the atmosphere that the results of linear theory provides a reasonably realistic basis for comparisons with observations.

We present initial results of a project aimed at a consistent calculation of linear oscillations in the solar interior and atmosphere, non-radial as well as radial, with as detailed a treatment of the radiation field as permitted by limitations in the available computational resources. It is hoped eventually to include the radiation field at a sufficient number of frequency points to give a realistic description of the variation in the continuum, and to study the effect of spectral lines on the oscillations; the formalism we have developed is sufficiently general to cover these cases. However in this initial paper results are only given for the grey case, where the absorption coefficients are assumed to be frequency independent, and a major concern is to test the accuracy of the Eddington approximation. Section 2 presents the equations of radial oscillations in the solar atmosphere, and discusses the use of variable Eddington factors (e.g. Auer and Mihalas, 1970) to treat the directional dependence of the radiation. In Section 3 we describe the setting up of the equilibrium solar model which consists of a semi-empirical atmosphere model matched to an envelope model. Section 4 presents results on the radial oscillations of this model. In Section 5 is discussed the generalization of the variable Eddington factor technique to non-radial oscillations, and preliminary results are given which indicate that departures from the standard Eddington approximation may become quite important at large horizontal wavenumbers. Finally Section 6 contains a discussion of the results and considers future directions for our work.

2. Equations and Boundary Conditions for Radial Oscillations

We consider small-amplitude radial oscillations about an equilibrium state and linearize the equations in the perturbations. For simplicity only the plane-parallel case with constant gravity is treated here; the full set of spherical equations are presented in the Appendix.

As usual (e.g. Cox, 1980) the perturbations must satisfy the continuity equation which in the plane-parallel case may be written

$$\frac{\mathrm{d}\delta r}{\mathrm{d}r} = -\frac{\delta\rho}{\rho}\,; \tag{2.1}$$

here r is a vertical coordinate increasing outwards, δr is the displacement, ρ the density and δ denotes the amplitude of the Lagrangian perturbation (i.e. the perturbation following the motion; e.g. Cox, 1980).

The equation of motion requires a little more care. The atmospheric model includes a turbulent contribution p_{turb} to the total pressure which may thus be written $p_{\mathrm{tot}} = p + p_{\mathrm{turb}}$, where p is the gas pressure; p_{turb} is determined from the turbulent velocity (the so-called microturbulence) introduced to account for the observed line spectrum. The equation of hydrostatic support is supposed to involve p_{turb}, so that

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -g\rho - \frac{\mathrm{d}p_{\mathrm{turb}}}{\mathrm{d}r} = -g\rho + f_{\mathrm{turb}}\rho\,, \tag{2.2}$$

which defines the turbulent body force f_{turb} (e.g. Mihalas and Toomre, 1981); here g is gravity. Thus the perturbed equation of motion is

$$\rho\frac{\partial^2\delta\mathbf{r}}{\partial t^2} = -\nabla p' - \rho'(g - f_{\mathrm{turb}})\mathbf{a}_r - \rho(\mathbf{g}' - \mathbf{f}'_{\mathrm{turb}})\,, \tag{2.3}$$

where the prime denotes the Eulerian perturbation and $\mathbf{a}_r$ is a unit vector directed vertically upwards. (We have neglected the effects of radiation pressure.)

In the absence of a definite physical model for f_{turb} the specification of $\mathbf{f}'_{\text{turb}}$ is largely arbitrary. Here we shall assume that the Lagrangian perturbation $\delta\mathbf{f}_{\text{turb}}$ vanishes; in this we differ from Mihalas and Toomre (1981) who took instead $\mathbf{f}'_{\text{turb}} = 0$. Taking $\delta\mathbf{f}_{\text{turb}} = 0$ ensures that an infinitely slow uniform vertical displacement of the layer is a solution to the resulting oscillation equations, as it certainly should be; on the other hand this clearly does not guarantee the validity of the approximation at higher frequencies. Then

$$\mathbf{f}'_{\text{turb}} = -\mathbf{a}_r \frac{\mathrm{d}\mathbf{f}_{\text{turb}}}{\mathrm{d}r}\,\delta r\,; \tag{2.4}$$

if the perturbation is assumed to depend on time as $\exp(-i\omega t)$ and we return to the plane-parallel, constant-gravity case it is easy to show that

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta p}{p}\right) = \frac{\rho}{p}\left(\tilde{g}\frac{\delta p}{p} + \omega^2\,\delta r\right), \tag{2.5}$$

where $\tilde{g} \equiv g - f_{\text{turb}}$ is an effective gravity. We neglect the turbulent pressure in the convection zone.

The energy equation may be written in the general form

$$i\omega\left(\frac{\delta T}{T} - \nabla_{\text{ad}}\frac{\delta p}{p}\right) = \frac{1}{\rho c_p T}\,\delta(\operatorname{div}\mathscr{F}_{\text{tot}}), \tag{2.6}$$

where T is temperature, $\nabla_{\text{ad}} = (\partial \ln T/\partial \ln p)_s$, s being specific entropy and c_p is the specific heat at constant pressure. $\mathscr{F}_{\text{tot}}$ is the total energy flux, which may be written as

$$\mathscr{F}_{\text{tot}} = \mathscr{F}_R + \mathscr{F}_C + \mathscr{F}_M\,, \tag{2.7}$$

where $\mathscr{F}_R$ is the radiative flux, integrated over frequency, $\mathscr{F}_C$ is the convective flux, and $\mathscr{F}_M$ is a 'mechanical' flux which must be invoked to explain the heating of the chromosphere. The perturbation of the radiative flux is the major subject of the present paper, and we return to it shortly. The perturbation of $\mathscr{F}_C$ causes problems because no definitive theory of time-dependent convection exists; furthermore even the physical mechanism responsible for $\mathscr{F}_M$ is uncertain, and so here the difficulties are even greater. Although expressions for the perturbation in the convective flux have been derived in the mixing-length formalism (e.g. Unno, 1967; Gough, 1977) we have chosen here to neglect the contributions to Equation (2.6) from $\mathscr{F}_C$ as well as $\mathscr{F}_M$, i.e. to assume

$$\delta(\operatorname{div}\mathscr{F}_C) = \delta(\operatorname{div}\mathscr{F}_M) = 0\,. \tag{2.8}$$

This represents a major uncertainty in the present work, which must be kept in mind when evaluating the results. Nevertheless one may hope that it does not invalidate at least the overall features of the behaviour of the oscillations in the solar atmosphere.

To determine the perturbation in the radiative flux we must consider the equation of transfer. The radiation field in the equilibrium state satisfies

$$\mu \frac{\partial I_\nu(r, \mu)}{\partial r} = \rho \kappa_{s,\nu} J_\nu + \rho \kappa_{a,\nu} B_\nu - \rho(\kappa_{s,\nu} + \kappa_{a,\nu}) I_\nu \tag{2.9}$$

(e.g. Mihalas, 1978); for simplicity we have assumed isotropic scattering and local thermodynamical equilibrium (LTE). Here $I_\nu(r, \mu)$ is the specific intensity at position r and frequency ν at the angle $\vartheta = \cos^{-1}\mu$ from the vertical; $\kappa_{s,\nu}$ and $\kappa_{a,\nu}$ are the monochromatic scattering and absorption coefficients, respectively, B_ν the Planck function and

$$J_\nu = \frac{1}{4\pi} \oint I_\nu \, \mathrm{d}\Omega \tag{2.10}$$

is the mean intensity, the integration being over solid angle Ω. We also introduce the higher moments

$$H_\nu = \frac{1}{4\pi} \oint \mu I_\nu \, \mathrm{d}\Omega , \tag{2.11}$$

$$K_\nu = \frac{1}{4\pi} \oint \mu^2 I_\nu \, \mathrm{d}\Omega ; \tag{2.12}$$

here H_ν is related to the radiative flux by

$$\mathcal{F}_R = 4\pi \int_0^\infty H_\nu \, \mathrm{d}\nu \, \mathbf{a}_r , \tag{2.13}$$

and K_ν is related to the radiation pressure. By integrating Equation (2.9) over μ and ν we obtain

$$\operatorname{div} \mathcal{F}_R = \frac{\mathrm{d}\mathcal{F}_{R,r}}{\mathrm{d}r} = 4\pi\rho \int_0^\infty \kappa_{a,\nu}(B_\nu - J_\nu) \, \mathrm{d}\nu , \tag{2.14}$$

where $\mathcal{F}_{R,r}$ is the radial component of $\mathcal{F}_R$.

We shall assume that the motion is non-relativistic. Then the time derivative in the complete equation of transfer (e.g. Mihalas, 1980) may be neglected. If we furthermore neglect the effect of lines and other regions with sharp gradients in I_ν as a function of ν (where otherwise Doppler-shift effects caused by velocity gradients might become important) Equation (2.14) may be perturbed in a straightforward manner; substituting the result into the energy equation (2.6), using Equations (2.7) and (2.8), we finally get

$$i\omega\left(\frac{\delta T}{T} - \nabla_{\text{ad}}\frac{\delta p}{p}\right) = \frac{1}{\rho c_p T}\,\mathrm{div}\,\mathscr{F}_R\frac{\delta\rho}{\rho} + \\ + \frac{4\pi}{c_p T}\int_0^\infty [\delta\kappa_{a,\nu}(B_\nu - J_\nu) + \kappa_{a,\nu}(\delta B_\nu - \delta J_\nu)]\,\mathrm{d}\nu. \tag{2.15}$$

Here $\delta\rho/\rho$ and $\delta\kappa_{a,\nu}$ can be expanded in terms of $\delta T/T$ and $\delta p/p$ as

$$\frac{\delta\rho}{\rho} = \rho_T\frac{\delta T}{T} + \rho_p\frac{\delta p}{p}, \qquad \frac{\delta\kappa_{a,\nu}}{\kappa_{a,\nu}} = (\kappa_{a,\nu})_T\frac{\delta T}{T} + (\kappa_{a,\nu})_p\frac{\delta p}{p}, \tag{2.16}$$

where e.g., $\rho_T = (\partial\ln\rho/\partial\ln T)_p$ and may be determined from the equation of state. Furthermore B_ν is an explicit function of T and so δB_ν can immediately be found as

$$\delta B = \frac{\mathrm{d}B_\nu}{\mathrm{d}T}\delta T. \tag{2.17}$$

Thus Equation (2.15) can be written as

$$\left(i\omega - \frac{4\pi}{c_p T}\int_0^\infty\left\{[(\kappa_{a,\nu})_T + \rho_T](B_\nu - J_\nu) + T\frac{\mathrm{d}B_\nu}{\mathrm{d}T}\right\}\kappa_{a,\nu}\,\mathrm{d}\nu\right)\frac{\delta T}{T} = \\ = \left\{i\omega\nabla_{\text{ad}} + \frac{4\pi}{c_p T}\int_0^\infty [(\kappa_{a,\nu})_p + \rho_p](B_\nu - J_\nu)\,\kappa_{a,\nu}\,\mathrm{d}\nu\right\}\frac{\delta p}{p} - \\ - \frac{4\pi}{c_p T}\int_0^\infty \kappa_{a,\nu}\delta J_\nu\,\mathrm{d}\nu. \tag{2.18}$$

This equation can clearly be used to eliminate $\delta T/T$ in terms of $\delta p/p$ and the δJ_ν.

To close the system of equations we therefore need equations for the δJ_ν. By perturbing Equation (2.9) it follows that

$$\mu\frac{\partial\delta I_\nu}{\partial r} = \rho[\delta\kappa_{s,\nu}J_\nu + \delta\kappa_{a,\nu}B_\nu - (\delta\kappa_{s,\nu} + \delta\kappa_{a,\nu})I_\nu + \\ + \kappa_{s,\nu}\delta J_\nu + \kappa_{a,\nu}\delta B_\nu - (\kappa_{s,\nu} + \kappa_{a,\nu})\delta I_\nu]; \tag{2.19}$$

here δI_ν and δJ_ν are related by

$$\delta J_\nu(r) = \frac{1}{4\pi} \oint \delta I_\nu(r, \mu)\, \mathrm{d}\Omega. \tag{2.20}$$

Thus one might in principle include $\delta I_\nu(r, \mu)$, at suitable discrete points in ν and μ, as variables, and solve the corresponding Equation (2.19) together with Equations (2.1) and (2.5). It is evident that reasonable resolution in ν and μ may result in a very large set of equations.

As is common in the theory of stellar atmospheres, it is more efficient to work in terms of moments of the radiation field. We introduce δH_ν and δK_ν as

$$\delta H_\nu(r) = \frac{1}{4\pi} \oint \mu \delta I_\nu(r, \nu)\, \mathrm{d}\Omega, \tag{2.21}$$

$$\delta K_\nu(r) = \frac{1}{4\pi} \oint \mu^2 \delta I_\nu(r, \nu)\, \mathrm{d}\Omega. \tag{2.22}$$

The zeroth and first moments of Equation (2.19) then yield

$$\frac{\mathrm{d}\delta H_\nu}{\mathrm{d}r} = \rho[\delta\kappa_{a,\nu}(B_\nu - J_\nu) + \kappa_{a,\nu}(\delta B_\nu - \delta J_\nu)], \tag{2.23}$$

$$\frac{\mathrm{d}\delta K_\nu}{\mathrm{d}r} = -\rho[\delta\kappa_\nu H_\nu + \kappa_\nu \delta H_\nu], \tag{2.24}$$

where $\kappa_\nu = \kappa_{s,\nu} + \kappa_{a,\nu}$ is the total opacity. To close the system of equations we now use the variable Eddington-factor technique (e.g. Auer and Mihalas, 1970), by introducing the Eddington-factors

$$f_{\mathrm{osc},\nu} = \delta K_\nu(r)/\delta J_\nu(r). \tag{2.25}$$

Had $f_{\mathrm{osc},\nu}$ been known the equations would clearly form a closed system. This is not the case, but we may determine $f_{\mathrm{osc},\nu}$ by iteration. As an initial gues we take $f_{\mathrm{osc},\nu} = f_{\mathrm{eq}} \equiv K_\nu/J_\nu$ which is known from the equilibrium solution. Equations (2.1), (2.5), (2.23), and (2.24) are then solved, to give a first estimate of the perturbations. This solution is substituted into the right-hand side of Equation (2.19) which may then be integrated for δI_ν; this integration can be done by quadrature, independently at each point in ν and μ, and so it is extremely fast. Finally a new estimate of $f_{\mathrm{osc},\nu}$ is obtained from Equations (2.20), (2.22), and (2.25). This iteration may be performed simultaneously with the iteration for ω which is in general an eigenvalue of the problem, and so it is quite efficient. In this way the number of radiation variables is reduced by a factor corresponding to the number of μ-points, at the expense of a few additional iterations. It might be noticed that the standard Eddington approximation corresponds to taking $f_{\mathrm{osc},\nu} = \frac{1}{3}$.

We still have to specify the boundary conditions. The outer surface of the model, at $r = r_s$ say, is assumed to correspond to the lower boundary of an isothermal corona; by making its temperature different from that of the last point in the model we can simulate the rapid temperature rise in the outer part of the transition region. A condition on δr and $\delta p/p$ at this point is then obtained by requiring the solution to match continuously to an outward propagating, adiabatic wave in the corona (e.g. Cox, 1980). The conditions on the perturbations in the radiation field are that there be no incoming radiation at the surface, i.e.

$$\delta I_\nu(r_s, \mu) = 0 \quad \text{for} \quad \mu < 0\,. \tag{2.26}$$

When using the variable Eddington factors these are replaced by the conditions

$$\delta H_\nu(r_s) = g_{\text{osc},\,\nu}\,\delta J_\nu(r_s)\,, \tag{2.27}$$

where the $g_{\text{osc},\,\nu}$ form another set of variable Eddington factors that may be determined in the iteration for $f_{\text{osc},\,\nu}$; the conditions (2.26) are then applied in the quadrature to determine δI_ν from Equation (2.19).

In a complete stellar model (where one must clearly use the spherical equations) the solution has to satisfy regularity conditions at the centre and these, together with the surface conditions, determine ω as an eigenvalue of the problem. Here we shall be concerned with models truncated at a finite distance from the centre, and so we need a separate set of conditions. From the energy Equation (2.15) follows that the solution becomes increasingly adiabatic with increasing depth, and we shall assume that at the bottom boundary, $r = r_b$, the solution is exactly adiabatic, i.e.

$$\frac{\delta T}{T} = \nabla_{\text{ad}} \frac{\delta p}{p} \quad \text{at} \quad r = r_b\,. \tag{2.28}$$

A more realistic condition that takes into account departures from adiabaticity has been formulated by Dziembowski (1977), but if the lower boundary is at a sufficient depth the condition (2.28) is adequate. It may also be shown (as in the equilibrium case, e.g. Mihalas, 1978) that the radiation field tends to the diffusion approximation at great depths. This determines the variation of δH_ν with ν, leaving the ν-integrated flux perturbation

$$\delta H = \int_0^\infty \delta H_\nu \,\mathrm{d}\nu \tag{2.29}$$

undetermined, and thus provides an additional $N_f - 1$ conditions, where N_f is the number of points in ν.

The conditions specified so far permit a solution for any value of ω, because no mechanical condition has been invoked at the lower boundary. However it is of interest to look for standing wave solutions in the truncated model, and to do so we have to impose an, essentially artificial, mechanical condition. Here we shall take the condition that the displacement vanish, i.e.

$$\delta r = 0 \quad \text{at} \quad r = r_b\,. \tag{2.30}$$

Alternatively one might look for solutions at given period (or real part of ω), regarding the growth or damping rate (i.e. the imaginary part of ω) as a real eigenvalue. This requires one real condition at the lower boundary which we take to be that the part of the model interior to r_b perform no work on the part exterior to r_b, i.e.

$$\mathrm{Im}\,(\delta r^* \delta p) = 0 \quad \text{at} \quad r_b\,. \tag{2.31}$$

Finally let us consider the grey case, where $\kappa_{a,\nu} = \kappa_a$ and $\kappa_{s,\nu} = \kappa_s$ are assumed to be independent of ν. Then Equation (2.18) becomes

$$\{i\omega - \omega_R[4 - \Delta_c(\kappa_{a,T} + \rho_T)]\}\,\frac{\delta T}{T} = $$

$$= [i\omega\nabla_{\mathrm{ad}} - \omega_R\Delta_c(\kappa_{a,p} + \rho_p)]\,\frac{\delta p}{p} - \omega_R\,\frac{\delta J}{B}\,, \tag{2.32}$$

where B is the frequency-integrated Planck function, $\delta J = \int_0^\infty \delta J_\nu\,\mathrm{d}\nu$ and we have introduced

$$\Delta_c = J/B - 1 = -\,\frac{\operatorname{div}\mathscr{F}_R}{4\pi\rho\kappa_a B} \tag{2.33}$$

as a measure of the departure from radiative equilibrium, J being the integrated mean intensity; furthermore

$$\omega_R = \frac{4\pi B\,\kappa_a}{c_p T} \tag{2.34}$$

is a characteristic radiative relaxation rate in the optically thin limit related to the one introduced by Spiegel (1957). Equation (2.32) is similar to Equation (12) of Ando and Osaki (1975), but as they assumed that $J = B$ everywhere they did not have the terms in Δ_c. This is equivalent to assuming that the equilibrium model is in radiative equilibrium everywhere, an assumption that is invalid not only in a mechanically heated atmosphere but also in the upper part of the convection zone where there is a transition from radiative to convective energy transport (deeper down in the solar convection zone energy is carried almost exclusively by convection and so once again Δ_c is negligible). As shown in Section 4 the inclusion of the terms in Δ_c appears to have a fairly strong stabilizing effect on acoustic modes in the Sun.

After integration over ν Equations (2.19) and (2.24) are unchanged, apart from the removal of the subscripts ν, whereas Equation (2.23) may be written, using

Equation (2.32), as

$$\frac{\mathrm{d}\delta H}{\mathrm{d}r} = \rho B \kappa_a \left\{ \frac{4[i\omega\nabla_{\mathrm{ad}} - \omega_R \Delta_c(\kappa_{a,p} + \rho_p)]\dfrac{\delta p}{p} - [i\omega + \omega_R \Delta_c(\kappa_{a,T} + \rho_T)]\dfrac{\delta J}{B}}{i\omega - \omega_R[4 - \Delta_c(\kappa_{a,T} + \rho_T)]} - \right.$$

$$\left. - \Delta_c \frac{\delta\kappa_a}{\kappa_a} \right\}. \tag{2.35}$$

(By writing the equation in this apparently more complicated form we avoid difficulties that might otherwise be caused by the near cancellation of δJ and δB in the interior of the model.) The only inner thermal boundary condition is now Equation (2.28). The variable Eddington factor method may be used as before, to iterate for $f_{\mathrm{osc}}(r) = \delta K(r)/\delta J(r)$ and $g_{\mathrm{osc}} = \delta H(r_s)/\delta J(r_s)$. Notice that there are now only four perturbation quantities (e.g. δr, $\delta p/p$, δH, and δK) and so the order of the system is the same as when the Eddington approximation, or the diffusion approximation, is used.

3. The Equilibrium Model

The solar model we have considered consists of a deep envelope matched smoothly to a semi-empirical atmospheric model. The envelope model is constructed by integrating the equations of hydrostatic support, continuity and radiative or convective energy transport inwards, assuming constant luminosity, from boundary conditions determined at the bottom of the atmosphere. The physics of the envelope is, with a few exceptions discussed below, as in the solar evolution calculation of Christensen-Dalsgaard (1982): the equation of state of Eggleton *et al.* (1973), opacity tables from Cox and Tabor (1976) interpolated with stretched splines (Cline, 1974) and convection treated with mixing length theory in the form of Baker and Temesváry (1966).

The atmospheric model is based on a model similar, but not identical, to model C of Vernazza *et al.* (1981) and was kindly supplied by A. Skumanich. This model, however, had to be somewhat modified to make it suitable for oscillation calculations (Mihalas and Toomre, 1981). In particular the turbulent body force f_{turb}, defined by Equation (2.2), had erratic variations in some parts of the original model above the temperature minimum. These were removed by resetting f_{turb} smoothly where needed and redetermining p by integrating Equation (2.2), keeping p fixed at the top and bottom of the atmosphere. All number densities were scaled with the ratio β between the new and the old p, thus keeping all degrees of ionization the same; the value of β was always between 0.9 and 1.1.

The mesh in the original model had only about 50 points from the bottom of the atmosphere to the base of the corona. This is insufficient to resolve adequately the variations in the oscillation quantities, and so the mesh had to be reset by interpolation. The distribution of the new mesh was based largely on the variation in density and mass, but with additional points in the region of rapid temperature rise from 10000 K to

24 000 K and at the temperature plateau at 24 000 K. To keep $f_{\rm turb}$ smooth p was reset from the interpolated $f_{\rm turb}$; this required changes in p of less than 1.2% from the directly interpolated value. The final mesh had 300 points in the atmosphere.

It might be argued that the fluctuations in $f_{\rm turb}$ represent a physical effect and that accordingly they should not be eliminated. However the fluctuations are inadequately resolved in the original model and so they cannot be reliably interpolated. We have therefore preferred to remove them. Future, more detailed, atmospheric models are needed to demonstrate their reality.

In the final, interpolated model the maximum deviation in pressure from the original model caused by the smoothing of $f_{\rm turb}$ is about 9% and occurs at the 24 000 K temperature plateau. For temperatures less than 9000 K the deviation is less than 5% everywhere. Thus the smoothing has a relatively small effect on the thermodynamics of the atmosphere.

A number of thermodynamic derivatives, not provided in the original atmosphere, are needed in the oscillation calculation. Of course the notion of an equation of state is not well-defined in the upper part of the atmosphere where LTE is no longer a good approximation. Here, ideally, one should perturb the statistical equilibrium equations (e.g. Mihalas, 1978) which provide a direct coupling between the radiation field and the thermodynamic state of the gas. However this would greatly increase the complexity of the calculation, and in the present work we have avoided such complications. Following Mihalas (1979) we have taken some account of the effect of NLTE by introducing departure coefficients, found from the equilibrium model, in the Saha and Boltzman's equations; these were assumed to be unaffected by the oscillation. The equation of state was otherwise as in Christensen-Dalsgaard (1982), with complete calculation of the ionization of C, N, and O; however here we also include the ionization of the first electron of the 7 most abundant among the remaining heavy elements.

The departure coefficient $d_{\rm H}$ for hydrogen ionization was taken from the original model. The ionization of He and heavy elements was calculated with a ficticious mean departure coefficient d_Z, assumed to be the same for all elements and all levels of ionization; d_Z was found by iteration, to make the density found from the equation of state calculation, at given temperature and electron density, the same as the density in the original atmosphere. In a limited region around a temperature of 10^4 K d_Z was smoothed to remove fairly drastic fluctuations in the opacity derivatives. This smoothing caused a maximum deviation in density of about 0.4%.

The composition of the atmosphere model assumed a ratio $\alpha_{\rm He}$ between the number densities of He and H of 0.1. A direct, spectroscopic determination of $\alpha_{\rm He}$ is of course difficult; it may be that evolution calculations, where $\alpha_{\rm He}$ is regarded as a free parameter which is adjusted until the luminosity of the model of the present Sun agrees with the observed value, offer a more reliable determination of $\alpha_{\rm He}$. The values obtained are generally somewhat smaller than 0.1. In the present calculation we have preferred to use the value 0.0858 determined from the evolution calculation of Christensen-Dalsgaard (1982), to obtain an envelope that is as closely as possible consistent with the complete solar model. The atmosphere therefore had to be adjusted for the change in composition.

This was done by regarding the relation between temperature and the total hydrogen number density n_{H} as fixed; d_{H} and d_Z were determined as before, using the original α_{He}, and the electron number density was then found by iteration, keeping d_{H} and d_Z fixed, to get the density corresponding to the new composition. By assuming that p_{turb}/ρ had the same dependence on n_{H} as in the original model we could then determine the total pressure and hence the height and mass scales, using the equations of hydrostatic support and continuity. This procedure is clearly not strictly correct, but in view of the relatively small change in α_{He} it is probably adequate.

The opacity in the atmosphere was calculated using a programme described by Auer *et al.* (1972), taking into account contributions from H, H^-, H_2^+, C, Si, and Mg. In the present, grey, calculation the value $\kappa_{A,5000}$ at 5000 Å was used. This is fairly close to, and much faster to compute than, the Rosseland mean; furthermore it agrees with the Rosseland mean κ_{CT} found from the tables of Cox and Tabor (1976) at a temperature of about 6300 K, over a wide range in density. In the photosphere a smooth matching between $\kappa_{A,5000}$ and κ_{CT} was therefore accomplished by computing

$$\log \kappa^* = \phi(y) \log \kappa_{CT} + [1 - \phi(y)] \log \kappa_{A,5000}\,, \tag{3.1}$$

where $y = \Delta_m^{-1} \log(T/T_m)$,

$$\phi(y) = \begin{cases} 0 & \text{for} \quad y < -1 \\ \frac{1}{2} + \frac{3}{4} y(1 - \frac{1}{3}y^2) & \text{for} \quad |y| \le 1 \\ 1 & \text{for} \quad y > 1\,, \end{cases} \tag{3.2}$$

and T_m and Δ_m were 6300 K and 0.1, respectively. This ensures that κ^* and its first derivatives with respect to T and ρ are continuous. Above the temperature minimum κ^* was set to $\kappa_{A,5000}$. To compensate for the fact that $\kappa_{A,5000}$ is different from the opacity used in constructing the atmospheric model the final atmospheric opacity κ was obtained as $\kappa = \alpha_{\mathrm{op}} \kappa^*$, where α_{op} in the atmosphere is a constant determined such that the optical depth at the bottom of the atmosphere, at 5000 Å, was the same as in the original model.

In the integration in the interior we neglect the turbulent pressure, so that $f_{\mathrm{turb}} = 0$. We furthermore assume strict LTE, by setting all departure coefficients to unity, use the opacity as given by Equation (3.1) (i.e. take $\alpha_{\mathrm{op}} = 1$) and assume no mechanical (as distinct from convective) energy flux, i.e. $\mathscr{F}_M = 0$. To get a smooth transition from the atmosphere to the interior we determine, rather arbitrarily, the values of f_{turb}, d_{H}, d_Z, α_{op}, and $\mathscr{F}_M$ in the upper part of the interior as

$$A(\tau) = [1 - \phi(\hat{y})] A_{\mathrm{bot}} + \phi(\hat{y}) A_{\mathrm{int}}, \tag{3.3}$$

where $A_{\mathrm{bot}} = A(\tau_{\mathrm{bot}})$ is the value of any of these quantities at the bottom of the atmosphere, A_{int} is its value in the interior and τ is optical depth; ϕ is defined in Equation (3.2) and $\hat{y} = (\tau - \tau_{\mathrm{bot}})/\Delta\tau - 1$. The bottom of the atmosphere was chosen to be at $\tau_{\mathrm{bot}} = 0.44$ (corresponding to $T = 5840$ K), and $\Delta\tau = 1$ was used in the smoothing.

As the envelope integration starts at a fairly small optical depth we cannot treat the radiative flux in the diffusion approximation. Instead we use the grey moment equation

$$\frac{\mathrm{d}K}{\mathrm{d}r} - \frac{1}{r}\left(\frac{1}{f_{\mathrm{eq}}} - 3\right)K = -\kappa\rho H, \tag{3.4}$$

together with

$$B = \frac{K}{f_{\mathrm{eq}}(1 + \Delta_c)} \tag{3.5}$$

which follows from Equation (2.32) and the definition of f_{eq}. Thus the temperature gradient is

$$\nabla \equiv \frac{\mathrm{d}\ln T}{\mathrm{d}\ln p} = $$

$$= \frac{1}{4}\frac{P}{\rho\tilde{g}}\left[\frac{\kappa\rho H}{K} - \frac{1}{r}\left(\frac{1}{f_{\mathrm{eq}}} - 3\right) + \frac{\mathrm{d}\ln f_{\mathrm{eq}}}{\mathrm{d}r} + (1 + \Delta_c)^{-1}\frac{\mathrm{d}\Delta_c}{\mathrm{d}r}\right], \tag{3.6}$$

where Equation (2.2) was used. To make a smooth transition in ∇ between the atmosphere and the interior we replaced H in Equations (3.4) and (3.6) by $\Phi(\tau)H$, where $\Phi(\tau_{\mathrm{bot}})$ was determined from the condition that ∇ be continuous at the bottom of the atmosphere, and $\Phi(\tau)$ was found as in Equation (3.3), with $\Phi_{\mathrm{int}} = 1$; $\Phi(\tau_{\mathrm{bot}})$ was always close to 1. H is found from Equation (2.7) as

$$H = \frac{1}{4\pi}(\mathscr{F}_{\mathrm{tot}} - \mathscr{F}_M - \mathscr{F}_C). \tag{3.7}$$

Finally the mixing length treatment of convection was modified appropriately.

The equations for the structure of the envelope thus depend on f_{eq} and Δ_c, and hence on the radiation field. Rather than making a full model atmosphere calculation we have determined the radiation quantities iteratively. Given f_{eq} and Δ_c as functions of τ and the value of $\mathscr{F}_{M,\mathrm{bot}}$, we can determine the envelope structure and hence integrate the equation of transfer in the envelope and atmosphere, to get new $f_{\mathrm{eq}}(\tau)$, $\Delta_c(\tau)$ and $\mathscr{F}_{M,\mathrm{bot}}$. As initial values were used $f_{\mathrm{eq}} = \frac{1}{3}$, $\Delta_c(\tau) = \mathscr{F}_{M,\mathrm{bot}} = 0$. The convergence of this iteration was rather slow; in the calculation we stopped after three iterations, where the mean change in f_{eq} and Δ_c was less than 0.3%.

The ratio α of the mixing length to the pressure scale height was determined such that the depth of the convection zone was the same as in model 1 of Christensen-Dalsgaard (1982); this required $\alpha = 1.8556$. The difference between this value and the value of 1.6364 used by Christensen-Dalsgaard reflects the change in the surface boundary condition. With this calibration the differences between the envelope model and model 1 in p, ρ, and T, compared at fixed mass, was less than 1% at the base of the convection zone. The remaining difference is probably still an effect of the difference in the surface boundary condition, and could presumably be removed by suitably modifying the boundary condition in the evolution calculation. In this way it should be possible to construct a complete model of the present Sun, matched smoothly to a detailed atmospheric model.

4. Results for Radial Oscillations with Grey Radiation

The equilibrium model described in the preceding section was truncated at a fractional radius $x = r/r_{\text{phot}}$ of 0.24, where r_{phot} is the photospheric radius, here taken to correspond to the point where $T = T_{\text{eff}}$, the effective temperature. The temperature at the bottom of the envelope was 8.2×10^6 K; at this temperature the luminosity in a complete solar model is about 0.97 times the surface value, and hence to compute a deeper envelope one would have to take into account the variation in luminosity caused by nuclear energy generation. The outer boundary of the model was taken at a temperature of 49000 K in the transition region, and a coronal temperature of 1.6×10^6 K was assumed in the outer mechanical boundary condition. The complete equilibrium model, including the atmosphere, had 992 mesh points.

The spherical oscillation equations (cf. the Appendix) were solved in the grey case using a second order centred difference scheme similar to the one used by Baker and Kippenhahn (1965) (see also Baker *et al.*, 1971). The Eddington factors were calculated using 10 point Gaussian quadrature in μ. We iterated for ω or its imaginary part until the mechanical boundary conditions (2.30) or (2.31) were satisfied.

Some indication of the possible eigenmodes of the model can be obtained by writing the equations of adiabatic oscillation in the form

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\tau_a^2} + \Omega^2\psi = 0 \tag{4.1}$$

(Christensen-Dalsgaard *et al.*, 1983), where $\psi = r(\rho c)^{1/2}\,\delta r$, c being the sound speed, and

$$\tau_a = \int_r^{r_s} \frac{\mathrm{d}r}{c} \tag{4.2}$$

is the 'acoustical depth'. Ω^2 is given by Equation (4.4) of Christensen-Dalsgaard *et al.* and may be written as

$$\Omega^2 = 4\pi^2(\nu^2 - \mathscr{V}), \tag{4.3}$$

where $\nu = \omega/2\pi$ is the cyclic frequency; this equation defines an effective 'potential' $\mathscr{V}$ for radial oscillations, such that the mode is oscillatory when $\nu^2 > \mathscr{V}$ and evanescent when $\nu^2 < \mathscr{V}$.

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The behaviour of $\mathscr{V}$ in the upper part of the model, as a function of the height $h = r - r_{\text{phot}}$ above the photosphere, is illustrated on Figure 1. $\mathscr{V}$ increases with decreasing temperature in the convection zone until close to its upper boundary. Here the rapid change in the temperature gradient associated with the strong super-adiabaticity causes a dip in $\mathscr{V}$, followed by a gradual increase until near the temperature minimum, where $\mathscr{V}$ has a maximum of height $\mathscr{V}_{\max} = \nu_c^2$, $\nu_c \approx 5.9$ mHz corresponding roughly to Lamb's (1909) acoustical cut-off frequency at the temperature minimum. Further out $\mathscr{V}$ decreases again due to the chromospheric temperature rise until the base of the transition region where the rapid temperature increase causes a rapid rise in $\mathscr{V}$. The sharp peak in $\mathscr{V}$ at zero height is probably an artifact of the matching between the atmosphere and the interior.

There is a possibility of modes with $\nu < \nu_c$ predominantly trapped in the interior of the model; these are the subphotospheric acoustic modes responsible for the 5 min oscillations. In addition to this subphotospheric cavity, however, there is also a cavity between the temperature minimum and the transition region; as first shown by Ando and Osaki (1977) and Ulrich and Rhodes (1977) in the non-radial, and R. Scuflaire (private communication) in the radial case, it is possible to trap a single chromospheric mode in this cavity. Modes with $\nu > \nu_c$ are essentially free to propagate in the entire atmosphere, and so they might be expected to experience stronger damping than modes

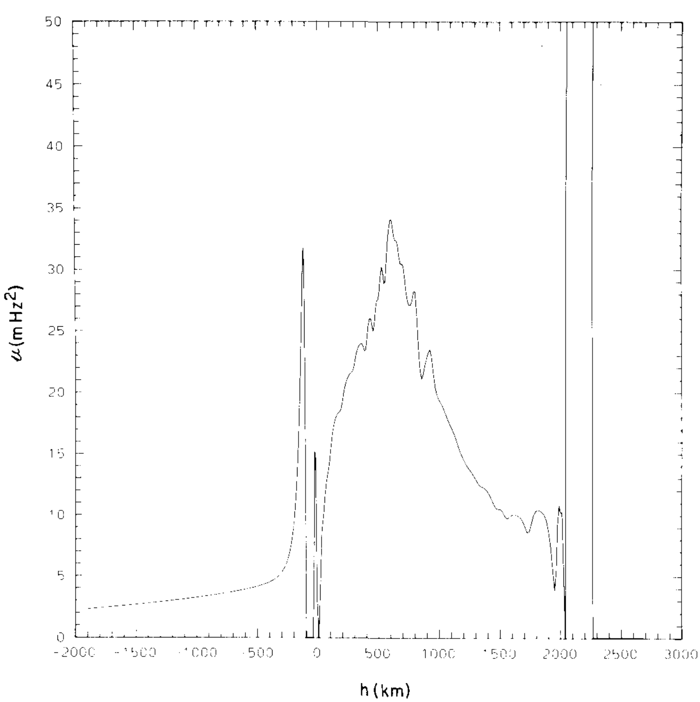

Fig. 1. The effective 'potential' $\mathscr{V}$ for radial adiabatic oscillations (cf. Equations (4.1) and (4.3)), as a function of the height h above the point where $T = T_{\text{eff}}$.

TABLE I

Results for selected eigenmodes of the complete model

(I)		(II)		(III)		(IV)		(V)	
n	ν	ν	η	ν	η	ν	η	ν	η
3	0.586 508	0.586508	-5.75×10^{-9}	0.586 508	-5.79×10^{-9}	0.586 508	-5.69×10^{-9}	0.586 508	1.80×10^{-9}
7	1.211 290	1.211289	-8.45×10^{-7}	1.211 289	-8.43×10^{-7}	1.211 289	-8.35×10^{-7}	1.211 289	1.56×10^{-7}
11	1.812 284	1.812263	-3.08×10^{-5}	1.812 264	-3.05×10^{-5}	1.812 263	-3.04×10^{-5}	1.812 261	3.06×10^{-6}
15	2.399 463	2.399172	-2.87×10^{-4}	2.399 169	-2.82×10^{-4}	2.399 158	-2.83×10^{-4}	2.399 126	2.72×10^{-6}
19	2.987 472	2.986223	-9.06×10^{-4}	2.986 160	-8.91×10^{-4}	2.986 128	-8.96×10^{-4}	2.985 939	-8.41×10^{-5}
23	3.583 508	3.580353	-1.73×10^{-3}	3.580 003	-1.70×10^{-3}	3.579 969	-1.71×10^{-3}	3.579 587	-3.36×10^{-4}
28	4.187 176	4.181323	-2.73×10^{-3}	4.180 675	-2.80×10^{-3}	4.180 649	-2.82×10^{-3}	4.179 942	-1.27×10^{-3}
32	4.793 326	4.782402	-4.34×10^{-3}	4.780 914	-4.55×10^{-3}	4.781 012	-4.57×10^{-3}	4.782 005	-3.14×10^{-3}
37	5.422 156	5.405070	-7.02×10^{-3}	5.405 052	-6.89×10^{-3}	5.405 001	-6.89×10^{-3}	5.415 151	-3.13×10^{-3}
41	6.010 853	5.977390	-5.83×10^{-3}	5.973 994	-5.50×10^{-3}	5.974 366	-5.52×10^{-3}	5.984 504	-6.25×10^{-3}
45	6.564 577	6.550810	-3.49×10^{-3}	6.550 168	-3.41×10^{-3}	6.552 798	-3.69×10^{-3}	6.541 162	-3.85×10^{-3}
49	7.124 885	7.116820	-7.31×10^{-3}	7.113 316	-8.19×10^{-3}	7.116 736	-8.01×10^{-3}	7.114 371	-5.18×10^{-3}
C	3.802 524	3.772163	-4.27×10^{-2}	3.774 059	-4.19×10^{-2}	3.774 702	-4.18×10^{-2}	3.800 963	-2.15×10^{-2}

(I) is for the adiabatic case, n being the number of modes in the eigenfunction and ν the cyclic frequency. (II) was obtained using consistently iterated Eddington factors; here η is the relative growth rate (a negative value indicating that the mode is damped). (III) used the equilibrium Eddington factors. (IV) assumed the Eddington approximation, as did (V) where in addition the terms in Δ_c were neglected (corresponding to the calculations of Ando and Osaki, 1975, 1977). The line marked 'C' gives results for the chromospheric mode. To display the fairly small differences between the different cases the results are presented with greater precision than warranted by their absolute accuracy.

with $\nu < \nu_c$; but there is still an appreciable degree of reflection in the transition region (see also Bahng and Schwarzschild, 1963).

Some results on ν and the relative damping (or growth) rate η for selected modes of the complete envelope are shown in Table I; here $\eta = \omega_i/\omega_r$ where $\omega_r = \mathrm{Re}(\omega)$ and $\omega_i = \mathrm{Im}(\omega)$. The modes were computed by imposing the boundary condition (2.30). For comparison we have included adiabatic results obtained by solving just Equations (A.1) and (A.2) and relating $\delta\rho/\rho$ and $\delta p/p$ adiabatically; these are shown in the first column of the table. The second set of values were obtained with consistently iterated Eddington factors. The remaining results show the effects of various approximations; the third set was obtained by replacing f_{osc} and g_{osc} by f_{eq} and g_{eq}, the equilibrium Eddington factors, the fourth by using the standard Eddington approximation, i.e. $f_{\mathrm{osc}} = \frac{1}{3}$, $g_{\mathrm{osc}} = \frac{1}{2}$, and the final set used the Eddington approximation and in addition assumed $J = B$ as did Ando and Osaki (1975, 1977). In addition to the internal acoustic modes, which have been labelled by the number of nodes in the adiabatic case, results are also shown for the chromospheric mode.

It is evident from the table that the treatment of radiation has a very small effect on the cyclic frequencies of oscillation, at most about 0.05%; thus until the observational accuracy has been improved and the uncertainty caused by other approximations in the theory is reduced the Eddington approximation is certainly adequate for the computation of eigenfrequencies. On the other hand the differences caused by assuming adiabatic oscillations, up to about 0.5%, are quite significant. The effects on the growth rates are considerably larger, up to a few per cent, but here the effects of other theoretical uncertainties are correspondingly larger, and so once again the Eddington approximation is probably adequate.

The Ando and Osaki approximation has a strong effect on the damping rates. In contrast to when this approximation is not made modes with frequencies less than about 2.4 mHz are found to be unstable, and for the remaining modes the damping rate is generally considerably smaller. There is also a significant effect on ν. We return to this question in more detail at the end of the section.

The behaviour of the consistent Eddington factors f_{osc} is of some interest. Figure 2 shows the difference $f_{\mathrm{osc}} - f_{\mathrm{eq}}$ for the modes of Table I. Had our treatment of the convective flux perturbation been consistent f_{osc} should have tended to f_{eq} as ω tended to zero; although this is not exactly true, $f_{\mathrm{osc}} - f_{\mathrm{eq}}$ is nevertheless small at low frequencies. With increasing ν f_{osc} departs increasingly from f_{eq}, although the departure is fairly small for frequencies less than 4.5 mHz, where most of the power in the 5 min oscillations is concentrated. The rapid variation in f_{osc} at $\nu = 6.55$ mHz is caused by δJ nearly vanishing here. In fact points where $\delta J = 0$ and $\delta K \neq 0$ are clearly singularities of f_{osc}, and so here the variable Eddington factor method formally breaks down; in practice, however, this has not caused convergence problems. More serious are zeros in δK at points where $\delta J \neq 0$; here $f_{\mathrm{osc}} = 0$, giving rise to singularities in the moment equations as they have been formulated here. In the present calculation such behaviour caused convergence problems in narrow frequency ranges around $\nu = 6.25$ and 7.6 mHz.

(a)

(b)

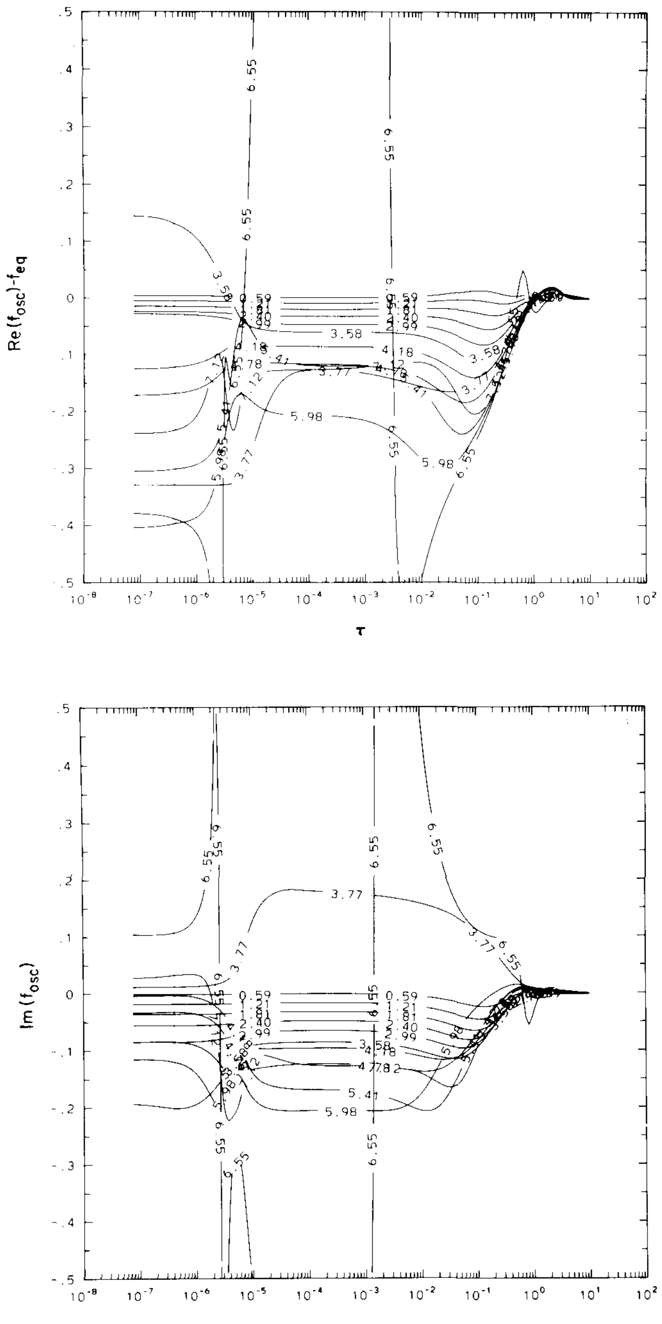

Fig. 2. Real part of $f_{\mathrm{osc}} - f_{\mathrm{eq}}$ (a) and imaginary part of f_{osc} (b), where f_{osc} is the consistently iterated Eddington factor for the oscillation and f_{eq} the equilibrium Eddington factor, as functions of the optical depth τ, for the modes of Table I. The curves are labelled with the cyclic frequencies ν of the modes, in mHz. For $\nu = 6.55$ mHz f_{osc} is almost singular at $\tau = 2 \times 10^{-3}$ and at $\tau = 3 \times 10^{-6}$, and $\mathrm{Re}(f_{\mathrm{osc}}) - f_{\mathrm{eq}}$ has a plateau at a value of about 1 between $\tau \approx 10^{-5}$ and $\tau \approx 5 \times 10^{-4}$.

The integration of Equation (2.19) for the radiation field and the evaluation of f_{osc} occupied less than 10% of the total computation time. Thus consistent treatment of the radiation field only requires a modest increase in the numerical work.

Some effects on the oscillation eigenfunctions of the treatment of the radiation field

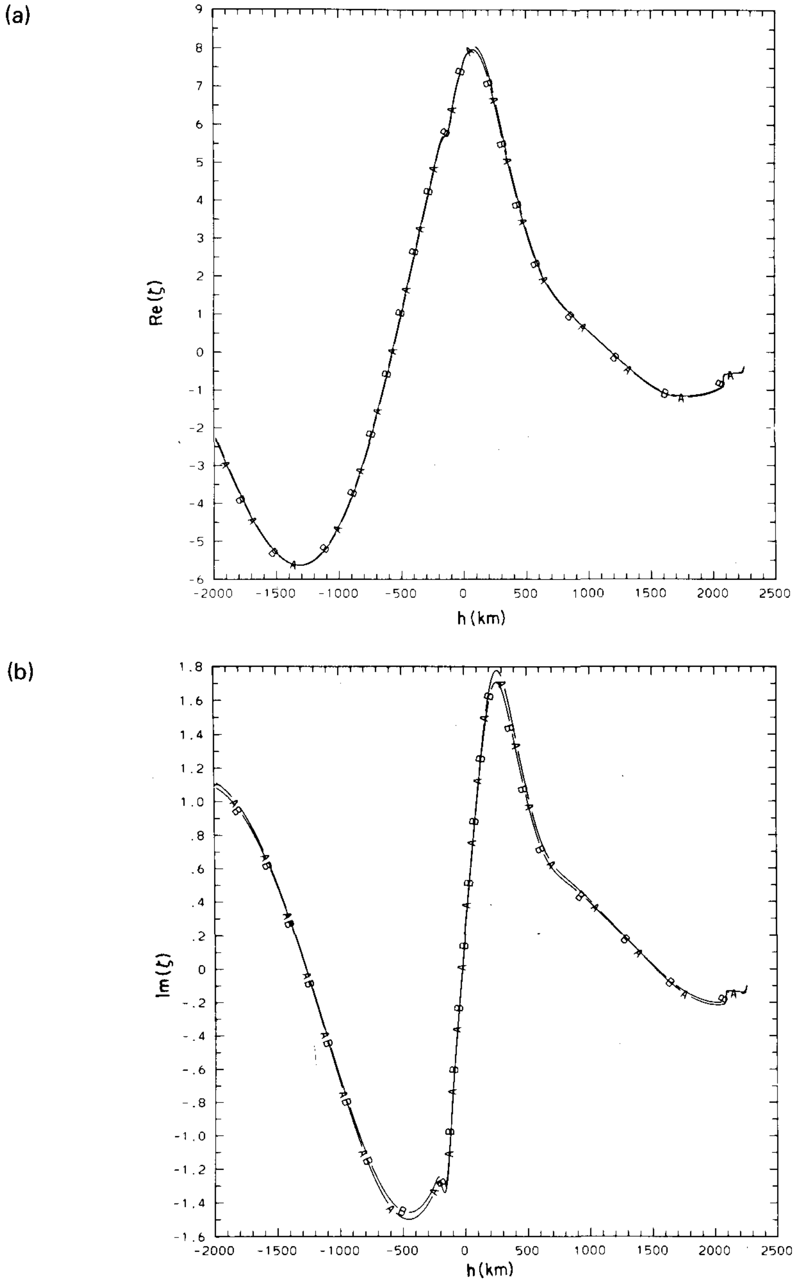

Fig. 3a–b.

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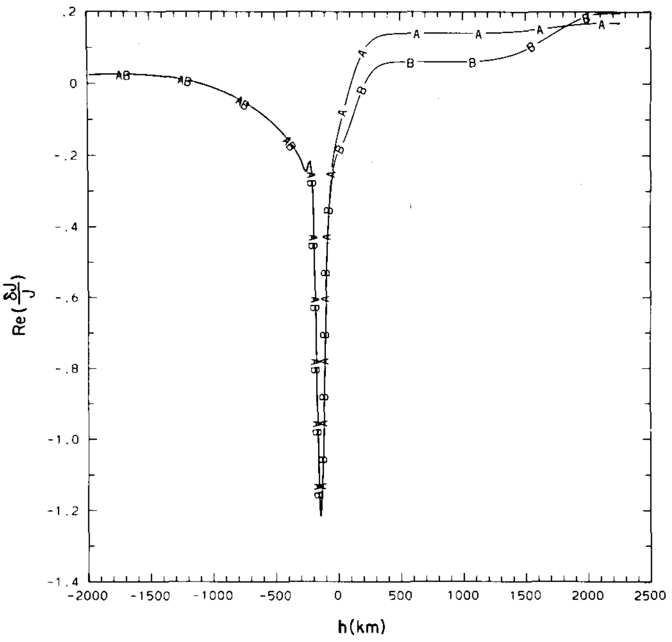
(c)



(d)

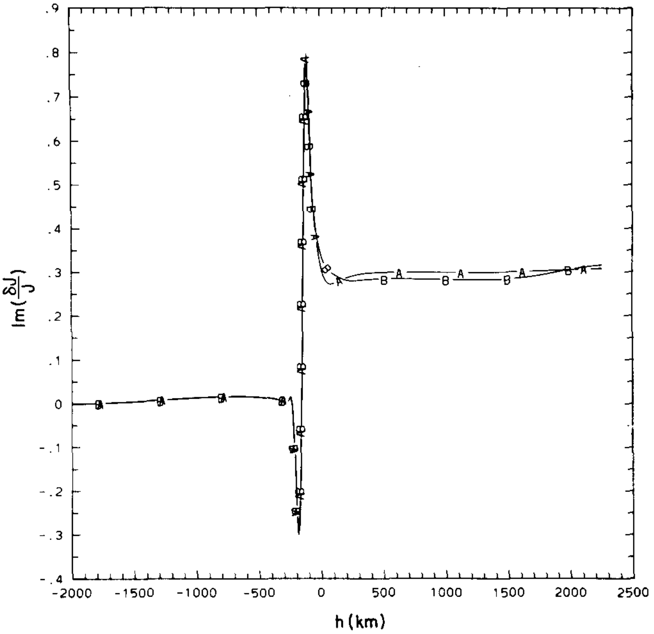


Fig. 3c–d.

Fig. 3a–d. Eigenfunctions in the Eddington approximation (A–A–A–) and with consistently iterated Eddington factors (B–B–B–) for the mode with frequency 4.18 mHz. Figures (a) and (b) show real and imaginary parts of the scaled displacement ζ, defined in Equation (4.4), and (c) and (d) real and imaginary parts of the relative perturbation $\delta J/J$ in the mean intensity.

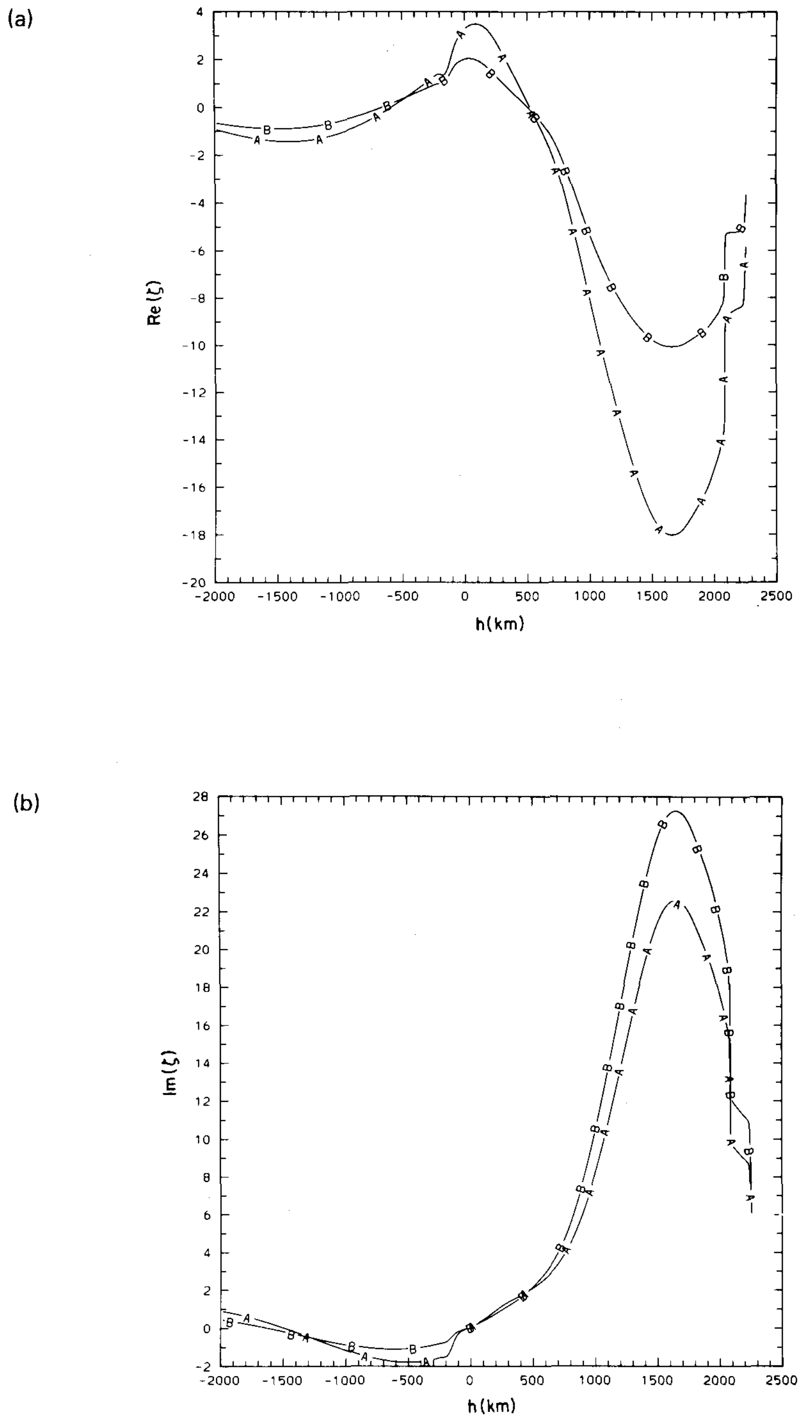


Fig. 4a–b.

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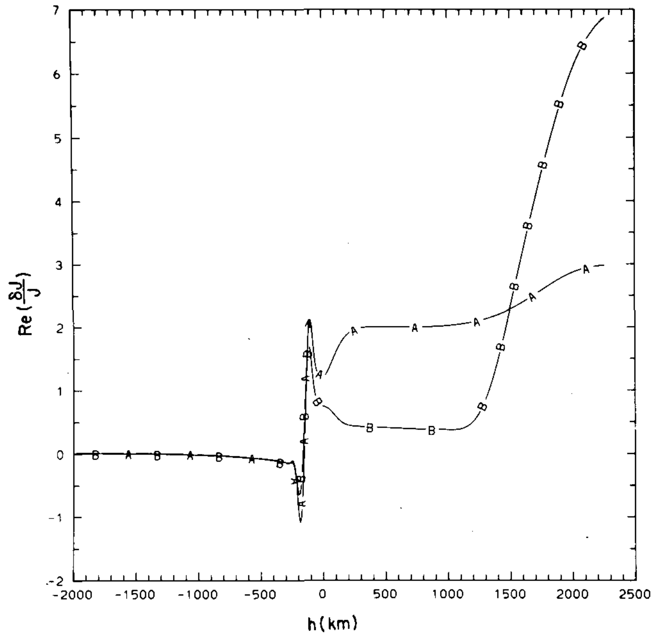
(c)



(d)

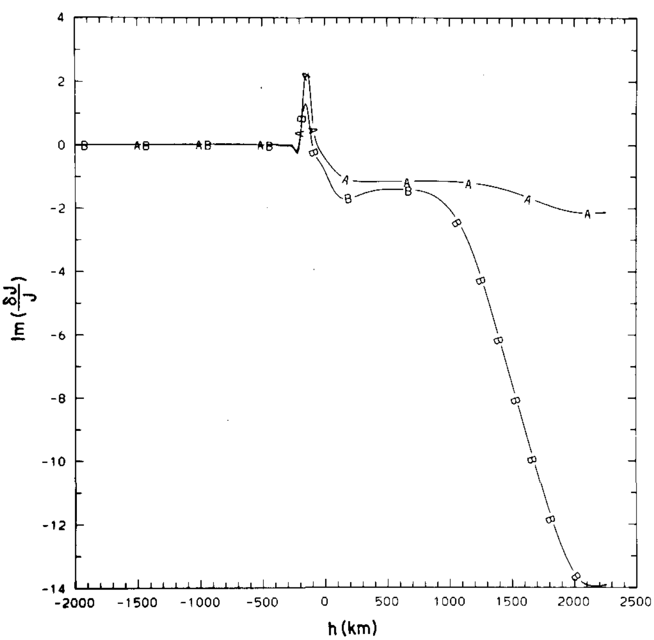


Fig. 4c–d.

Fig. 4a–d. Eigenfunctions for the chromospheric mode. See caption to Figure 3.

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are presented on Figures 3 and 4, for the interior mode with frequency $\nu = 4.18$ mHz and the chromospheric mode, respectively. To indicate the distribution of the kinetic energy of pulsation we plot in each case

$$\zeta = \frac{\rho^{1/2}\delta r}{r_{\text{phot}}^{-3/2} E^{1/2}} , \tag{4.4}$$

where

$$E = \int_{r_b}^{r_s} r^2 \rho |\delta r|^2 \, \mathrm{d}r \tag{4.5}$$

is related to the kinetic energy of pulsation. In addition we show $\delta J/J$. The solution was normalized such that $\delta r/r_e = 1$ at $r = r_{\text{phot}}$, where

$$r_e = \frac{\mathscr{R} T_{\text{eff}}}{g_s} \tag{4.6}$$

is related to the photospheric pressure scale height, $\mathscr{R}$ being the gas constant and g_s the surface gravity of the model.

For the interior mode shown on Figure 3 most of the pulsational energy is clearly associated with the inner cavity, the maximum in ζ being close to the turning point at about $h = 120$ km predicted on the basis of Figure 1. The effect of the treatment of radiation on ζ is so small as to be almost unnoticeable. A somewhat larger, but still fairly small, effect is seen for $\delta J/J$.

The chromospheric mode shown on Figure 4 has a pronounced maximum in ζ in the chromospheric cavity shown on Figure 1 and is evidently strongly reflected in the transition region. Not surprisingly, for this mode ζ is significantly affected by the treatment of the radiation field. Furthermore there is now a large difference in $\delta J/J$ between the two cases, which could conceivably have some observational effect. Significant differences are also found for the remaining perturbations.

Finally we may consider in more detail the results on the damping of the modes. It is easy to show from the oscillation equations that

$$2\omega_i \omega_r E = \operatorname{Im}(r^2 \delta r^* \delta p) \Big|_{r_b}^{r_s} - \\ - \operatorname{Re}\left\{ \frac{1}{\omega} \int_{r_b}^{r_s} (\Gamma_3 - 1) \frac{\delta\rho^*}{\rho} \, \delta(\operatorname{div} \mathscr{F}_R) r^2 \, \mathrm{d}r \right\} , \tag{4.7}$$

where $\Gamma_3 - 1 = (\partial \ln T/\partial \ln \rho)_s$. Here the integrated term corresponds to the work done from the outside on the region considered or, equivalently, the acoustical energy flux across the boundaries, whereas the integral gives the contribution form the internal (positive or negative) dissipation. If the no-work condition (2.31) is assumed, Equation (4.7) may be written as

$$\eta = \eta_s + W(r_s)\,, \tag{4.8}$$

where

$$\eta_s = \frac{\mathrm{Im}\,(r^2 \delta r^* \delta p)}{2\omega_r^2 E} \tag{4.9}$$

comes from the loss of acoustical energy at the outer boundary, and

$$W(r) = -\frac{\mathrm{Re}\left[\dfrac{1}{\omega}\displaystyle\int_{r_b}^{r} (\Gamma_3 - 1)\frac{\delta\rho^*}{\rho}\,\delta(\mathrm{div}\,\mathscr{F}_R)\,r'^2\,\mathrm{d}r'\right]}{2\omega_r^2 E} \tag{4.10}$$

is the normalized work integral.

If the bottom boundary condition (2.31) is used we may compute η as a continuous function of ν. The results are shown on Figure 5, both for the consistent treatment of the radiation field and using the Ando and Osaki approximation. For the exact case the figure also shows the ratio η_s/η, i.e. the relative contribution of the leakage of wave energy at the outer boundary to the total damping rate of the oscillation. As was also found by Ando and Osaki (1977) this contribution is never dominant for interior modes; for the chromospheric mode $\eta_s/\eta \approx 0.32$, and so here wave leakage accounts for about $\frac{1}{3}$ of the total damping.

The very large peak in η at around 5.2 mHz corresponds to a pronounced minimum in the normalized pulsational energy,

$$\mathscr{E}_{\mathrm{osc}} = \frac{E}{M\delta r(r_s)^2} \tag{4.11}$$

(M being the total mass of the Sun), so that here the energy of the mode is predominantly in the atmosphere; this also causes the peak in η_s/η. These frequencies are close to or above ν_c, and so there is no longer strict trapping in the chromosphere; but the shape of the eigenfunctions indicates that there may still be some resonance with the first overtone of the chromospheric cavity, and this is probably the reason for the enhanced damping. Similarly the small peak in η_s/η at 3.8 mHz might be caused by resonance with the fundamental chromospheric mode, and there is some indication that the peak in η at $\nu = 7.2$ mHz is associated with resonance with the second chromospheric overtone.

The apparent abrupt cut-off of power in the Birmingham whole-disk spectra at about 6 mHz presented by Isaak (1983) might be related to the resonance close to this frequency. There is clearly no sudden increase in damping at an 'acoustical cut-off frequency', and from the shape of the 'potential' in Figure 1 non would be expected; but

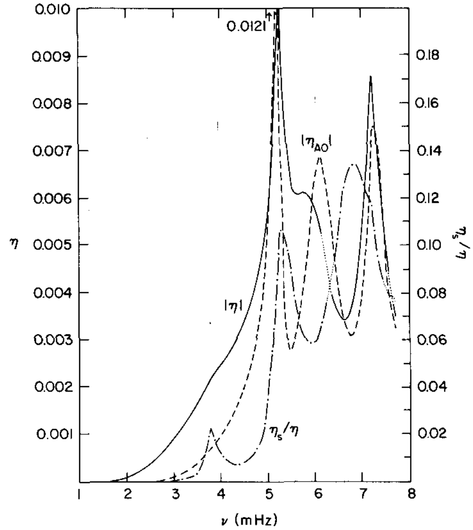

Fig. 5. Relative damping rates for oscillations extending over the entire envelope, as functions of the cyclic frequency ν. The curve labelled η (———) was obtained with consistent treatment of the radiation field, whereas for the curve labelled η_{AO} (– – –) we used the Eddington approximation and assumed $J = B$ in the equilibrium model (as did Ando and Osaki, 1975, 1977). These two curves use the left-hand scale. Finally the curve labelled η_s/η (–·–·–), using the right-hand scale, shows the relative contribution to the total damping rate from the leakage of wave energy into the corona. The dotted parts of the curves correspond to frequency regions where the near-vanishing of δK at points in the atmosphere caused convergence problems in the Eddington factor iteration.

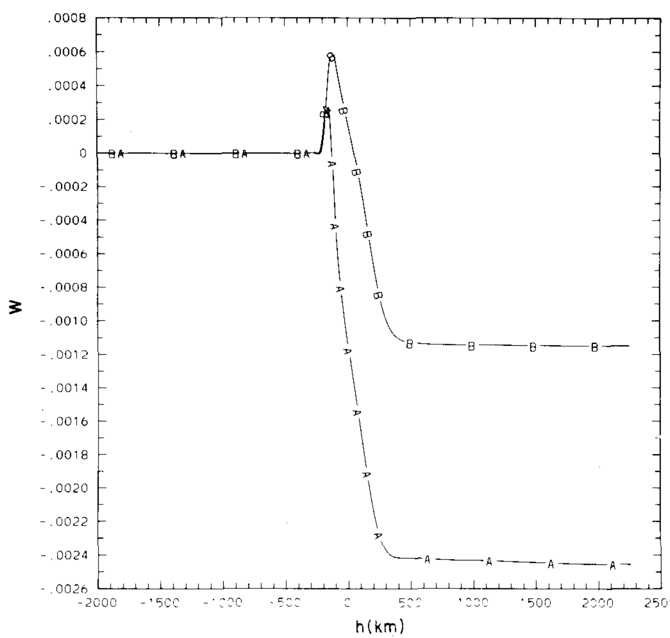

Fig. 6. The normalized work integral W (cf. Equation (4.10)) for the mode of frequency 4.18 mHz, with consistent treatment of the radiation field (A–A–A–) and in the Ando and Osaki approximation (B–B–B–).

it is still true, as pointed out by Isaak, that the position of the power cut-off might eventually be used as a diagnostic for the structure of the solar atmosphere, although not quite in the simple form envisaged by Isaak.

The damping rates found using the Ando and Osaki approximation are generally considerably smaller than those obtained when the approximation is not made. To illustrate this Figure 6 shows the behaviour of the work integral in the two cases. In both there is some excitation (which Ando and Osaki attributes to the κ mechanism) close to the outer edge of the convection zone. However this is weaker in the exact case, and here the damping in the lower atmosphere is also much stronger. The difference is solely an effect of assuming $\Delta_c = 0$; results obtained in the Eddington approximation but keeping the terms in Δ_c, are virtually indistinguishable form the exact case.

In the region of large damping $|\Delta_c|$ is quite small, less than about 0.05, and so it might at first seem surprising that it has such a significant effect on the stability of the oscillation. While there appears to be no obvious physical reason why this is so, it should be noticed that in Equation (2.35) Δ_c occurs in the combination $\omega_R \Delta_c$, which is to be compared with ω; in the region considered ω_R is much larger than ω (see also Figure 6 of Ando and Osaki, 1975), and so the terms in Δ_c may in fact dominate. The physical significance of this effect may, however, be questionable; it is certainly linked to our neglect of the perturbation in the convective and mechanical cooling rates (cf. Equation (2.8)), and so it should perhaps be regarded as a measure of the uncertainties introduced by this assumption.

5. Radiative Transfer in Non-Radial Oscillations

The general equations of non-radial, non-adiabatic oscillation are given by e.g. Ando and Osaki (1975) (see also Christensen-Dalsgaard, 1981), and will not be presented here; instead we concentrate on the part of the problem involving radiative transfer. To facilitate comparison with earlier work the spherical terms in the moment equations are kept; however the treatment of the radiation field is essentially plane-parallel.

The angular and temporal variation of the perturbations may be separated as, e.g.,

$$\delta p(r, \vartheta, \phi, t) = \mathrm{Re}[\delta p(r) Y_l^m(\vartheta, \phi) e^{-i\omega t}], \tag{5.1}$$

where r, ϑ, and ϕ are spherical polar coordinates; here Y_l^m is a spherical harmonic, and for simplicity we have used the same symbol for the perturbation and its amplitude. The displacement vector, no longer purely radial, may be written as

$$\delta \mathbf{r} = \mathrm{Re}\left\{\left[\xi_r(r) Y_l^m(\vartheta, \phi)\mathbf{a}_r + \right.\right.$$
$$\left.\left. + \xi_h(r)\left(\frac{\partial Y_l^m}{\partial \vartheta}\mathbf{a}_\vartheta + \frac{1}{\sin\vartheta}\frac{\partial Y_l^m}{\partial \phi}\mathbf{a}_\phi\right)\right]\exp(-i\omega t)\right\}, \tag{5.2}$$

where $\mathbf{a}_\vartheta$ and $\mathbf{a}_\phi$ are unit vectors in the ϑ and ϕ directions. In addition to the 'global' coordinates (r, ϑ, ϕ) it is useful to introduce a local Cartesian system of coordinates

(x, y, z) in a given point $(r_0, \vartheta_0, \phi_0)$; here $z = r - r_0$ and the x-axis may be chosen to be in the direction of the local horizontal component of the wave vector, so that, e.g.,

$$\delta p(r, \vartheta, \phi, t) = \mathrm{Re}\,\{\delta\tilde{p}(z)\exp[i(kx - \omega t)]\}\ , \tag{5.3}$$

where the local horizontal wave number k is related to l by

$$k = r_0^{-1}[l(l+1)]^{1/2}\,. \tag{5.4}$$

For radial oscillations the equations for the perturbation in the radiative intensity, Equations (2.19) or (A.4), were relatively simple, because of the rotational symmetry around the radial direction. For non-radial oscillations this symmetry is clearly destroyed, and we must consider the general equation of transfer

$$\mathbf{n}\cdot\nabla I(\mathbf{r}, \mathbf{n}) = \rho\kappa_s J + \rho\kappa_a B - \rho(\kappa_s + \kappa_a)I \equiv \rho A\ , \tag{5.5}$$

where $I(\mathbf{r}, \mathbf{n})$ is the specific intensity in the direction of the unit vector $\mathbf{n}$. For simplicity we have dropped the subscript ν. In the spherical case the components of $\mathbf{n}$, relative to a local coordinate system, changes with $\mathbf{r}$, and so $\mathbf{n}\cdot\nabla I$ involves derivatives of I with respect to $\mathbf{n}$ as well as $\mathbf{r}$. However the solar atmosphere is thin compared with the solar radius, and so the terms involving $\mathbf{n}$-derivatives may be expected to be small. If (x, y, z) is the local Cartesian system introduced above, and (n_x, n_y, n_z) are the components of $\mathbf{n}$ in this system, we shall neglect the derivatives of I with respect to n_x and n_y; but we keep the derivative with respect to n_z, as this gives rise to terms also found when the 3-dimensional Eddington approximation (Unno and Spiegel, 1966) is used. Then the left-hand side of Equation (5.5) may be written

$$\mathbf{n}\cdot\nabla I = n_z\frac{\partial I}{\partial z} + n_x\frac{\partial I}{\partial x} + n_y\frac{\partial I}{\partial y} + \frac{1}{r}(1 - n_z^2)\frac{\partial I}{\partial n_z}\ . \tag{5.6}$$

Assuming that all perturbations, including the Lagrangian perturbation δI of I, are of the form given in Equation (5.3), it is straightforward to find the perturbation of Equation (5.5):

$$\begin{aligned} n_z\frac{\partial\delta\tilde{I}}{\partial z} + ikn_x\delta\tilde{I} + \frac{1 - n_z^2}{r}\frac{\partial\delta\tilde{I}}{\partial n_z} &= \\ = \delta(\widetilde{\rho A}) + \rho A\frac{\mathrm{d}\tilde{\xi}_r}{\mathrm{d}z} - (1 - n_z^2)\frac{\partial}{\partial z}\left(\frac{\xi_r}{r}\frac{\partial I}{\partial\mu}\right) &+ ikn_x\frac{\partial I}{\partial z}\tilde{\xi}_r\,. \end{aligned} \tag{5.7}$$

We now replace $\partial/\partial z$ by $\partial/\partial r$, drop the tildes over the perturbed quantities and use the equation of continuity

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\xi_r) - \frac{l(l+1)}{r}\xi_h = -\frac{\delta\rho}{\rho} \tag{5.8}$$

to eliminate $\mathrm{d}\xi_r/\mathrm{d}r$. The perturbed equation of transfer is then finally

$$n_z \frac{\partial \delta I}{\partial r} + ikn_x \delta I + \frac{1-n_z^2}{r} \frac{\partial \delta I}{\partial n_z} = \rho \delta A + \rho A \left[\frac{l(l+1)}{r} \xi_h - \frac{2}{r} \xi_r \right] +$$

$$+ \frac{1}{r}(1-n_z^2) \frac{\partial I}{\partial n_z} \left[\frac{\delta \rho}{\rho} + \frac{3}{r} \xi_r - \frac{l(l+1)}{r} \xi_h \right] + ikn_x \frac{\partial I}{\partial r} \xi_r , \quad (5.9)$$

where

$$\delta A = \delta \kappa_s J + \kappa_s \delta J + \delta \kappa_a B + \kappa_a \delta B - (\delta \kappa_s + \delta \kappa_a) I - (\kappa_s + \kappa_a) \delta I. \quad (5.10)$$

It should be noticed that the derivation of the perturbed energy equation, Equation (2.15), did not require that the oscillations be radial. Thus this equation, as well as Equation (2.18), are valid also for non-radial oscillations, provided the assumptions (2.8) are still made.

It is convenient again to introduce moments of the radiation field. The mean intensity is, as in Equation (2.20),

$$\delta J(r) = \frac{1}{4\pi} \oint \delta I(r, \mathbf{n}) \, \mathrm{d}\Omega . \quad (5.11)$$

The first moment is now a vector with components (in the local coordinate system)

$$\delta H_i(r) = \frac{1}{4\pi} \oint n_i \delta I(r, \mathbf{n}) \, \mathrm{d}\Omega , \quad (5.12)$$

and the second moment is a tensor with components

$$\delta K_{ij}(r) = \frac{1}{4\pi} \oint n_i n_j \delta I(r, \mathbf{n}) \, \mathrm{d}\Omega ; \quad (5.13)$$

here the indices i and j take the values x, y, and z. From Equation (5.3) it is obvious that

$$\delta H_y = 0 , \qquad \delta K_{xy} = \delta K_{zy} = 0 ; \quad (5.14)$$

in addition we clearly have

$$\delta K_{ij} = \delta K_{ji} \quad \text{for all} \quad i, j ,$$

$$\delta K_{xx} + \delta K_{yy} + \delta K_{zz} = \delta J . \quad (5.15)$$

The three-dimensional Eddington approximation of Unno and Spiegel (1966) corresponds to assuming that δK_{ij} is isotropic, i.e.

$$\delta K_{ij} = 0 \quad \text{for} \quad i \neq j ,$$

$$\delta K_{xx} = \delta K_{yy} = \delta K_{zz} = \tfrac{1}{3} \delta J . \quad (5.16)$$

We define the generalized Eddington factors by

$$f_{\rm osc} = \delta K_{zz}/\delta J\,, \tag{5.17}$$

$$\phi_{xx} = \delta K_{xx}/\delta K_{zz}\,, \tag{5.18}$$

and

$$\phi_{xz} = \delta K_{xz}/(ikr\delta K_{zz})\,. \tag{5.19}$$

The factor ikr was included in the definition of ϕ_{xz} because this quantity, as shown below, tends to an approximately real, non-zero constant in the limit of radial oscillations. It is evident that in the Eddington approximation $f_{\rm osc} = \frac{1}{3}$, $\phi_{xx} = 1$, and $\phi_{xz} = 0$.

By taking moments of Equation (5.9), using Equation (5.10), we now obtain

$$\begin{aligned}\frac{\mathrm{d}\delta K_{zz}}{\mathrm{d}r} = {} & -\rho\kappa\delta H_z + \rho\kappa\left[\frac{2}{r}\xi_r - \frac{l(l+1)}{r}\xi_h - \frac{\delta\kappa}{\kappa}\right]H + \\ & \underline{+\left[\frac{l(l+1)}{r}\phi_{xz} - \frac{1}{r}\left(3 - \frac{1}{f_{\rm osc}}\right)\right]\delta K_{zz} +} \\ & \underline{+\frac{J}{r}(3f_{\rm eq} - 1)\left[\frac{3}{r}\xi_r - \frac{l(l+1)}{r}\xi_h + \frac{\delta\rho}{\rho}\right]}\,,\end{aligned} \tag{5.20}$$

and

$$\begin{aligned}\frac{\mathrm{d}\delta H_z}{\mathrm{d}r} = {} & -\frac{2}{r}\delta H_z + \kappa_a\rho(J - B)\left[\frac{2}{r}\xi_r - \frac{l(l+1)}{r}\xi_h - \frac{\delta\kappa_a}{\kappa_a}\right] + \\ & + \frac{H}{r}\left[2\frac{\delta\rho}{\rho} + 6\frac{\xi_r}{r} - \frac{l(l+1)}{r}(2\xi_h + \xi_r)\right] + \rho\kappa_a(\delta B - \delta J) - \\ & - \frac{l(l+1)}{r^2\kappa\rho}\underline{\phi_{xx}}\delta K_{zz} - \\ & \underline{-\frac{l(l+1)}{r^2\kappa\rho}\left[r\phi_{xz}\frac{\mathrm{d}\delta K_{zz}}{\mathrm{d}r} + \left(r\frac{\mathrm{d}\phi_{xz}}{\mathrm{d}r} + 4\phi_{xz}\right)\delta K_{zz} +\right.} \\ & \underline{\left. + \left\{\frac{J}{r}(3f_{\rm eq} - 1) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}r}[J(3f_{\rm eq} - 1)]\right\}\xi_r\right]}\,,\end{aligned} \tag{5.21}$$

where Equation (5.20) may be substituted for $\mathrm{d}\delta K_{zz}/\mathrm{d}r$. When the underlined factors or terms are omitted, Equations (5.20) and (5.21) reduce to those obtained in the Eddington approximation (e.g. Christensen-Dalsgaard, 1981).

If the Eddington factors were known, Equations (5.20) and (5.21), together with the energy equation and the mechanical equations, would form a closed system. Thus, exactly as in the radial case, one may solve the problem by iterating for the Eddington factors. Given the solution for a set of trial Eddington factors Equation (5.9) may be integrated for δI on a suitable grid of directions $\mathbf{n}_j$, and new values of the Eddington factors obtained from Equations (5.11)–(5.13) and (5.17)–(5.19). This procedure is then iterated simultaneously with the iteration for the eigenfrequency.

The integration for the radiation field is now, however, considerably more complicated than in the radial case. To make it more efficient we generalize a technique first used by Logan and Hill (1980) to find the perturbation in the mean intensity. Neglecting the spherical term in $\partial\delta I/\partial n_z$, Equation (5.9) may be written as

$$n_z \frac{\partial \delta I}{\partial r} + ikn_x \delta I = -\rho\kappa(\delta I - \delta S), \tag{5.22}$$

which defines δS. This equation has the solution

$$\delta I(r, \mathbf{n}) = \int_{\tau}^{\mathscr{T}(n_z)} \exp\{[\tau - \tau' + in_x k(r - r')]/n_z\} \, \delta S(\tau', \mathbf{n}) \, \mathrm{d}\tau'/n_z, \tag{5.23}$$

$$\mathscr{T}(n_z) = \begin{cases} \tau_s & \text{for} \quad n_z < 0, \\ \infty & \text{for} \quad n_z > 0, \end{cases} \tag{5.24}$$

where τ is the optical depth defined by

$$\frac{\mathrm{d}\tau}{\mathrm{d}r} = -\kappa\rho, \quad \tau \to 0 \quad \text{as} \quad r \to \infty, \tag{5.25}$$

$\tau' = \tau(r')$, τ_s is the value of τ at the surface of the model, and we have explicitly indicated that δS depends on $\mathbf{n}$; in fact

$$\delta S = \delta S_0 + \delta S_1 I + \frac{ikn_z}{\kappa\rho} \frac{\partial I}{\partial r} \xi_r, \tag{5.26}$$

where δS_0 and δS_1 are then independent of $\mathbf{n}$, whereas I is of course a function of n_z.

In spherical coordinates the vector n has components

$$n_z = \cos\hat{\vartheta}, \qquad n_x = \sin\hat{\vartheta}\cos\hat{\phi}, \qquad n_y = \sin\hat{\vartheta}\sin\hat{\phi}, \tag{5.27}$$

where $\hat{\vartheta}$ is the angle between the vertical and $\mathbf{n}$, and ϕ the angle between the projection of $\mathbf{n}$ on the horizontal plane and the horizontal wave vector. We now introduce moments of δI with respect to ϕ as

$$\overline{\delta I}_j(r, \mu) = \frac{1}{2\pi} \int_0^{2\pi} \cos^j\phi \, \delta I(r, \mathbf{n}) \, \mathrm{d}\hat{\phi}, \tag{5.28}$$

where, in accordance with the notation in Section 2, we have written μ for n_z. From Equations (5.23) and (5.26) now follows that

$$\overline{\delta I}_j(r,\mu) = \int_{\tau}^{\mathcal{T}(\mu)} e^{(\tau-\tau')/\mu}\left[(\delta S_0 + \delta S_1 I)\mathcal{J}_j(x) + \right.$$
$$\left. + \frac{ik}{\kappa\rho}\frac{\partial I}{\partial r}\xi_r\sqrt{1-\mu^2}\,\mathcal{J}_{j+1}(x)\right]\mathrm{d}\tau/\mu\,, \tag{5.29}$$

where

$$x = k(r-r')\tan\hat{\vartheta}\,, \tag{5.30}$$

and

$$\mathcal{J}_j = \frac{1}{2\pi}\int_0^{2\pi}\cos^j\phi\, e^{ix\cos\phi}\,\mathrm{d}\phi\,. \tag{5.31}$$

The $\mathcal{J}_j$ may clearly be expressed in terms of Bessel functions. For j less than 4 (which is all we shall need) the result is

$$\left.\begin{aligned}
\mathcal{J}_0(x) &= J_0(x)\,,\\
\mathcal{J}_1(x) &= iJ_1(x)\,,\\
\mathcal{J}_2(x) &= \tfrac{1}{2}[J_0(x) - J_2(x)]\,,\\
\mathcal{J}_3(x) &= \frac{i}{4}[3J_1(x) - J_3(x)]\,.
\end{aligned}\right\} \tag{5.32}$$

Finally the moments of δI with respect to $\mathbf{n}$ may be obtained as

$$\left.\begin{aligned}
\delta J(r) &= \tfrac{1}{2}\int_{-1}^{1}\overline{\delta I}_0(r,\mu)\,\mathrm{d}\mu\,,\\
\delta H_z(r) &= \tfrac{1}{2}\int_{-1}^{1}\mu\,\overline{\delta I}_0(r,\mu)\,\mathrm{d}\mu\,,\\
\delta H_x(r) &= \tfrac{1}{2}\int_{-1}^{1}\sqrt{1-\mu^2}\,\overline{\delta I}_1(r,\mu)\,\mathrm{d}\mu\,,\\
\delta K_{zz}(r) &= \tfrac{1}{2}\int_{-1}^{1}\mu^2\,\overline{\delta I}_0(r,\mu)\,\mathrm{d}\mu\,,\\
\delta K_{xz}(r) &= \tfrac{1}{2}\int_{-1}^{1}\mu\sqrt{1-\mu^2}\,\overline{\delta I}_1(r,\mu)\,\mathrm{d}\mu\,,\\
\delta K_{xx}(r) &= \tfrac{1}{2}\int_{-1}^{1}(1-\mu^2)\,\overline{\delta I}_2(r,\mu)\,\mathrm{d}\mu\,.
\end{aligned}\right\} \tag{5.33}$$

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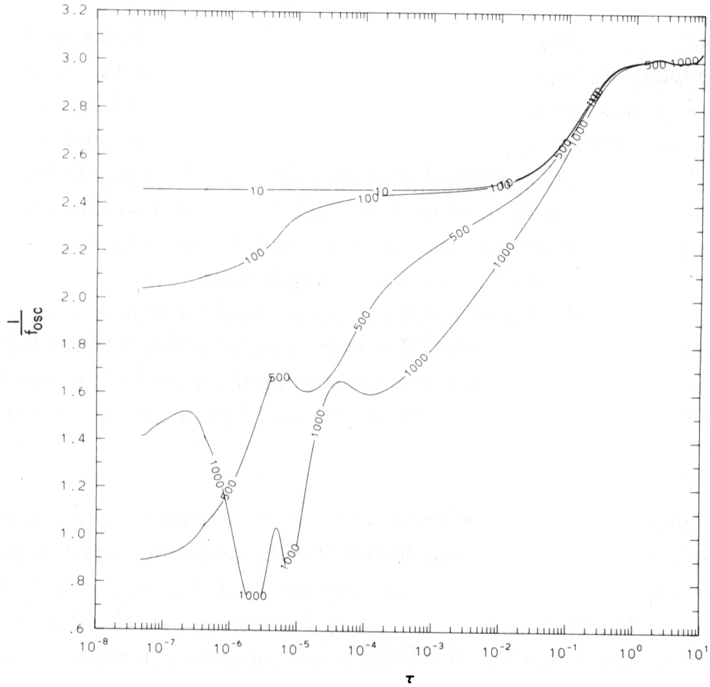

Fig. 7. The generalized Eddington factor $1/f_{\text{osc}} = \delta J/\delta K_{zz}$ calculated from the equilibrium source function, as a function of optical depth τ. The curves are labelled with the degree l.

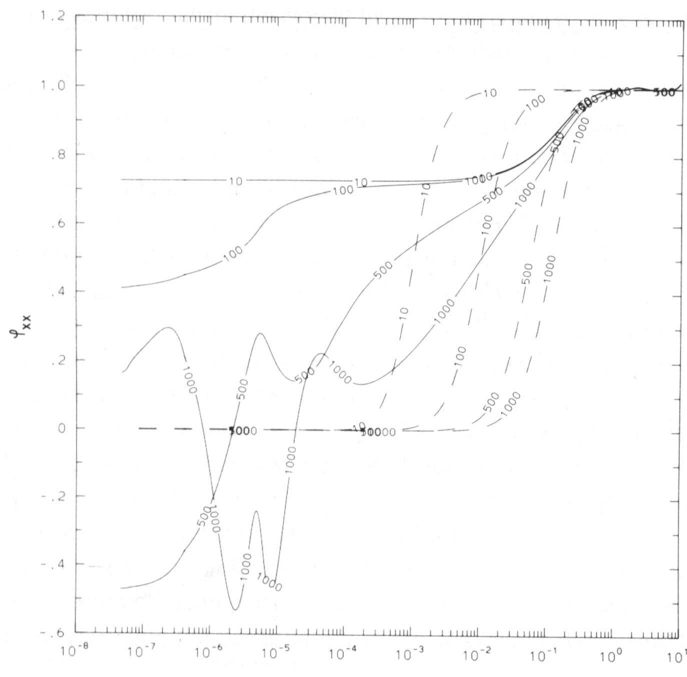

Fig. 8. The generalized Eddington factor $\phi_{xx} = \delta K_{xx}/\delta K_{zz}$. The dashed curves show the approximation suggested by Ando and Osaki (1977). See caption to Figure 7.

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It should be noticed that $\mathscr{J}_1(x) \sim ix$ as $x \to 0$, and so δK_{xz} goes as ik as k tends to zero. This is why the factor ikr was taken out in the definition of ϕ_{xz} in Equation (5.19).

The calculation is now quite efficient. The integrals in Equation (5.29) are approximated by a product between vectors, containing components of the source function at discrete points in τ, and matrices derived from the kernels $\mathscr{J}_j$. To compute the matrices one must clearly evaluate a large number of Bessel functions and so this is fairly time consuming; however these matrices need only be evaluated once, for given k, and when they are known the calculation of the moments is quite fast.

We have yet to implement the iteration for the non-radial Eddington factors, and so we cannot comment on its convergence. However an initial estimate of the Eddington factors, which in any case should be useful for starting the iteration, can be obtained by replacing in Equation (5.29) $\delta S_0 + I\delta S_1$ by the equilibrium source function

$$S = \kappa^{-1}(\kappa_a B + \kappa_s J) \tag{5.34}$$

and neglecting the term in ξ_r. In the radial case, i.e. $k = 0$, the resulting f_{osc} is identical to f_{eq}, the equilibrium Eddington factor. Results for the model described in Section 3 are presented on Figure 7, 8, and 9 showing, respectively, $1/f_{\text{osc}}$, ϕ_{xx}, and ϕ_{xz} for four different values of l. The main effect of increasing l is seen to be a decrease of $1/f_{\text{osc}}$, ϕ_{xx}, and ϕ_{xz}. The reason is that for $l \neq 0$ oblique rays pass through regions of different phase, so that the contribution to the intensity is to some extent averaged out. Thus with

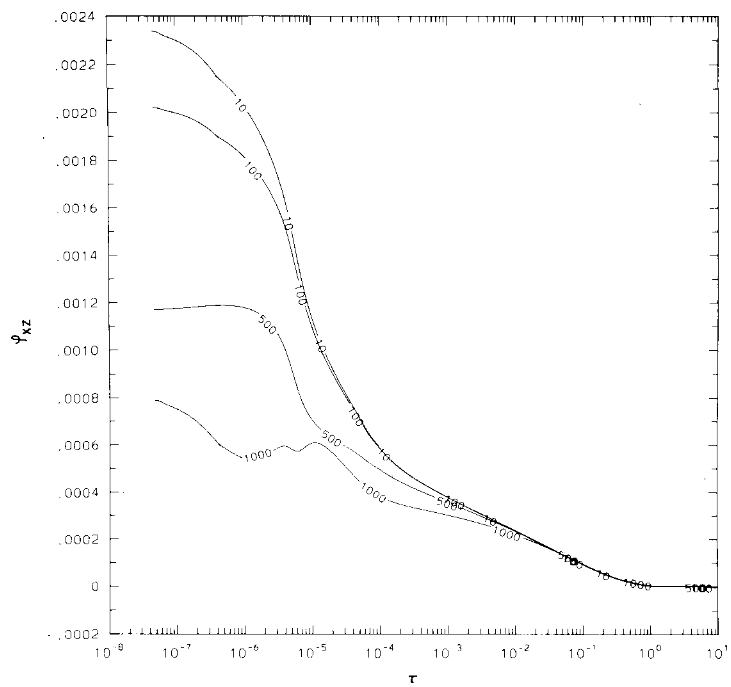

Fig. 9. The generalized Eddington factor $\phi_{xz} = \delta K_{xz}/(ikr\delta K_{xx})$. See caption to Figure 7.

increasing l δI becomes more strongly peaked in the radial direction, and this increases δK_{zz} relative to δJ, δK_{xx}, δK_{xz}. The reduction of the horizontal heat exchange relative to the Eddington approximation, corresponding to the reduction in ϕ_{xx}, was anticipated by Ando and Osaki (1977). They proposed a modification to the equations which corresponds, in our notation, essentially to approximating ϕ_{xx} by

$$\phi_{xx}^{(\mathrm{AO})} = \frac{1}{1 + l(l+1)/(\kappa \rho r)^2} ; \tag{5.35}$$

these values are shown on Figure 7, 8, and 9 as dashed lines. Although the approximation leads to a decrease in ϕ_{xx} with decreasing τ or increasing l, the effect is clearly in general too large. In fact better approximations that are still computationally simple could presumably be derived from the ϕ_{xx} found here. However the full moment equations contain a number of additional terms, and detailed calculations, with a consistent iteration for the Eddington factors, will be needed to determine the effects of the departures from the Eddington approximation.

6. Discussion

The principal result of the present work is probably that for grey radiative transport in radial solar oscillations the Eddington approximation is generally adequate. In that sense the consistent formalism developed here is not required, at least in the grey case; however a detailed calculation was needed to demonstrate this.

The numerical results obtained in Section 4 only strictly apply to radial oscillations. For non-radial oscillations of low degree, however, the optical thickness across a horizontal wavelength is large, and the effect of the horizontal variations therefore small; thus here the results obtained in the radial case are probably still valid. The preliminary estimates for non-radial oscillations presented in Section 5 showed that when l is greater than about 100 the radiation field is increasingly affected by the horizontal variation, and so here the full non-radial case must be treated. We have so far not attempted to implement the iteration for the generalized, non-radial Eddington factors in the calculation; but it seems reasonable to hope that the convergence properties of this iteration will not be much worse than in the radial case.

Similar results were obtained by Kneer and Heasley (1979) for static perturbations in a simplified, grey atmosphere. As here the Eddington approximation was adequate for plane-parallel perturbations, but was found to be increasingly inaccurate with decreasing horizontal wavelength of the perturbation.

Perhaps the most serious among the approximations made here is that only radiation contributes to the Lagrangian perturbation in the local heating rate. We are now in the process of incorporating the perturbation in the convective flux, using the formalisms of Gough (1977) and Unno (1967). The mechanical heating in the atmosphere presents a greater challenge, as even the physical mechanism responsible is not definitively known. It might be possible to parameterize and then perturb the weak shock theory

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of, e.g., Ulmschneider (1971), and we intend to consider this; but the mechanical heating is likely to remain a serious difficulty in the theory of atmospheric solar oscillations. A related problem concerns the perturbation of the 'turbulent pressure'. As shown by Gough (1977) and Baker and Gough (1979), in the convection zone this can be treated within the framework of mixing length theory (although an equilibrium model which consistently incorporates turbulent pressure has yet to be constructed); but in the atmosphere we lack even a proper physical model for the turbulent pressure.

We have assumed that the model is spherically symmetric, and thus neglected the inhomogeneous nature of the solar atmosphere. This becomes an increasingly bad approximation with increasing height in the atmosphere. For the interior acoustic modes, which are trapped below the temperature minimum, the effects of inhomogeneities are probably not serious; in fact the observed sharp ridges in the $k - \omega$ diagram of, e.g., Deubner *et al.* (1979) and Rhodes *et al.* (1983) are evidence that for these modes the reflection in the atmosphere is not significantly affected by the inhomogeneities. These are almost certain to be important, however, in the chromosphere, and may invalidate the notion of a well defined chromospheric cavity as shown on Figure 1, and hence of a chromospheric mode. On the other hand it may be that modes whose horizontal wavelength is much longer than the scale of the inhomogeneities feel only the mean structure of the atmosphere, and that therefore our results may apply to such modes. More work on the behaviour of waves in an inhomogeneous medium is certainly needed to investigate this question.

In contrast to Ando and Osaki (1975, 1977) we found that all acoustic modes were stable. As shown in Section 4 this difference is largely caused by the fact that Ando and Osaki assumed $J = B$ in the equilibrium model. When making this approximation we also find instability of a number of modes, but in a smaller frequency range than Ando and Osaki. Although they only consider non-radial oscillations with $l \geq 10$ this is unlikely to be the cause of the difference; essentially all the damping and excitation of these modes take place very close to the surface where the vertical scale of the modes is small compared with the horizontal scale. Thus the difference is most likely caused by differences in the equilibrium model. Goldreich and Keeley (1977) also found stability of the acoustic modes of the Sun, although in their case an important contribution to the damping came from turbulent viscosity, treated in a highly simplified way (like Ando and Osaki they assumed $J = B$ and so they found instability when no turbulent viscosity was included). A similar conclusion was reached by Baker and Gough (cf. Gough, 1980), using Gough's (1977) treatment of the convective flux perturbation. Thus there now seems to be fairly strong evidence that acoustic modes are not self-excited in the Sun. However the calculated damping rates seem to be in conflict with the observed lifetimes of modes in the 5 min range of a few days (Grec *et al.*, 1980) or possibly up to a month (Claverie *et al.*, 1981; Woodard and Hudson, 1983). In fact the values of the relative damping rates η shown on Figure 4 correspond to a natural line width (full width at half maximum) of about 6 μHz at a frequency of 3 mHz, which is considerably larger than the observed line widths. Thus there appears to be problems in our understanding of the excitation and damping of these modes.

We intend to further extend the calculations. We have developed a programme to solve the non-grey equations in the atmosphere using the method devised by Rybicki (1971) which allows the inclusion of a fairly large number of frequency points; the atmospheric solution is then matched to a grey solution in the interior of the model. This procedure furthermore allows for the inclusion of at least a few spectral lines. Not only are these important from a diagnostic point of view but they may also have important dynamic effects; in fact radiative losses in the chromosphere occur predominantly in spectral lines (e.g. Giovanelli, 1978; Athay, 1981) and hence are probably severly underestimated in the present, grey calculation. We hope shortly to be able to report initial results of these calculations. By comparing the predictions of such increasingly detailed calculations with the large body of observational evidence that is becoming available it may become possible to determine how our theoretical treatment of the oscillations, and perhaps eventually the underlying atmospheric model, should be improved.

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Appendix: The Equations for Radial Oscillations in Spherical Geometry

The continuity equation is

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}(r^2\delta r) = -\frac{\delta\rho}{\rho}\,. \tag{A.1}$$

Furthermore it is easy to show, by generalizing the analysis of e.g. Cox (1980) that the equation of motion is now

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\delta p}{p}\right) = \frac{\rho\tilde{g}}{p}\frac{\delta p}{p} + \frac{\rho}{p}\left[\omega^2 + \frac{2}{r}g(1+\lambda)\right]\delta r\,, \tag{A.2}$$

where $\lambda = \tilde{g}/g$ and we have assumed Equation (2.4).

The static equation of transfer in the spherical case is

$$\mu\frac{\partial I_\nu}{\partial r} + \frac{1}{r}(1-\mu^2)\frac{\partial I_\nu}{\partial \mu} = \rho[\kappa_{s,\nu}J_\nu + \kappa_{a,\nu}B_\nu - (\kappa_{s,\nu} + \kappa_{a,\nu})I_\nu] \tag{A.3}$$

(e.g. Hummer and Rybicki, 1971); under the same conditions as in Section 2 its Lagrangian perturbation is

$$\mu\frac{\partial \delta I_\nu}{\partial r} + \frac{1}{r}(1-\mu^2)\frac{\partial \delta I_\nu}{\partial \mu} + \rho\kappa_\nu \delta I_\nu =$$

$$= \rho[\kappa_{s,\nu}\delta J_\nu + \kappa_{a,\nu}\delta B_\nu + \delta\kappa_{s,\nu}B_\nu - (\delta\kappa_{s,\nu} + \delta\kappa_{a,\nu})I_\nu] -$$

$$- \frac{2}{r}\rho[\kappa_{s,\nu}J_\nu + \kappa_{a,\nu}B_\nu - (\kappa_{s,\nu} + \kappa_{a,\nu})I_\nu]\,\delta r +$$

$$+ \frac{1}{r}\left(\frac{\delta\rho}{\rho} + 3\frac{\delta r}{r}\right)(1-\mu^2)\frac{\partial I_\nu}{\partial \mu}\,, \tag{A.4}$$

where Equation (A.1) was used to eliminate the derivative of δr. By taking moments of Equation (A.4) we obtain

$$\frac{\mathrm{d}\delta H_\nu}{\mathrm{d}r} + \frac{2}{r}\delta H_\nu = \rho[\kappa_{a,\nu}(\delta B_\nu - \delta J_\nu) + \delta\kappa_{a,\nu}(B_\nu - J_\nu)] -$$

$$- \frac{2}{r}\rho\kappa_{a,\nu}(B_\nu - J_\nu)\,\delta r + \left(\frac{\delta\rho}{\rho} + 3\frac{\delta r}{r}\right)H_\nu\,, \tag{A.5}$$

and

$$\frac{\mathrm{d}\delta K_\nu}{\mathrm{d}r} + \frac{1}{r}(3\delta K_\nu - \delta J_\nu) = -\rho[(\kappa_\nu \delta H_\nu + \delta\kappa_\nu H_\nu] +$$

$$+ \frac{2}{r}\rho\kappa_\nu H_\nu \delta r + \frac{1}{r}\left(\frac{\delta\rho}{\rho} + 3\frac{\delta r}{r}\right)(3K_\nu - J_\nu)\,. \tag{A.6}$$

The energy equation, in the form of Equation (2.15) or (2.18), is unchanged.

As shown by Hummer and Rybicki (1971) in the static case, it is possible to generalize the variable Eddington factor method to spherical geometry; for the oscillations this could be done by integrating Equation (A.4), with the right-hand side found from a trial solution, and determining the Eddington factors from the moments of the resulting δI_ν. On the other hand the effort involved in solving the transfer equation is considerably greater in spherical geometry than in the plane-parallel case; furthermore the dominant spherical term in Equation (A.4) appears to be the last, especially at high oscillation frequencies where $\delta\rho/\rho$ and $\delta r/r$ increase rapidly with height in the atmosphere. We have approximated this term by finding $\partial I_\nu/\partial\mu$ from the plane-parallel equilibrium equation of transfer; from Equation (2.9) it follows that

$$\mu\frac{\partial}{\partial r}\left(\frac{\partial I_\nu}{\partial \mu}\right) = -\frac{\rho}{\mu}[\kappa_{s,\nu}J_\nu + \kappa_{a,\nu}B_\nu - (\kappa_{s,\nu} + \kappa_{a,\nu})I_\nu] - (\kappa_{s,\nu} + \kappa_{a,\nu})\frac{\partial I_\nu}{\partial \mu}\,, \tag{A.7}$$

and this equation can be solved by quadrature once I_ν has been determined. On the other hand we neglected the spherical term on the left-hand side of Equation (A.4). Although this procedure is not quite consistent, it seems to ensure, at least in the grey case, approximate agreement between the δH and δK calculated from δI, and the δH and δK resulting from solving the moment equations with the iterated Eddington factors; thus it may be adequate. We intend in future to make a consistent investigation of the effects of spherical geometry on the Eddington factors for the oscillations.

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