J. Austral. Math. Soc. **24** (Series A) (1977), 362–374.

IDEALS IN AUTOMETRIZED ALGEBRAS

K. L. N. SWAMY and N. P. RAO

(Received 8 June 1976; revised 24 November 1976)

Communicated by Hans Lausch

Abstract

A notion of a normal autometrized algebra is introduced which generalises the concepts of Boolean geometry, Brouwerian geometry, autometrized lattice ordered groups, semi-Brouwerian geometry, etc. The notions of ideals and congruence relations are introduced in normal autometrized algebras and a one to one correspondence between ideals and congruence relations is established. Some other common properties of the above geometries are also obtained for normal autometrized algebras.

Introduction

Swamy (1964a) introduced the notion of an autometrized algebra to obtain a unified theory of the then known autometrized algebras:—(1) Boolean algebras (Blumenthal (1952) and Ellis (1951)), (2) Brouwerian algebras (Nordhaus and Lapidus (1954)), (3) Newman algebras (Kamala Ranjan (1960)), (4) autometrized lattices (Nordhaus and Lapidus (1954)) and (5) commutative lattice ordered groups or l-groups (Swamy (1964)). Later commutative dually residuated lattice ordered semigroups (D.R.l. semigroups) (Swamy (1965), (1965a), (1966)), Boolean l-algebras (Rama Rao (1972)) and semi-Brouwerian algebras (Ramana Murty (1974)) were all shown to possess a natural metric namely the "symmetric difference" which makes all of them into autometrized algebras. Of the above examples D.R.l. semigroups contain Brouwerian algebras and hence Boolean algebras, l-groups and Boolean l-groups as special cases and these are all lattices whereas semi-Brouwerian algebras which are not lattices contain Brouwerian algebras as special cases. Excepting Newman algebras which are not even partially ordered, all the above examples are autometrized algebras in the following sense: A system $(A, +, \leqq, *)$ is called an *autometrized algebra* if

(1.1) $(A, +)$ is a commutative semigroup with identity '0',

(1.2) $\leqq$ is a partial ordering on A such that the semigroup translations are invariant under inclusion, that is, $x \leqq y$ implies $a + x \leqq a + y$ in A,

(1.3) $*$ is a metric operation on A, that is, $*$ is a mapping from $A \times A$ to A such that
(1) $a * b \geqq 0$ for all a, b in A with equality if and only if $a = b$
(2) $a * b = b * a$
and (3) $a * c \leqq a * b + b * c$.

In this paper, by an autometrized algebra, we mean a system satisfying (1.1) through (1.3). Even this definition of an autometrized algebra is too general and so it may not be possible to obtain many of the common results shared by the above examples in a unified manner. So we introduce the notion of a normal autometrized algebra (see definition 1 below). D.R.l.-semigroups, semi-Brouwerian algebras are all examples of normal autometrized algebras. The example (7) below shows that these conditions of normality alone (1.4) through (1.7) on an autometrized algebra need not degenerate it into any of the above special cases.

It is known that in D.R.l. semigroups and hence in l-groups and Boolean algebras the ideals (l-ideals in the case of l-groups) correspond one to one to congruence relations. Also it is well known that in lattice ordered groups the closed ideals form a Boolean algebra (Birkhoff (1973) p. 308). Most of the above examples share these properties and so in this paper we introduce the notions of ideals and congruence relations in normal autometrized algebras and extend the above results to them.

1

We begin with the following:

DEFINITION 1. *An autometrized algebra* $(A, +, \leqq, *)$ *is called normal if and only if*

(1.4) $a \leqq a * 0$ *for all* a *in* A,

(1.5) $(a + c) * (b + d) \leqq (a * b) + (c * d)$ *for all* a, b, c, d *in* A,

(1.6) $(a * c) * (b * d) \leqq (a * b) + (c * d)$ *for all* a, b, c, d *in* A *and*

(1.7) *For any* a *and* b *in* A, $a \leqq b$ *implies there exists an* $x \geqq 0$ *such that* $a + x = b$.

REMARK. We remark that in an autometrized algebra $(A, +, \leqq, *)$ the conditions (1.5) and (1.6) are equivalent to:

(1.5)′ For every a, the mapping $x \to a + x$ is a contradiction (that is,
$$(a+x)*(a+y) \leqq x*y).$$

(1.6)′ For every a, the mapping $x \to a * x$ is a contradiction (that is,
$$(a*x)*(a*y) \leqq x*y).$$

The following examples show that semi-Brouwerian algebras, commutative D.R.l. semigroups and consequently Boolean algebras, Brouwerian algebras, commutative lattice ordered groups, Boolean l-algebras are all normal autometrized algebras endowed with the natural metric: the symmetric difference: $a * b = (a-b) \cup (b-a)$ in the case of D.R.l semigroups and $a * b = (a-b)+(b-a)$ in the case of semi-Brouwerian algebras.

EXAMPLE 1. Every commutative D.R.l. semigroup $(A, +, \leqq, -)$ is a normal autometrized algebra with distance $a * b = (a-b) \cup (b-a)$. By definition, a commutative D.R.l. semigroup $(A, +, \leqq, -)$ is a system where (1) $(A, +, \leqq)$ is a commutative lattice ordered semigroup with identity element '0', (2) Given a, b in A, there exists a least x in A such that $b + x \geqq a$ (we denote this x by $a - b$), (3) $(a-b) \cup 0 + b \leqq a \cup b$ for all a, b in A and (4) $(a-a) \geqq 0$. By theorem 9 of Swamy (1965), $(A, +, \leqq, *)$ is an autometrized algebra. Obviously (1.4) is satisfied in A. For the verification of (1.5), see Swamy (1966). For the verification of (1.7), if $a \leqq b$ take $x = b - a$. Now it is enough to verify that (1.6) is satisfied. Now, $(a*c)*(b*d) - (a*b + c*d)$ $= \{(a*c)*(b*d) - a*b\} - c*d = [\{(a*c - b*d) \cup (b*d - a*c)\} - a*b] - c*d = [\{(a*c - b*d) - a*b\} \cup \{(b*d - a*c) - a*b\}] - c*d = [\{a*c - (a*b + b*d)\} \cup \{b*d - (a*c + a*b)\}] - c*d \leqq \{(a*c - a*d) \cup (b*d - b*c)\} - c*d = \{(a*c - a*d) - c*d\} \cup \{(b*d - b*c) - c*d\} = \{a*c - (a*d + c*d)\} \cup \{b*d - (b*c + c*d)\} \leqq (a*c - a*c) \cup (b*d - b*d) = 0$ which shows (1.6) is also true in A.

EXAMPLE 2. Let $A = (A, +, -, 0)$ be any semi-Brouwerian algebra i.e. A is an algebra with two binary operations $+, -$ and a nullary operation 0 satisfying (1) $a + a = a$, (2) $a + b = b + a$, (3) $a - a = 0$, (4) $(a-b)+b = a + b$ and (5) $(a-b)-c = a-(c+b)$ for all a, b, c in A. By definition $a \leqq b$ if and only if $a + b = b$. Put $a * b = (a-b)+(b-a)$. By theorem 6 of Ramana Murty (1974), $(A, +, \leqq, *)$ is an autometrized algebra. The verifications of (1.4) and (1.7) are obvious. The verification of (1.6) follows in a routine manner as in example 1. Now we verify (1.5): Note that in A, $\{(a+c)$

$- (b+d)\} - a*b = \{(a+c) - (b+d)\} - \{(a-b) + (b-a)\} = [\{(a+c) - (b+d)\} - (a-b)] - (b-a) = [(a+c) - \{(a-b) + (b+d)\}] - (b-a) = [(a+c) - \{a+(b+d)\}] - (b-a) = [c - \{a+(b+d)\}] - (b-a) = c - [(b-a) + \{a+(b+d)\}] = c - \{(a+b) + d\}$. Similarly $\{(b+d) - (a+c)\} - a*b = d - \{(a+b) + c\}$. Now, $\{(a+c)*(b+d)\} - \{(a*b) + (c*d)\} = \{(a+c)*(b+d) - a*b\} - c*d = [\{\{(a+c) - (b+d)\} + \{(b+d) - (a+c)\}\} - a*b] - c*d = [\{c - \{(a+b) + d\}\} + \{d - \{(a+b) + c\}\}] - c*d = [\{(c-d) - (a+b)\} + \{(d-c) - (a+b)\}] - c*d = [\{(c-d) + (d-c)\} - (a+b)] - c*d = \{c*d - (a+b)\} - c*d = (c*d - c*d) - (a+b) = 0 - (a+b) = 0$. Hence $(a+c)*(b+d) \leqq (a*b) + (c*d)$ holds in A.

The following examples (3) through (6) show that on an autometrized algebra the conditions (1.4) through (1.7) are independent.

EXAMPLE 3. Let A be the set of all nonnegative integers other than 1 together with an element ∞ outside the set of nonnegative integers. For any nonnegative integers x and y in A let $x+y$ be the usual integer addition and for any x in A, let $x+\infty = \infty + x = \infty$. For nonnegative integers x and y in A let $x \leqq y$ mean the usual inclusion and for any nonnegative integer x in A define $x < \infty$. For any x and y in A put $x*y = \infty$ if $x \neq y$ and $x*y = 0$ otherwise.In this (1.1) through (1.6) are satisfied and (1.7) fails.

EXAMPLE 4. Let A be the set of nonnegative integers. Let $+$ be the usual addition and $\leqq$ be the natural order. For any x and y put $x*y = 2|x-y|$. (1.6) fails in this with $a=0$, $c=3$, $b=0$, $d=1$ and all the other postulates are satisfied.

EXAMPLE 5. Let $(A, +, \leqq)$ be the same as in example (4). For any x and y in A, put $x*y =$ maximum of $\{x, y\}$ if $x \neq y$ and $x*y = 0$ otherwise. With $a=2$, $c=3$, $b=2$, $d=4$, (1.5) fails and all the other postulates hold.

EXAMPLE 6. Let $(A, +, \leqq)$ be the same as in example (4). For any x and y in A put $x*y = 5$ if $x \neq y$ and $x*y = 0$ otherwise. Obviously (1.4) fails and all the remaining postulates hold.

The following is an example of a normal autometrized algebra $(A, +, \leqq, *)$ in which $(A, \leqq)$ is not a lattice and in which $+$ is not idempotent. Consequently this example is neither a D.R.l. semigroup nor a semi-Brouwerian algebra.

EXAMPLE 7. Let $(G_1, +, \leqq)$ and $(G_2, +, \leqq)$ be abelian lattice ordered groups where G_1 is not a chain. Let $G = G_1 \circ G_2$ be the lexicographic product of G_1 and G_2. Then $(G, +, \leqq)$ is a partially ordered group and is not a lattice with this inclusion (see Birkhoff (1973) p. 289). For (a_1, a_2), (b_1, b_2) in G, put $(a_1, a_2)*(b_1, b_2) = (|a_1 - b_1|, |a_2 - b_2|)$. Clearly $(G, +, \leqq, *)$ is a normal autometrized algebra.

2

We begin this section with the following definition of an ideal in a normal autometrized algebra. Throughout this section $A = (A, +, \leqq, *)$ denotes a normal autometrized algebra.

DEFINITION 2. *A nonempty subset I of a normal autometrized algebra $A = (A, +, \leqq, *)$ is called an ideal if and only if*

(2.1) $$a \in I, b \in I \text{ imply } a + b \in I$$

(2.2) $$a \in I, b \in A, b*0 \leqq a*0 \text{ imply } b \in I.$$

In the case where $(A, +, \leqq, *)$ is a D.R.l. semigroup (i.e. when $a * b = (a - b) \cup (b - a)$ in the D.R.l. semigroup $(A, +, \leqq, -)$) this notion of ideal coincides with the notion of ideal in the D.R.l. semigroup. Consequently this notion of ideal coincides with the notion of l-ideal in a commutative l-group and also with that ideal in a Boolean algebra when commutative lattice ordered groups and Boolean algebras are treated as normal autometrized algebras.

Now we have:

THEOREM 1. *The set of ideals of a normal autometrized algebra A is a complete algebraic lattice ordered by set inclusion.*

We need the following lemmas.

LEMMA 1. *The intersection of any nonempty collection of ideals is an ideal.*

PROOF. Obvious.

LEMMA 2. *For any nonempty subset S of A, the set $\{x \in A \mid x*0 \leqq m_1(a_1*0) + \cdots + m_k(a_k*0)$ for some positive integers $m_1, \cdots, m_k$ and $a_1, \cdots, a_k \in S\}$ is the smallest ideal containing S.*

PROOF. Let $G = \{x \in A \mid x*0 \leqq m_1(a_1*0) + \cdots + m_k(a_k*0)$ for some positive integers $m_1, \cdots, m_k$ and $a_1, \cdots, a_k \in S\}$. For x, y in G, from (1.5) we get $(x + y)*0 \leqq x*0 + y*0$ and hence it follows that $x + y$ is in G. If $x \in G$ and $y*0 \leqq x*0$, clearly $y \in G$. Hence G is an ideal. Clearly $S \subset G$. Let I be any ideal containing S. Let $x \in G$. Then $x*0 \leqq m_1(a_1*0) + \cdots + m_k(a_k*0)$ for some positive integers $m_1, \cdots, m_k$ and $a_1, \cdots, a_k \in S$. Hence by (1.4) it follows that $x*0 \leqq \{m_1(a_1*0)+\cdots m_k(a_k*0)\}*0$ for some positive integers $m_1,\cdots,m_k$ and $a_1,\cdots,a_k \in S \subset I$. Since I is an ideal, $m_1(a_1*0)+\cdots+m_k(a_k*0) \in I$. Hence $x \in I$. So $G \subset I$. Hence G is the smallest ideal containing S.

REMARK. From the above it is clear that the principal ideal generated by 'a' denoted by (a) i.e. the smallest ideal containing a is $\{x \in A \mid x*0 \leqq m(a*0)$ for some positive integer $m\}$. Also from the lemmas (1) and (2) it is clear that the join of any collection of ideals $\{I_\alpha\}_{\alpha\in\Delta}$ is the ideal generated by $\bigcup_{\alpha\in\Delta} I_\alpha$.

In particular we have the following:

COROLLARY 1. *If I and J are ideals of A, $\{a \in A \mid a*0 \leqq x+y$ for some $x \in I$ and $y \in J\}$ is the join of I and J.*

LEMMA 3. *The join of any two principal ideals in A is a principal ideal.*

PROOF. Let (a) and (b) be principal ideals. We show that $(a)\vee(b) = (a*0+b*0)$. Clearly $(a*0+b*0)$ contains (a) and (b). Let J be any ideal containing (a) and (b). Let $x \in (a*0+b*0)$. Then $x*0 \leqq m\{(a*0+b*0)*0\}$ for some positive integer m. Now $a \in (a) \subset J$ and $b \in (b) \subset J$ show that $a \in J$ and $b \in J$. Hence $a*0$ and $b*0$ belong to J so that $t = m\{(a*0+b*0)*0\} \in J$. Now $x*0 \leqq t \leqq t*0$ (by 1.4) and $t \in J$. Hence it follows that $x \in J$. Hence $(a*0+b*0) \subset J$. So $(a)\vee(b) = (a*0+b*0)$. Hence $(a)\vee(b)$ is a principal ideal.

LEMMA 4. *In A, an ideal I is compact if and only if it is a principal ideal.*

PROOF. Let (a) be the principal ideal generated by a. Suppose $(a) \subset \bigvee_{\alpha\in\Delta} I_\alpha$ where $\{I_\alpha\}_{\alpha\in\Delta}$ is a family of ideals. Now $a \in (a)$ implies that $a \in \bigvee_{\alpha\in\Delta} I_\alpha$ which implies that $a*0 \leqq m_1(a_1*0)+\cdots+m_k(a_k*0)$ for some positive integers $m_1,\cdots,m_k$ and $a_1 \in I_{\alpha_1},\cdots,a_k \in I_{\alpha_k}$ for some $\alpha_1,\cdots,\alpha_k$. Hence $a \in I_{\alpha_1} \vee\cdots\vee I_{\alpha_k}$ so that $(a) \subset I_{\alpha_1} \vee\cdots\vee I_{\alpha_k}$. Hence it follows that (a) is a compact ideal.

Let I be any compact ideal. Since $I \subset \bigvee_{a\in I}(a)$, it follows that $I \subset (a_1)\vee\cdots\vee(a_n)$ for some $a_1,\cdots,a_n \in I$. Hence $I = (a_1)\vee\cdots\vee(a_n)$. From lemma (3) it follows that I is a principal ideal. Hence the lemma.

From lemmas (1), (2), (3) and (4) the proof of theorem (1) is clear.

COROLLARY 2. *The ideal lattice of any* D.R.l. *semigroup (consequently the lattice of l-ideals of any commutative l-group, the ideal lattice of a Boolean algebra, the ideal lattice of a Brouwerian algebra) and the ideal lattice of a semi-Brouwerian algebra are algebraic.*

Now we introduce the following:

DEFINITION 3. *An equivalence relation θ on A is called a congruence relation if and only if*

(3.1) $$a \equiv b(\theta), c \equiv d(\theta) \text{ imply that } a + c \equiv b + d(\theta)$$ *for any a, b, c, d in A,*

(3.2) $$a \equiv b(\theta), c \equiv d(\theta) \text{ imply that } a * c \equiv b * d(\theta)$$ *for any a, b, c, d in A,*

(3.3) $$a \equiv b(\theta) \text{ and } x * y \leqq a * b \text{ imply that } x \equiv y(\theta)$$ *for any a, b, x, y, in A.*

In the case where the autometrized algebra is either a D.R.l. semigroup or a Boolean algebra or a l-group, this notion of congruence relation coincides with the notion of congruence relation in those algebras. Also since semi-Brouwerian algebras form an equationally definable class of algebras there is also the notion of congruence relation as an equivalence relation having the substitution property with $+, -$. When we treat any semi-Brouwerian algebra as a normal autometrized algebra, the notion of congruence relation which we introduced coincides with the usual notion of congruence relation in a semi-Brouwerian algebra.

Now we have the following:

THEOREM 2. *The ideals of any normal autometrized algebra A correspond one to one to its congruence relations.*

We need a lemma.

LEMMA 5. *For any a and b in A, $(a * b) * 0 = a * b$.*

PROOF. $a * b \leqq (a * b) * 0$ (by 1.4). Now $(a * b) * 0 = (a * b) * (b * b) \leqq a * b + b * b$ (by 1.6) $= a * b$. Hence the lemma.

PROOF OF THEOREM 2. Let S be an ideal of A. Define $a \equiv b(\theta)$ if and only if $a * b \in S$. Since $0 \in S$, $a \equiv a(\theta)$. Let $a \equiv b(\theta)$. Since $a * b = b * a$ it follows that $b \equiv a(\theta)$. Let $a \equiv b(\theta)$ and $b \equiv c(\theta)$. Then $a * b \in S$ and $b * c \in S$. Now $(a * c) * 0 = a * c$ (by lemma 5) $\leqq a * b + b * c \leqq (a * b + b * c) * 0$ (by 1.4). $a * b + b * c \in S$ shows that $a * c \in S$. Hence $a \equiv c(\theta)$. Hence θ is an equivalence relation.

Let $a \equiv b(\theta)$ and $c \equiv d(\theta)$. Then $a * b \in S$ and $c * d \in S$. Now $\{(a + c) * (b + d)\} * 0 = (a + c) * (b + d) \leqq a * b + c * d$ (by 1.5) $\leqq (a * b + c * d) * 0$ (by 1.4). $a * b + c * d \in S$ and S is an ideal show that $(a + c) * (b + d) \in S$. Hence $a + c \equiv b + d(\theta)$. Also, $\{(a * c) * (b * d)\} * 0 =$

https://doi.org/10.1017/S1446788700020383 Published online by Cambridge University Press

$(a*c)*(b*d) \leqq a*b+c*d$ (by 1.6) $\leqq (a*b+c*d)*0$ (by 1.4). Since $a*b+c*d \in S$, this implies that $(a*c)*(b*d) \in S$. Hence $a*c \equiv b*d(\theta)$. Let $a \equiv b(\theta)$ and $x*y \leqq a*b$. Then $(x*y)*0 = x*y \leqq a*b = (a*b)*0$ and $a*b \in S$. Since S is an ideal, it follows that $x*y \in S$. Hence $x \equiv y(\theta)$. Hence θ is a congruence relation.

Suppose θ is a congruence relation on A. Let $S = \{x/x \equiv 0(\theta)\}$. Since $0 \in S$, S is nonempty. Let $a \in S$ and $b \in S$. Then $a \equiv 0(\theta)$ and $b \equiv 0(\theta)$. Since θ is a congruence relation, it follows that $a+b \equiv 0(\theta)$. Hence $a+b \in S$. Let $a \in S$ and $b*0 \leqq a*0$. Now $a \in S$ implies that $a \equiv 0(\theta)$. Since θ is a congruence relation, it follows that $b \equiv 0(\theta)$. Hence $b \in S$. Hence S is an ideal.

Let θ be a congruence relation. Let S be the ideal obtained from θ and let θ' be the congruence relation obtained from S, as above. Then $x \equiv y(\theta')$ if and only if $x*y \in S$ which holds if and only if $x*y \equiv 0(\theta)$ and which again holds if and only if $x \equiv y(\theta)$. Hence $\theta = \theta'$.

Let S be an ideal of A. Let θ be the congruence relation obtained from S and let S' be the ideal obtained from θ. Then $x \in S'$ if and only if $x \equiv 0(\theta)$ which holds if and only if $x*0 \in S$ and which again holds if and only if $x \in S$. Hence $S = S'$. Hence the theorem.

Corollary 3. *In each of* D.R.l. *semigroups (consequently in commutative l-groups, Boolean algebras, Brouwerian algebras) and semi-Brouwerian algebras, ideals correspond one to one to its congruence relations.*

If θ is any equivalence relation on a set A, as usual A/θ denotes the quotient set (i.e. the set of all θ-equivalence classes) and for any x in A, $\bar{x}$ denotes the equivalence class containing x.

Theorem 3. *Let θ be a congruence relation on A. For any $\bar{a}, \bar{b}$ in A/θ, define $\bar{a}+\bar{b} = \overline{a+b}$; $\bar{a}*\bar{b} = \overline{a*b}$; $\bar{a} \leqq \bar{b}$ if there is an $x \geqq 0$ such that $a+x \equiv b(\theta)$. Then $(A/\theta, +, \leqq, *)$ is a normal autometrized algebra if and only if θ has the following property*:

For any a, b and $z_1 \geqq 0, z_2 \geqq 0$; $a+z_1 \equiv b(\theta)$ and $b+z_2 \equiv a(\theta)$ implies that $a \equiv b(\theta)$.

Proof. Suppose θ has the property: For any a, b and $z_1 \geqq 0, z_2 \geqq 0$; $a+z_1 \equiv b(\theta)$ and $b+z_2 \equiv a(\theta)$ imply that $a \equiv b(\theta)$. Now we show that $(A/\theta, +, \leqq, *)$ is a normal autometrized algebra.

It is routine to verify that $+, *$ are well defined. Let $\bar{a} \leqq \bar{b}$ and $\bar{a} = \overline{a'}, \bar{b} = \overline{b'}$. Hence there exists an $x \geqq 0$ such that $a+x \equiv b(\theta)$ and $a \equiv a'(\theta), b \equiv b'(\theta)$. Now $a'+x \equiv a+x \equiv b \equiv b'(\theta)$. Hence $\overline{a'} \leqq \overline{b'}$. Hence $\leqq$ is well defined. It is routine to verify that $(A/\theta, +)$ is a binary commutative algebra with additive identity $\bar{0}$.

$\bar{a} \leqq \bar{a}$ is obvious. Let $\bar{a} \leqq \bar{b}$ and $\bar{b} \leqq \bar{a}$. Then there exists $z_1 \geqq 0$ and $z_2 \geqq 0$ such that $a + z_1 \equiv b(\theta)$ and $b + z_2 \equiv a(\theta)$. Hence $a \equiv b(\theta)$ by our assumption so that $\bar{a} = \bar{b}$. Let $\bar{a} \leqq \bar{b}$ and $\bar{b} \leqq \bar{c}$. Then there exist $z_1 \geqq 0, z_2 \geqq 0$ such that $a + z_1 \equiv b(\theta)$ and $b + z_2 \equiv c(\theta)$. Now $c \equiv b + z_2 \equiv a + z_1 + z_2(\theta)$. Since $z_1 + z_2 \geqq 0$ it follows that $\bar{a} \leqq \bar{c}$. Hence $\leqq$ is a partial ordering on A/θ. Let $\bar{x} \leqq \bar{y}$. Then there exist a $z \geqq 0$ such that $x + z \equiv y(\theta)$. Hence for any a in $A, x + a + z \equiv y + a(\theta)$ which shows that $\bar{x} + \bar{a} = \overline{x + a} \leqq \overline{y + a} = \bar{y} + \bar{a}$. Hence $(A/\theta, \leqq)$ is a partially ordered set such that the semigroup translations are invariant under inclusion.

$a * b \geqq 0$ and since $0 + a * b \equiv a * b(\theta)$ follows that $\bar{0} \leqq \overline{a * b} = \bar{a} * \bar{b}$ for any $\bar{a}, \bar{b}$ in A/θ. Clearly $\bar{a} * \bar{a} = \overline{a * a} = \bar{0}$. Let $\bar{a} * \bar{b} = \bar{0}$. Then $\overline{a * b} = \bar{0}$ so that $a * b \equiv 0(\theta)$ which implies $a \equiv b(\theta)$. Hence $\bar{a} = \bar{b}$. Also $\bar{a} * \bar{b} = \overline{a * b} = \overline{b * a} = \bar{b} * \bar{a}$. Since $a * c \leqq a * b + b * c$, there exists an $x \geqq 0$ such that $a * c + x = a * b + b * c$ (by 1.7). Hence $\bar{a} * \bar{c} = \overline{a * c} \leqq \overline{a * b + b * c} = \bar{a} * \bar{b} + \bar{b} * \bar{c}$. Hence $*$ satisfies the formal properties of a metric.

$a \leqq a * 0$ implies that there exists an $x \geqq 0$ such that $a + x = a * 0$ (by 1.7). Hence $\bar{a} \leqq \overline{a * 0} = \bar{a} * \bar{0}$ so that (1.4) is true in A/θ. $(a + c) * (b + d) \leqq (a * b) + (c * d)$ implies that there exists an $x \geqq 0$ such that $(a + c) * (b + d) + x = a * b + c * d$. Hence $(\bar{a} + \bar{c}) * (\bar{b} + \bar{d}) = \overline{(a + c) * (b + d)} \leqq \overline{a * b + c * d} = (\bar{a} * \bar{b}) + (\bar{c} * \bar{d})$. Hence (1.5) is true in A/θ. On similar lines we can show (1.6) is also true in A/θ. Let $\bar{a} \leqq \bar{b}$. Then there exists an $x \geqq 0$ such that $a + x \equiv b(\theta)$ which implies that $\bar{x} \geqq \bar{0}$ and that $\bar{a} + \bar{x} = \overline{a + x} = \bar{b}$. Hence (1.7) is also true in A/θ. Hence A/θ is a normal autometrized algebra.

Suppose A/θ is a normal autometrized algebra. Suppose for any a, b and $z_1 \geqq 0, z_2 \geqq 0$; $a + z_1 \equiv b(\theta)$ and $b + z_2 \equiv a(\theta)$. Then $\bar{a} \leqq \bar{b}$ and $\bar{b} \leqq \bar{a}$. Since $\leqq$ is a partial ordering on A/θ it follows that $\bar{a} = \bar{b}$ so that $a \equiv b(\theta)$. Hence the theorem.

We remark here that all congruence relations in all the normal autometrized algebras mentioned earlier satisfy the above property.

Now we introduce the notion of a homomorphism between normal autometrized algebras and we obtain a fundamental theorem of homomorphisms in normal autometrized algebras.

DEFINITION 4. *Let A and B be normal autometrized algebras. We say that a mapping $f: A \to B$ is a homomorphism from A to B if and only if*

(4.1) $f(a + b) = f(a) + f(b)$ *for any a, b in A,*

(4.2) $f(a * b) = f(a) * f(b)$ *for any a, b in A and*

(4.3) $a \leqq b$ *implies* $f(a) \leqq f(b)$ *for any a, b in A.*

We immediately have the following:

THEOREM 4. *In any normal autometrized algebra, ideals are just the kernels of epimorphisms.*

PROOF. Let f be an epimorphism from A to B. By definition $\ker f = \{x \in A / f(x) = \bar{0}$ where $\bar{0}$ is the zero element of $B\}$. Note that $f(0) = f(0 * 0) = f(0) * f(0) = \bar{0}$. Let $x \in \ker f, y \in \ker f$. Then $f(x) = \bar{0}, f(y) = \bar{0}$. Hence $f(x+y) = f(x) + f(y) = \bar{0} + \bar{0} = \bar{0}$. Hence $x + y \in \ker f$. Let $x \in \ker f$ and $y * 0 \leqq x * 0$. Then $f(x) = \bar{0}$ and $f(y * 0) \leqq f(x * 0)$. Now $\bar{0} \leqq f(y) * \bar{0} = f(y) * f(0) = f(y * 0) \leqq f(x * 0) = f(x) * f(0) = \bar{0} * \bar{0} = \bar{0}$. Hence $f(y) * \bar{0} = \bar{0}$. Hence $f(y) = \bar{0}$ so that $y \in \ker f$. Hence $\ker f$ is an ideal of A.

Let I be an ideal of a normal autometrized algebra A. Let θ be the congruence relation corresponding to I. The mapping $f : A \to A/\theta$ defined by $f(a) = \bar{a}$ for $a \in A$ is clearly onto and obviously f satisfies (4.1) and (4.2). Let $a \leqq b$. Then there exists an $x \geqq 0$ such that $a + x = b$. Hence $a + x \equiv b(\theta)$ which implies that $\bar{a} \leqq \bar{b}$ i.e. $f(a) \leqq f(b)$. Hence f is a homomorphism so that f is an epimorphism from A to A/θ. Also $x \in \ker f$ if and only if $f(x) = \bar{0}$ which holds if and only if $\bar{x} = \bar{0}$ which again holds if and only if $x \equiv 0(\theta)$ i.e. $x \in \ker f$ if and only if $x * 0 \in I$ which holds if and only if $x \in I$. Hence $\ker f = I$. Hence the theorem.

We conclude this section with the following fundamental theorem:

THEOREM 5. *If f is an epimorphism from a normal autometrized algebra A to a normal autometrized algebra B then $A/\ker f$ is isomorphic with B.*

PROOF. From the above theorem, $\ker f$ is an ideal of A. Let θ be the congruence relation corresponding to $\ker f$. Define $g : A/\theta \to B$ by $g(\bar{a}) = f(a)$ for $\bar{a} \in A/\theta$. Now $\bar{a} = \bar{b}$ if and only if $a \equiv b(\theta)$ which holds if and only if $a * b \in \ker f$ which again holds if and only if $f(a * b) = \bar{0}$ i.e. $\bar{a} = \bar{b}$ if and only if $f(a) * f(b) = \bar{0}$ which holds if and only if $f(a) = f(b)$. Hence g is well defined and also one to one. Obviously g is onto. It is routine to verify that g satisfies (4.1) and (4.2). Let $\bar{a} \leqq \bar{b}$. Then there exists an $x \geqq 0$ such that $a + x \equiv b(\theta)$. Hence $\overline{a + x} = \bar{b}$ so that $g(\overline{a + x}) = g(\bar{b})$. Hence $f(a) + f(x) = f(a + x) = f(b)$. Hence $f(a) \leqq f(b)$ since $f(x) \geqq \bar{0}$. So $\bar{a} \leqq \bar{b}$ implies that $g(\bar{a}) \leqq g(\bar{b})$. Hence g satisfies (4.3) so that g is an isomorphism from A/θ to B. Hence the theorem.

3

Swamy (1965a) introduced the notion of a lattice ordered autometrized algebra as a system $(A, +, \leqq, *)$ satisfying

(5.1) $(A, +, \leqq)$ is an abelian lattice ordered semigroup with

https://doi.org/10.1017/S1446788700020383 Published online by Cambridge University Press

identity '0' i.e. $(A, +)$ is a commutative semigroup with identity '0' and $(A, \leqq)$ is a lattice with lattice operations $\cup, \cap$ such that $a + (b \cup c) = (a + b) \cup (a + c)$ and $a + (b \cap c) = (a + b) \cap (a + c)$.

(5.2) $*$ is metric operation on A.

D.R.l. semigroups and consequently l-groups, Brouwerian algebras, Boolean algebras, autometrized lattices are all lattice ordered autometrized algebras. According to Swamy (1965) an autometrized algebra is regular if $a * 0 = a$ for all a. Even though Brouwerian algebras, semi-Brouwerian algebras etc., are regular, nonzero lattice ordered groups are never regular. Consequently, in general D.R.l. semigroups are not regular autometrized algebras with symmetric difference as distance but the symmetric difference in D.R.l. semigroup has the property that $a * 0 = a$ if $a \geqq 0$. Thus we introduce the following:

DEFINITION 5. *An autometrized algebra $(A, +, \leqq, *)$ is called semiregular if and only if $a \geqq 0$ implies $a * 0 = a$.*

Now we have the following:

LEMMA 6. *The lattice of ideals of a semiregular normal lattice ordered autometrized algebra A is distributive.*

PROOF. It is enough to show that $I \wedge (J \vee K) \subset (I \wedge J) \vee (I \wedge K)$ for ideals I, J, K of A. Let $x \in I \wedge (J \vee K)$. Then $x \in I$ and $x * 0 \leqq a + b$ for some $a \in J$ and $b \in K$. Put $x * 0 = y$. Now

$$\begin{aligned}(y \cap a) + (y \cap b) &= \{(y \cap a) + y\} \cap \{(y \cap a) + b\} \\ &= (y + y) \cap (a + y) \cap (y + b) \cap (a + b) \geqq y \cap (a + b) = y.\end{aligned}$$

Now $y \geqq 0$, $a \geqq 0$ show that $y \cap a \geqq 0$ so that $(y \cap a) * 0 = y \cap a \leqq a * 0$. Since $a \in J$, it follows that $y \cap a \in J$. Since $x \in I$, $y = x * 0 \in I$. Hence $(y \cap a) * 0 = y \cap a \leqq y = x * 0 = (x * 0) * 0$ which implies that $y \cap a \in I$. Hence $y \cap a \in I \wedge J$. Also $y \cap b \in I \wedge K$. Now $x * 0 = y \leqq (y \cap a) + (y \cap b)$ where $y \cap a \in I \wedge J$ and $y \cap b \in I \wedge K$. Hence $x \in (I \wedge J) \vee (I \wedge K)$. Hence $I \wedge (J \vee K) \subset (I \wedge J) \vee (I \wedge K)$. Hence the lemma.

Brouwerian logics (= Brouwerian algebras) considered and autometrized with the symmetric difference by Nordhaus and Lapidus (1954) are just the duals of what Birkhoff calls Brouwerian lattices. According to Birkhoff (1973), a Brouwerian lattice is a lattice L in which for any given elements a and b, the set of all x belonging to L such that $a \cap x \leqq b$ contains a greatest element $b : a$, the relative pseudocomplement of a in b. With this definition it is well known (Birkhoff (1973) p. 304) that the lattice of l-ideals of any l-group is a complete algebraic Brouwerian lattice. This result is also true in Boolean algebras. In fact we have the following theorem:

THEOREM 6. *The lattice of ideals of a semiregular normal lattice ordered autometrized algebra A is a complete algebraic Brouwerian lattice.*

PROOF. By theorem V.24 of Birkhoff (1973), it is enough if we show that $I \wedge (\vee I_\alpha) = \vee (I \wedge I_\alpha)$ where $I, \{I_\alpha\}_{\alpha \in \Delta}$ are ideals of A. $\vee (I \wedge I_\alpha) \subset I \wedge (\vee I_\alpha)$ is clear. Let $x \in I \wedge (\vee I_\alpha)$. Then $x \in I$ and $x * 0 \leqq m_1(a_1 * 0) + \cdots + m_k(a_k * 0)$ for some positive integers $m_1, \cdots, m_k$ and $a_1 \in I_{\alpha_1}, \cdots, a_k \in I_{\alpha_k}$ for some $\alpha_1, \cdots, \alpha_k$. Hence $x \in I_{\alpha_i} \vee \cdots \vee I_{\alpha_k}$ so that $x \in I \wedge (I_{\alpha_1} \vee \cdots \vee I_{\alpha_k}) = (I \wedge I_{\alpha_1}) \vee \cdots \vee (I \wedge I_{\alpha_k}) \subset \vee (I \wedge I_\alpha)$. Hence $x \in \vee (I \wedge I_\alpha)$. Hence $I \wedge (\vee I_\alpha) \subset \vee (I \wedge I_\alpha)$. Hence the theorem.

It is well known (Birkhoff (1973) p. 308) that in a lattice ordered group G, for any elements a and b if we define $a \perp b$ to mean $|a| \cap |b| = 0$, then the sets closed under the polarity defined by the symmetric relation $a \perp b$ form a Boolean lattice. One can observe that this is also true in Boolean algebras, Brouwerian algebras and D.R.l. semigroups. We extend this result to semiregular normal lattice ordered autometrized algebras.

We introduce the following:

DEFINITION 6. *For elements a and b in a lattice ordered autometrized algebra we write $a \perp b$ to mean $(a * 0) \cap (b * 0) = 0$.*

We observe the following lemma:

LEMMA 7. *If J is any ideal of a semiregular normal lattice ordered autometrized algebra A, then $J^* = \{x \in A / x \perp a$ for all a in $J\}$ is an ideal in A and J^* is the pseudocomplement of J.*

PROOF. Let $x \in J^*, y \in J^*$. Then $(x * 0) \cap (a * 0) = 0$ and $(y * 0) \cap (a * 0) = 0$ for all a in J. Now $0 \leqq \{(x + y) * 0\} \cap (a * 0) \leqq (x * 0 + y * 0) \cap (a * 0) \leqq \{(x * 0) \cap (a * 0)\} + \{(y * 0) \cap (a * 0)\} = 0$. Hence $\{(x + y) * 0\} \cap (a * 0) = 0$ for all a in J so that $x + y \in J^*$. Let $x \in J^*$ and $y * 0 \leqq x * 0$. $x \in J^*$ implies that $(x * 0) \cap (a * 0) = 0$ for all a in J. Now $(y * 0) \cap (a * 0) = (y * 0) \cap (x * 0) \cap (a * 0) = (y * 0) \cap 0 = 0$ for all a in J. Hence $y \in J^*$. Hence J^* is an ideal of A.

Let $x \in J \wedge J^*$. Then $x \in J$ and $x \in J^*$ which implies $(x * 0) \cap (x * 0) = 0$. Hence $x * 0 = 0$ so that $x = 0$. Hence $J \wedge J^* = \{0\}$. Let I be any ideal of A such that $J \wedge I = \{0\}$. Let $x \in I$. Take any $a \in J$. Since $(x * 0) \cap (a * 0) \geqq 0, \{(x * 0) \cap (a * 0)\} * 0 = (x * 0) \cap (a * 0) \leqq x * 0$. Since $x \in I$, it follows that $(x * 0) \cap (a * 0) \in I$. Similarly we can show $(x * 0) \cap (a * 0) \in J$. Hence

$(x*0) \cap (a*0) \in I \wedge J = \{0\}$ so that $(x*0) \cap (a*0) = 0$. Hence $x \in J^*$ which implies that $I \subset J^*$. Hence J^* is the pseudocomplement of J. Hence the lemma.

By a theorem which is essentially due to Glivenko (Birkhoff (1973) p. 130, Theorem 26), we immediately have the following:

THEOREM 7. *If $\Theta(A)$ is the lattice of all ideals of a semiregular normal lattice ordered autometrized algebra A, then the mapping $J \to J^{**}$ is a closure operation on $\Theta(A)$ and a lattice epimorphism of $\Theta(A)$ onto the Boolean lattice of closed elements of $\Theta(A)$.*

The authors would like to thank Prof. N. V. Subrahmanyam whose interest in this topic has led the authors to write this. In conclusion the authors thank the referee whose comments led this paper into the present form.

References

G. Birkhoff (1973), *Lattice Theory*, (Amer. Math. Soc. Col. Publications, 3rd Edition, 2nd Printing, 1973).

L. M. Blumenthal (1952), 'Boolean Geometry I', *Rend. Contidel. Circ. Palermo Serie ii* **1**, 343–360.

D. Ellis (1951), 'Autometrized Boolean Algebras I', *Canad. J. Math.* **3**, 83–87.

Roy, Kamala Ranjan (1960), 'Newmanian Geometry I', *Bull. Calcutta Math. Soc.* **52**, 187–194.

E. A. Nordhaus and Leo Lapidus (1954), 'Brouwerian Geometry', *Canad. J. Math.* **6**. 217–229.

P. V. Ramana Murty (1974), 'Semi Brouwerian Algebras', *Jour. Aust. Math. Soc.* **18**, Part 3, 293–302.

V. V. Rama Rao (1972), 'On a Common Abstraction of Boolean Rings and Lattice Ordered Groups II', *Jour. Aust. Math. Soc.* **13**, Part 2, 185–195.

K. L. N. Swamy (1964), 'Autometrized Lattice Ordered Groups I', *Math. Annalen* **154**, 406–412.

K. L. N. Swamy (1964a), 'A general theory of Autometrized Algebras', Math. Annalen **157**, 65–74.

K. L. N. Swamy (1965), 'Dually Residuated Lattice Ordered Semigroups', *Math. Annalen* **159**, 105–114.

K. L. N. Swamy (1965a), 'Dually Residuated Lattice Ordered Semigroups II', *Math. Annalen* **160**, 64–71.

K. L. N. Swamy (1966), 'Dually Residuated Lattice Ordered Semigroups III', *Math. Annalen* **167**, 71–74.

Department of Mathematics,
Andhra University,
Waltair 530 003, India.

https://doi.org/10.1017/S1446788700020383 Published online by Cambridge University Press