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DISTANCE TO THE INTERSECTION OF TWO SETS

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We give sufficient conditions so that the distance of a point to the intersection of two sets agrees with the maximum of the distances to each of them. The results are established in several settings: complete metric spaces, Banach spaces and spaces of subsets of Banach spaces.

1. INTRODUCTION

Let E be a metric space and d its distance. Given a point z in E and a nonempty subset R of E, the *distance* of z to R is $d(z,R) := \inf\{d(z,u) : u \in R\}$. If $R \neq \emptyset$ and $S \neq \emptyset$ are subsets of E and $z \in E$, then

$$R \subset S \Rightarrow d(z,R) \geqslant d(z,S).$$

From this, we obtain, for $\emptyset \neq R, S \subset E$ and $z \in E$,

$$d(z, R \cap S) \geqslant \max\left\{d(z,R), d(z,S)\right\}.$$

It is very easy to find examples for the strict inequality. Take $E = \mathbf{R}^2$, the real space of two dimensions, endowed with the Euclidean distance, $R = \{(\alpha,\beta) : \beta > 0\}$, $S = \{(\alpha,\beta) : \alpha > 0\}$ and $z = (-3,-4)$; then $d(z, R\cap S) = 5$ and $\max\left\{d(z,R), d(z,S)\right\} = 4$. In this paper we are interested in finding sufficient conditions for the validity of the equality

$$d(z, R \cap S) = \max\left\{d(z,R), d(z,S)\right\}. \tag{1}$$

Taking into account that the inequality "$\geqslant$" is always correct, we look for conditions which assure the other inequality

$$d(z, R \cap S) \leqslant \max\left\{d(z,R), d(z,S)\right\}.$$

The following theorem refers to Banach spaces and it is the prototype of the theorems which we obtain in this paper. We give several proofs of it in the next sections, since it is obtained as corollary of other theorems.

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THEOREM 1. *Let X be a Banach space. If R and S are nonempty closed subsets of X such that $R \cup S$ is a convex set, then $R \cap S \neq \emptyset$ and*

$$d(z, R \cap S) = \max\{d(z, R), d(z, S)\},$$

for every $z \in X$.

The essential hypothesis in the above theorem is that $R \cup S$ is a convex subset of the Banach space. We want to obtain similar results to Theorem 1 in other settings. For this purpose we need to work with other notions of convexity.

First we consider a complete metric space E. The notions of metric segment and metrically convex subset turn out to be very useful. We show that if the union of the subsets R and S of E is metrically convex and the point z has a special position with respect to R and S, then the equality (1) is correct (Theorem 2). Theorem 1 is obtained by applying Theorem 2, and the proof is based on the fact that segments of a Banach space are metric segments, and convex subsets are metrically convex subsets.

We also study two spaces of subsets of a Banach space X: the space $bc(X)$ of all the nonempty bounded and closed subsets of X, endowed with the Hausdorff metric, and its subspace $bcx(X)$ of all the nonempty bounded closed and convex subsets of X. In this setting we work with the concept pseudo natural segment and with the related notion pseudo naturally convex subset.

In the space $bcx(X)$ we obtain the following result: if for the subsets R and S the union $R \cup S$ is pseudo naturally convex, then the equality (1) holds (Theorem 3). The proof is an application of Theorem 2 since pseudo natural segments in $bc(X)$ are metrically convex; moreover, Theorem 1 can to be deduced from it.

In $bc(X)$ also we achieve a similar result under the assumption that $R \cup S$ is pseudo naturally convex. Now it is necessary to demand some additional condition because pseudo natural segments in $bc(X)$ need not be metrically convex: either (1) the point z is a convex subset of X (Theorem 4); or (2) the intersection $R \cap S$ is a family of subsets which is closed under inclusion (Theorem 5). Theorem 1 also is deduced from Theorems 4 and 5: segments of a Banach space X are pseudo natural segments in $bc(X)$. Also Theorem 3 can to be obtained from Theorem 4.

Note that the logic order between the above theorems is the following: $2 \Rightarrow 3 \Rightarrow 1$, $4 \Rightarrow 3 \Rightarrow 1$ and $5 \Rightarrow 1$.

Other authors have studied the distance of a point to the intersection of a family of subsets. Thus using the distance of a point to two convex sets, Hoffmann [**6**] obtains an upper estimation of the distance of that point to the intersection of these two sets; his viewpoint is different from ours. On the other hand, Martínez-Legaz, Rubinov and Singer [**8, 11, 12, 13**] have proved in several papers the validity of the equality

$$d\Big(z, \bigcap_{i \in I} G_i\Big) = \sup_{i \in I} d(z, G_i), \tag{2}$$

where $(G_i)_{i\in I}$ is a certain family contained in some classes of subsets of $\mathbf{R^n}$, endowed with the ℓ_∞-norm:

$$\left\|(x_1, x_2, \dots, x_n)\right\|_\infty := \max_{1\leqslant k\leqslant n} |x_k|.$$

In $\mathbf{R^n}$ the natural order is considered:

$$(x_1, x_2, \dots, x_n) \leqslant (y_1, y_2, \dots, y_n) \iff x_i \leqslant y_i (0 \leqslant i \leqslant n).$$

A subset G of $\mathbf{R^n}_+ := \{x \in \mathbf{R^n} : 0 \leqslant x\}$ is said to be *normal* if $0 \leqslant x \leqslant y \in G \Rightarrow x \in G$ [**12, 13**]; then the equality (2) holds if all the G_i are closed normal subsets of $\mathbf{R^n}_+$ [**13**, Theorem 3.1]. Analogously, a subset G of $\mathbf{R^n}$ is called *downward* if $x \leqslant y \in G \Rightarrow x \in G$ [**8**, Definition 1]; the equality (2) has been proved in [**8**, Theorem 5.1] under the hypothesis that all the subsets G_i are closed and downward, putting $d(z, \emptyset) = \infty$. The notions of convex and downward subset of $\mathbf{R^n}$, and normal subset of $\mathbf{R^n}_+$ are related with the set of solutions of an inequality $f(x) \leqslant 0$ for f a convex or increasing real function on $\mathbf{R^n}$, or an increasing real function on $\mathbf{R^n}_+$, respectively. For more details see [**8, 11, 12, 13**]. In the setting of normed spaces a family $(G_i)_{i\in I}$ is said to be *linearly regular* if there exists a constant $C > 0$ such that

$$d\Big(z, \bigcap_{i\in I} G_i\Big) \leqslant C \sup_{i\in I} d(z, G_i),$$

for every point z. It is clear that if this inequality holds for $C = 1$ we obtain the equality (2).

2. In metric spaces and in Banach spaces

Throughout this section E denotes a metric space and d its distance. Now we consider a notion of metric convexity due to K. Menger, who first introduced it in its celebrated paper [**10**].

Definition 1: The metric space E is said to be metrically convex if for any points $u, v \in E$, with $u \neq v$, there exists $w \in E$, $u \neq w \neq v$, that satisfies

$$d(u, w) + d(w, v) = d(u, v);$$

it is said that w lies metrically between u and v.

Of course, the above definition is adequate for nonempty subsets of E (they are also metric spaces).

Definition 2: The subset T of the metric space E is called a metric segment with endpoints $u, v \in E$ if there exists an isometry $\varphi : \big[0, d(u, v)\big] \longrightarrow E$ such that $\varphi(0) = u$, $\varphi\big(d(u, v)\big) = v$ and $\varphi\Big(\big[0, d(u, v)\big]\Big) = T$.

Note that each metric segment is connected and compact. If E is complete, then we obtain the following well known Menger's theorem, which tell us that E is metrically convex if and only if given two points u and v in E there exists a metric segment with endpoints u and v.

LEMMA 1. (Menger's Theorem.) *If the metric space E is metrically convex and complete, then, given two points $u, v \in E$, there exists a metric segment with endpoints u and v.*

PROOF: [**3**, Theorem 14.1], [**4**, Theorem 6.2] or [**5**, p. 24]. ∎

From the above theorem we obtain that if a complete metric space is metrically convex, then it is connected: each segment is connected and the whole space is the union of segments with a endpoint in a fixed point. On the other hand, note that every metric segment in a complete metric space is a metrically convex subset.

Now we introduce a notion which is very useful in the next theorem.

DEFINITION 3: Let $T = \varphi\Big(\big[0, d(u,v)\big]\Big)$ be the metric segment defined by the isometry φ with endpoints u and v in E. Given $z \in E$ we say that z is a point with quasi convex distance to T if the function $f(z,T)$ defined by

$$\lambda \in \big[0, d(u,v)\big] \longrightarrow f(z,T)(\lambda) = d\big(z, \varphi(\lambda)\big).$$

is quasi convex; that is,

$$d\big(z, \varphi(\lambda)\big) \leqslant \max\big\{d(z,u), d(z,v)\big\},$$

for any $\lambda \in \big[0, d(u,v)\big]$.

In the following theorem we give sufficient conditions for that the distance to the intersection $R \cap S$ agrees with the maximum of the distances to R and S.

THEOREM 2. *Let E be a complete metric space. If R and S are nonempty closed subsets of E such that $R \cup S$ is a metrically convex set, then $R \cap S \neq \emptyset$ and*

$$d(z, R \cap S) = \max\big\{d(z,R), d(z,S)\big\},$$

for any $z \in E$ satisfying the following property: given $u \in R$ and $v \in S$, there exists a metric segment $T \subset R \cup S$ with endpoints u and v such that z is a point with quasi convex distance to T.

PROOF: Let $z \in E$ satisfy the property of the statement. Given $\varepsilon > 0$ there exist $u \in R$ and $v \in S$ such that

$$d(z,u) < d(z,R) + \varepsilon \quad \text{and} \quad d(z,v) < d(z,S) + \varepsilon \quad .$$

Assume $u \neq v$. There exists a metric segment $T = \varphi\Big(\big[0, d(u,v)\big]\Big) \subset R \cup S$ with endpoints u and v such that z is a point with quasi convex distance to T. We have that

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T is connected compact and, moreover, $T \cap R$ and $T \cap S$ are non empty closed subsets of T, hence there is $w \in T \cap R \cap S$. Then we obtain

$$d(z, R \cap S) \leqslant d(z, w) \leqslant \max \left\{d(z, u), d(z, v)\right\} < \max \left\{d(z, R), d(z, S)\right\} + \varepsilon .$$

These inequalities are obvious if $u = v$, since we may to take $w = u = v$. Because of the arbitrariness of ε we obtain the announced result. $\square$

Now we can deduce Theorem 1 from Theorem 2. Let is recall some definitions: given two vectors x, y in a linear space, the *segment*

$$[x, y] := \left\{\lambda x + (1 - \lambda) y : 0 \leqslant \lambda \leqslant 1\right\}$$

is defined; a subset C is *convex* if holds the implication $x, y \in C \Rightarrow [x, y] \subset C$. Note that every segment is a convex subset.

Proof 1 of Theorem 1: It is sufficient note some facts. As it is customary, the norm in the Banach space X is denoted by $\|.\|$, hence the distance is given by $d(u, v) = \|u - v\|$.

(1) Given two points u, v in the Banach space X, $u \neq v$, the function $\varphi : \left[0, \|u - v\|\right] \to X$ defined by $\varphi(\lambda) = (1 - \lambda')u + \lambda' v$, where $\lambda' = \lambda / \|u - v\| \in [0, 1]$, defines the segment $[u, v] = \varphi\Big(\left[0, \|u - v\|\right]\Big)$ with endpoints u and v.

(2) Every segment $[u, v]$ of the Banach space X is metrically convex. Indeed, with the notation of (1),

$$\left\|u - \varphi(\lambda)\right\| + \left\|\varphi(\lambda) - v\right\| = (1 - \lambda')\|u - v\| + \lambda'\|u - v\| = \|u - v\|.$$

Consequently, every convex subset of X is a metrically convex subset.

(3) Every $z \in X$ is a point with quasi convex distance to every usual segment $[u, v]$. Indeed, with the same notation of (1):

$$\left\|z - \varphi(\lambda)\right\| \leqslant (1 - \lambda')\|z - u\| + \lambda'\|z - v\| \leqslant \max\left\{\|z - u\|, \|z - v\|\right\}.$$

(4) Let R and S be nonempty closed subsets of the Banach space X such that $R \cup S$ is convex. By (2) $R \cup S$ is metrically convex and because of (3) every $z \in X$ is a point with quasi convex distance to every segment $[u, v]$ with $u \in R$ and $v \in S$. From Theorem 2 we obtain

$$d(z, R \cap S) = \max \left\{d(z, R), d(z, S)\right\} . \qquad \square$$

In part (2) of the above proof it is showed that

$$C \subset X \text{ is convex } \Rightarrow C \subset X \text{ is metrically convex,}$$

where X is a Banach space. In general, the converse implication is not true. Indeed, we take $X = \mathbf{R}^2$ with the ℓ_1-norm $\|(\alpha,\beta)\| := |\alpha| + |\beta|$; the set

$$C := \{(0,\beta) : 0 \leqslant \beta \leqslant 1\} \cup \{(\alpha,0) : 0 \leqslant \alpha \leqslant 1\}$$

is metrically convex, but it is not a convex subset. This example holds because the space X is not a strictly convex Banach space. Let us recall the definition now.

Definition 4: A Banach space X is strictly convex (or rotund) if given $x, y \in X, x \neq y, \|x\| = \|y\| = 1$, then $\|\alpha x + (1-\alpha)y\| < 1$ for every $0 < \alpha < 1$.

The reader interested can to find the main properties and examples of strictly convex spaces in [**9**].

Proposition 1. *Let X be a strictly convex Banach space. The subset C of X is metrically convex if and only if C is convex.*

Proof: Assume that C is metrically convex. Let $x, y \in X, x \neq y$, and let $w \in X$ lie metrically between x and y:

$$\|x - w\| + \|w - y\| = \|x - y\|.$$

Let $u = \tau(x - w)$, for $0 \leqslant \tau \leqslant 1$, and $v = \sigma(y - w)$, for $0 \leqslant \sigma \leqslant 1$. It is clear that u and v lie in the usual sense between $x - w$ and 0, and $w - y$ and 0, respectively. Moreover we choose τ and σ such that $\|u\| = \|v\|$. Then

$$\begin{aligned} \|x - y\| &\leqslant \|(x - w - u)\| + \|u - v\| + \|v - (y - w)\| \\ &\leqslant \|(x - w) - u\| + \|u\| + \|v\| + \|v - (y - w)\| \\ &= \|x - w\| + \|w - y\| = \|x - y\|, \end{aligned}$$

hence

$$\|u - v\| = \|u\| + \|v\| = 2\|u\|.$$

Since X is strictly convex, from

$$\|u + (-v)\| = 2\|u\|$$

we obtain $u = -v$, that is $\tau(x-w) = \sigma(w-y)$, hence $w = \eta x + (1-\eta)y$, with $\eta = \tau/(\tau+\sigma)$. Consequently w lies in the usual sense between x and y, hence C is convex. The other implication has been showed previously. ∎

3. In spaces of subsets of a Banach space

Throughout this section X denotes a Banach space, $\|.\|$ its norm and d the distance associated. Denote by B_X the closed unit ball of X; that is, $B_X = \{x \in X : \|x\| \leqslant 1\}$.

Now the space E is the complete metric space $bc(X)$ of all the nonempty bounded and closed subsets of X, endowed with the Hausdorff metric h, or its subspace $bcx(X)$ of all the nonempty bounded closed and convex subsets of X.

Given two nonempty bounded subsets U and V of X, the *nonsymmetric Hausdorff distance* is given by

$$h'(U,V) := \inf\{\varepsilon > 0 : U \subset V + \varepsilon B_X\}.$$

The *Hausdorff distance* between U and V is defined by

$$h(U,V) := \max\{h'(U,V), h'(V,U)\},$$

which is a pseudometric on the class of all the nonempty bounded subsets of X. Note that $h(U,V) = 0$ if and only if $\overline{U} = \overline{V}$ (the overline denotes closure). Then h is a metric on $bc(X)$ and, moreover, $bc(X)$ and $bcx(X)$ are complete metric spaces. As it is common, for $\mathcal{R} \subset bc(X)$ and $Z \in bc(X)$, we put

$$h'(Z,\mathcal{R}) := \inf\{h'(Z,U) : U \in \mathcal{R}\},$$
$$h(Z,\mathcal{R}) := \inf\{h(Z,U) : U \in \mathcal{R}\}.$$

Our goal is to find sufficient conditions for the equality (1) in this setting:

$$h(Z,\mathcal{R} \cap \mathcal{S}) = \max\{h(Z,\mathcal{R}), h(Z,\mathcal{S})\},$$

where $Z \in bc(X)$ and $\emptyset \neq \mathcal{R}, \mathcal{S} \subset bc(X)$, or $Z \in bcx(X)$ and $\emptyset \neq \mathcal{R}, \mathcal{S} \subset bcx(X)$. In the following we need some properties of the Hausdorff distance which we collect in the next lemma.

LEMMA 2. *Let T, U, V, W be nonempty bounded subsets of the Banach space X and α, β scalars. We put*

$$\|U\| = \sup\{\|x\| : x \in U\}.$$

Then the following properties hold:

1. $h(\alpha U, \beta U) \leqslant |\alpha - \beta|\, \|U\|$
2. $h(T + U, V + W) \leqslant h(T,V) + h(U,W)$

PROOF:

(1) It is proven that

$$\alpha U \subset \beta U + |\alpha - \beta|\, \|U\|\, B_X.$$

Given $x \in U, x \neq 0$, we have

$$\alpha x = \beta x + (\alpha - \beta)\|x\| \frac{x}{\|x\|} \in \beta U + |\alpha - \beta|\, \|U\|\, B_X.$$

If $x = 0$, then it is obvious.

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(2) In [**7**, 1.2] the statement is refered to compact subsets, but the proof is valid for our case. ∎

As $bc(X)$ and $bcx(X)$ are metric spaces the notion of metrically convex is defined and it is possible to apply Theorem 2.

Now introduce other notions of convexity which allow us to obtain new results about the validity of the equality (1).

Definition 5: We define

1. Let $\emptyset \neq U, V \subset X$. The natural segment with endpoints U and V is
$$[U,V] := \{\lambda U + (1-\lambda)V : 0 \leqslant \lambda \leqslant 1\}.$$
2. Let $\mathcal{T}$ a family of nonempty subsets of X. It says that $\mathcal{T}$ is naturally convex if, for every $U, V \in \mathcal{T}$, the natural segment $[U,V]$ is contained in $\mathcal{T}$.

The notions of natural segment and naturally convex set are introduced in [**7**, p. 237, p. 240], but these concepts are not the most adequate for our purpose in the spaces $bc(X)$ and $bcx(X)$. Indeed, if U and V are bounded, then
$$W(\lambda) := \lambda U + (1-\lambda)V$$
is also bounded; if U and V are convex, then $W(\lambda)$ is convex; but it is possible that U and V are closed and $W(\lambda)$ is not, because the sum of two closed subsets of a Banach space need not be closed.

However, in the case U, V convex subsets, we have that the natural segment $[U,V]$ is a naturally convex subset, because
$$\alpha W(\lambda) + (1-\alpha)W(\mu) = W\big(\alpha\lambda + (1-\alpha)\mu\big).$$

In order to define a useful notion of convexity in $bc(X)$ the above remarks lead us to consider the following concept of pseudo natural segment and pseudo naturally convex subset.

Definition 6: We define

1. Let $\emptyset \neq U, V \subset X$. The pseudo natural segment with endpoints U and V is
$$\big[[U,V]\big] := \{\overline{\lambda U + (1-\lambda)V} : 0 \leqslant \lambda \leqslant 1\}.$$
2. Let $\mathcal{T}$ a family of nonempty subsets of X. It is said that $\mathcal{T}$ is pseudo naturally convex if, for every $U, V \in \mathcal{T}$, the pseudo natural segment $\big[[U,V]\big]$ is contained in $\mathcal{T}$.

Note that if U (or V) is finite and V (or U) is closed, then $[U,V]$ and $\big[[U,V]\big]$ coincide. Moreover, if $\mathcal{T} \subset bc(X)$ is naturally convex, then it is pseudo naturally convex.

First we consider the space $bcx(X)$ and give some properties.

Proposition 2. *Let $U, V \in bcx(X)$.*

1. $\big[[U,V]\big] \subset bcx(X)$
2. $\big[[U,V]\big]$ *is a metric segment*
3. $\big[[U,V]\big]$ *is a pseudo naturally convex subset*

Proof: We put $W(\lambda) := \lambda U + (1-\lambda)V$ $(0 \leqslant \lambda \leqslant 1)$.

(1) Note that $W(\lambda)$ is bounded convex and its closure $\overline{W(\lambda)}$ is bounded closed convex. Hence $\big[[U,V]\big]$ is a subset of $bcx(X)$.

(2) In [**2**, Théorem 3.1] it is proved that

$$h\big(U, W(\lambda)\big) = (1-\lambda)h(U,V) \text{ and } h\big(W(\lambda), V\big) = \lambda h(U,V).$$

Moreover, $h\big(U, \overline{W(\lambda)}\big) = h\big(U, W(\lambda)\big) = (1-\lambda)h(U,V)$, hence $\big[[U,V]\big]$ is a metric segment.

(3) Let $\overline{W(\lambda)}, \overline{W(\mu)} \in \big[[U,V]\big]$ $(0 \leqslant \lambda, \mu \leqslant 1)$. We need to prove that, for $0 \leqslant \alpha \leqslant 1$,

$$\overline{\alpha\overline{W(\lambda)} + (1-\alpha)\overline{W(\mu)}} = \overline{W(\eta)}, \tag{3}$$

for certain $\eta \in [0,1]$. We have already showed that

$$\alpha W(\lambda) + (1-\alpha)W(\mu) = W(\eta),$$

where $\eta := \alpha\lambda + (1-\alpha)\mu$. Then

$$W(\eta) \subset \alpha\overline{W(\lambda)} + (1-\alpha)\overline{W(\mu)} \subset \overline{\alpha\overline{W(\lambda)} + (1-\alpha)\overline{W(\mu)}},$$

and we obtain "$\geqslant$" in the equality (3). On the other hand,

$$\alpha\overline{W(\lambda)} + (1-\alpha)\overline{W(\mu)} \subset \overline{\alpha W(\lambda) + (1-\alpha)W(\mu)} = \overline{W(\eta)},$$

hence "$\leqslant$" in (3). ∎

Theorem 3. *Let X be a Banach space. If $\mathcal{R}$ and $\mathcal{S}$ are nonempty closed subsets of $bcx(X)$ such that $\mathcal{R} \cup \mathcal{S}$ is pseudo naturally convex set, then $\mathcal{R} \cap \mathcal{S} \neq \emptyset$ and, for any $Z \in bcx(X)$,*

$$h(Z, \mathcal{R} \cap \mathcal{S}) = \max\big\{h(Z,\mathcal{R}), h(Z,\mathcal{S})\big\}.$$

Proof: Let $U \in \mathcal{R}$ and $V \in \mathcal{S}$. Now we show that every $Z \in bcx(X)$ is a point with quasi convex distance to the metric segment $\big[[U,V]\big]$. Let $W(\lambda) := \lambda U + (1-\lambda)V$ $(0 \leqslant \lambda \leqslant 1)$. As Z is convex we can to write $Z = \lambda Z + (1-\lambda)Z$. Then

$$h(Z, \overline{W(\lambda)}) = h(Z, W(\lambda)) = h(\lambda Z + (1-\lambda)Z, \lambda U + (1-\lambda)V)$$

Because of Lemma 2

$$h(Z, \overline{W(\lambda)}) \leqslant \lambda h(Z,U) + (1-\lambda)h(Z,V) \leqslant \max\big\{h(Z,U), h(Z,V)\big\}.$$

From Proposition 2(2), $\mathcal{R} \cup \mathcal{S}$ is metrically convex. Now we apply Theorem 2 and the proof is finished. ∎

The space $bc(X)$ has worse convexity properties than $bcx(X)$. Note that it is possible that $U, V \in bc(X)$, $[U,V] = \big[[U,V]\big] \subset bc(X)$ and $[U,V]$ is not a naturally convex family. Indeed, take $X = \mathbf{R}^2$ endowed with the Euclidean norm, $U = \{(0,-1),(0,1)\}$, $V = \{(1,0)\}$ and $W(\lambda) = \lambda U + (1-\lambda)V$; then $\big[W(1/3), W(2/3)\big]$ is not contained in $[U,V]$. Moreover, in general, natural segments and pseudo natural segments in $bc(X)$ are not metrically convex, as is showed in the next example.

EXAMPLE 1. *A natural segment which is not metrically convex.* Let $X = \mathbf{R}^2$ with the Euclidean norm. Let $C := \{c_1, c_2\}$ and $D := \{d_1, d_2\}$, where $c_1 = (-4,0)$, $c_2 = (0,3)$, $d_1 = (4,0)$ and $d_2 = (0,-3)$. For $0 \leqslant \lambda \leqslant 1$ we put $T(\lambda) := \lambda C + (1-\lambda)D$. Then the natural segment

$$\mathcal{T} := \{T(\lambda) : 0 \leqslant \lambda \leqslant 1\}$$

is not metrically convex, as it is proved in the sequel. It is clear that

$$h(C,D) = h'(C,D) = h'(D,C) = 5.$$

It is easy (but some tedious!) to prove that $h'\big(D, T(\lambda)\big) = 5\lambda$ and that

$$h(T(\lambda),D) = h'(T(\lambda),D) = \begin{cases} 8\lambda & \text{if } 0 \leqslant \lambda \leqslant 25/64 \\ (64\lambda^2 - 64\lambda + 25)^{1/2} & \text{if } 25/64 \leqslant \lambda \leqslant 1/2 \\ 6\lambda & \text{if } 1/2 \leqslant \lambda \leqslant 25/36 \\ (36\lambda^2 - 36\lambda + 25)^{1/2} & \text{if } 25/36 \leqslant \lambda \leqslant 1 \end{cases}$$

Using symmetry it is easy to calculate the distance between C and $T(\lambda)$; indeed,

$$\begin{aligned} h'\big(C, T(\lambda)\big) &= h'\big(D, T(1-\lambda)\big), \\ h\big(C, T(\lambda)\big) &= h'\big(T(\lambda), C\big) = h'\big(T(1-\lambda), D\big), \end{aligned}$$

hence $h'\big(C, T(\lambda)\big) = 5(1-\lambda)$ and

$$h\big(C,T(\lambda)\big) = h'\big(T(\lambda),C\big) = \begin{cases} (36\lambda^2 - 36\lambda + 25)^{1/2} & \text{if } 0 \leqslant \lambda \leqslant 11/36 \\ 6 - 6\lambda & \text{if } 11/36 \leqslant \lambda \leqslant 1/2 \\ (64\lambda^2 - 64\lambda + 25)^{1/2} & \text{if } 1/2 \leqslant \lambda \leqslant 39/64 \\ 8 - 8\lambda & \text{if } 39/64 \leqslant \lambda \leqslant 1 \end{cases}$$

The function $g(\lambda) = h(T(\lambda), D)$ is strictly increasing, $g(0) = 0$ and $g(1) = 5$; on the other hand, the function $k(\lambda) = h\big(C, T(\lambda)\big)$ is strictly decreasing, $k(0) = 5$ and $k(1) = 0$. Consequently, there is a unique λ such that

$$h\big(C, T(\lambda)\big) = k(\lambda) = g(\lambda) = h\big(T(\lambda), D\big).$$

The value $\lambda = 1/2$ satisfies

$$h\big(C, T(1/2)\big) = h\big(T(1/2), D\big) = 3,$$

hence

$$h(C, D) \neq h\big(C, T(1/2)\big) + h\big(T(1/2), D\big)$$

and, consequently, $\mathcal{T}$ is not a metric segment. Indeed, if $\mathcal{T}$ is metrically convex, then there would be an isometry $\varphi : [0,5] \to \mathcal{T}$ such that $\varphi(0) = C$ and $\varphi(5) = D$, hence $\varphi(3) = T(1/2)$ and $h\big(\varphi(3), \varphi(5)\big) = |3 - 5| = 2$, which contradicts $h\big(T(1/2), D\big) = 3$.

In order to prove the theorems about the equality (1) in $bc(X)$ we need a previous result.

PROPOSITION 3. *Given $U, V \in bc(X)$, the pseudo natural segment $\big[[U, V]\big]$ is connected and compact.*

PROOF: Consider the map $\varphi : [0,1] \longrightarrow bc(X)$ defined by $\varphi(\lambda) := \overline{\lambda U + (1-\lambda)V}$. From Lemma 2 we obtain, for $\lambda, \mu \in [0,1]$,

$$h\big(\varphi(\lambda), \varphi(\mu)\big) \leqslant h(\lambda U, \mu U) + h\big((1-\lambda)V, (1-\mu)V\big) \leqslant |\lambda - \mu| \,\big(\|U\| + \|V\|\big).$$

Hence φ is (uniformly) continuous and $\varphi\big([0,1]\big) = \big[[U, V]\big]$ is connected and compact. ∎

THEOREM 4. *Let X be a Banach space. If $\mathcal{R}$ and $\mathcal{S}$ are nonempty closed subsets of $bc(X)$ such that $\mathcal{R} \cup \mathcal{S}$ is a pseudo naturally convex set, then $\mathcal{R} \cap \mathcal{S} \neq \emptyset$ and*

$$h(Z, \mathcal{R} \cap \mathcal{S}) = \max\big\{h(Z, \mathcal{R}), h(Z, \mathcal{S})\big\},$$

for any convex set $Z \in bc(X)$.

PROOF: Let $Z \in bc(X)$ be a convex subset of X. Let $\varepsilon > 0$ satisfy

$$\varepsilon > \max\big\{h(Z, \mathcal{R}), h(Z, \mathcal{S})\big\}.$$

Then there exist $U \in \mathcal{R}$ and $V \in \mathcal{S}$ such that $h(Z, U) < \varepsilon$ and $h(Z, V) < \varepsilon$. The pseudo natural segment $\big[[U, V]\big]$ is connected compact and is contained in $\mathcal{R} \cup \mathcal{S}$. Moreover, $\big[[U, V]\big] \cap \mathcal{R}$ and $\big[[U, V]\big] \cap \mathcal{S}$ are nonempty closed subsets of $\big[[U, V]\big]$, hence there is $W \in \big[[U, V]\big] \cap \mathcal{R} \cap \mathcal{S}$. Note that $W \in \mathcal{R} \cap \mathcal{S}$ is the closure of $\mu U + (1-\mu)V$ for a certain $\mu \in [0,1]$. Since Z is convex we can argue as in the proof of Theorem 3 and we conclude $h(Z, W) < \varepsilon$. ∎

The proof of Theorem 3 that we have given has been obtained as a consequence of Theorem 2. Also we can to deduce Theorem 3 from Theorem 4 as a particular case.

Now we can to deduce Theorem 1 from Theorem 4 because the conditions "convex in X" and "naturally convex in $bc(X)$" are related.

PROOF 2 OF THEOREM 1: We can to consider the Banach space X as a subspace of $bcx(X)$ because the map $i : X \to bc(X)$, $i(x) := \{x\}$, is an isometry. Then we identify $R \subset X$ with $i(R) \subset bc(X)$. Moreover, $R \subset X$ is convex if and only if $i(R)$ is naturally convex; $\{x\}$ is also a convex subset of X. Then Theorem 1 is a particular case of Theorem 4. $\square$

Note that Theorem 1 can to be deduced from Theorem 3 in an analogous way to the above Proof 2.

In the following theorem we use the condition "$\mathcal{R} \cap \mathcal{S}$ is closed under inclusions" instead the condition "Z is convex" of Theorem 4.

DEFINITION 7: A family $\mathcal{F}$ of nonempty sets is said to be *closed under inclusions* if

$$\emptyset \neq U \subset V \in \mathcal{F} \Rightarrow U \in \mathcal{F}.$$

The following result is necessary in the proof of the next theorem.

LEMMA 3. *Let $\mathcal{T} \subset bc(X)$ be a family which is closed under inclusions. For any $Z \in bc(X)$ the following equality holds:*

$$h(Z, \mathcal{T}) = h'(Z, \mathcal{T}),$$

whenever h' is the nonsymmetric Hausdorff distance.

PROOF: [**1**, Theorem 1]. $\square$

THEOREM 5. *Let X be a Banach space. If $\mathcal{R}$ and $\mathcal{S}$ are nonempty closed subsets of $bc(X)$ such that $\mathcal{R} \cup \mathcal{S}$ is a pseudo naturally convex set, then $\mathcal{R} \cap \mathcal{S} \neq \emptyset$ and, if moreover $\mathcal{R} \cap \mathcal{S}$ is closed under inclusions, we have that*

$$h(Z, \mathcal{R} \cap \mathcal{S}) = \max\left\{h(Z, \mathcal{R}), h(Z, \mathcal{S})\right\}$$

for every $Z \in bc(X)$.

PROOF: Let $Z \in bc(X)$ and ε, U, V, W as in the proof of Theorem 4. Then

$$Z \subset U + \varepsilon B_X \text{ and } Z \subset V + \varepsilon B_X,$$

hence

$$Z \subset \lambda Z + (1-\lambda) Z \subset \lambda U + (1-\lambda) V + \varepsilon B_X \subset W + \varepsilon B_X,$$

so $h'(Z, W) < \varepsilon$. From Lemma 3 we obtain

$$h(Z, \mathcal{R} \cap \mathcal{S}) = h'(Z, \mathcal{R} \cap \mathcal{S}) \leqslant h'(Z, W) \leqslant \varepsilon$$

and the proof is finished. $\square$

PROOF 3 OF THEOREM 1. It is similar to Proof 2, but now taking into account that $i(T)$ is a family which is closed under inclusions, for every $T \subset X$. $\square$

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