But vector operators are to be treated in all respects like vectors, provided each be always kept *before* its subject.

Let $$\sigma = i\xi + j\eta + k\zeta,$$
where ξ, η, ζ are functions of x, y, z; and let
$$\nabla = i\frac{d}{dx} + j\frac{d}{dy} + k\frac{d}{dz},$$
as usual. Also let σ_1, ∇_1 be their values when x_1, y_1, z_1 are put for x, y, z.

Then by the first equation, attending to the rule for the place of an operator,
$$\mathrm{V}.\mathrm{V}\nabla\sigma\mathrm{V}\nabla_1\sigma_1 = \nabla\mathrm{S}.\sigma\nabla_1\sigma_1 - \mathrm{S}(\nabla_1\sigma_1\nabla)\sigma.$$

If we suppose the operations to be completed, and *then* make $x_1 = x$, $y_1 = y$, $z_1 = z$, the left-hand member must obviously vanish. So therefore must the right.

That is :— $$\nabla\mathrm{S}.\sigma\nabla_1\sigma_1 = \mathrm{S}(\nabla_1\sigma_1\nabla)\sigma\,;$$
if when the operations are complete, we put $\sigma_1 = \sigma, \nabla_1 = \nabla$.

In Cartesian co-ordinates this is equivalent to three equations, of the same type. I write only one, viz. :—
$$\frac{d}{dx}\begin{vmatrix} \xi & \eta & \zeta \\ \frac{d}{dx_1} & \frac{d}{dy_1} & \frac{d}{dz_1} \\ \xi_1 & \eta_1 & \zeta_1 \end{vmatrix} = \begin{vmatrix} \frac{d}{dx_1} & \frac{d}{dy_1} & \frac{d}{dz_1} \\ \xi_1 & \eta_1 & \zeta_1 \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \end{vmatrix}\xi,$$
if, *after* operating, we put $x_1 = x$, $\xi_1 = \xi$, &c., &c.

On a property of odd and even polygons.

By R. E. Allardice, M.A.

The property referred to comes to light on consideration of the problem, "To inscribe in a given n-gon the n-gon of minimum perimeter."

Triangle.

Let us consider first the case of the triangle (fig. 29). If ABC is the given triangle and DEF the inscribed triangle of minimum perimeter, it is obvious that we must have $\angle$ FDB = $\angle$ EDC($=\alpha$, say), $\angle$ DEC = $\angle$ FEA($=\beta$), $\angle$ EFA = $\angle$ DFB($=\gamma$). This condition is satisfied if D,E,F, are the feet of the perpendiculars from the oppo-

https://doi.org/10.1017/S0013091500030492 Published online by Cambridge University Press

site vertices; and it is usually assumed that the triangle so obtained, the pedal triangle, must therefore be the triangle required; in other words, the assumption is made that only one inscribed triangle may be constructed, whose sides shall be equally inclined to the sides of ABC. This assumption is as a matter of fact correct in the case of the triangle; but it can hardly be considered not to require proof, as the corresponding assumption would be wrong in the case of the quadrilateral.

We might proceed to determine DEF as follows :—

We have
$$\alpha+\beta+C=\pi,$$
$$\beta+\gamma+A=\pi,$$
$$\gamma+\alpha+B=\pi;$$
from which we get
$$\alpha=A,\ \beta=B,\ \gamma=C.$$

To calculate CD, put $CD=x$, $AE=y$, $BF=z$, then
$$x\sin\alpha=(b-y)\sin\beta,$$
$$y\sin\beta=(c-z)\sin\gamma,$$
$$z\sin\gamma=(a-x)\sin\alpha;$$
whence
$$x\sin\alpha=b\sin\beta-c\sin\gamma+(a-x)\sin\alpha;$$
from which we get $x=(a^2+b^2-c^2)/2a$; which shows that D is the foot of the perpendicular from A on BC.

QUADRILATERAL.

Consider next the case of the quadrilateral (fig. 30),

Let $\angle APS=\angle BPQ=\alpha$; $\angle BQP=\angle CQR=\beta$; etc.

If now we try to determine α as we did in the case of the triangle, we get the equations
$$\alpha+\beta+B=\pi, \qquad \beta+\gamma+C=\pi,$$
$$\gamma+\delta+D=\pi, \qquad \delta+\alpha+A=\pi,$$
whence
$$B-C+D-A=0;$$
and the problem is therefore impossible unless the given quadrilateral be cyclic; that is to say, it is in general impossible to inscribe in a given quadrilateral a quadrilateral each pair of consecutive sides of which shall be equally inclined to the side of the given quadrilateral in which they meet.

If the given quadrilateral be cyclic there are an infinite number of inscribed quadrilaterals satisfying the given condition.

The investigation that determined the position of the vertices in the case of the triangle determines the inclination of the sides in the case of the quadrilateral.

Using a corresponding notation in this case, we have as before

$$x\sin\alpha = (b-y)\sin\beta, \qquad z\sin\gamma = (d-w)\sin\delta,$$
$$y\sin\beta = (c-z)\sin\gamma, \qquad w\sin\delta = (a-x)\sin\delta;$$

whence $$a\sin\alpha - b\sin\beta + c\sin\gamma - d\sin\delta = 0.$$

Let now α' denote the angle subtended at the circumference of the circumcircle by the side a, β' the angle subtended by the side b; and so on. Then $a = 2\mathrm{R}\sin\alpha'$, and therefore

$$\sin\alpha'\sin\alpha - \sin\delta'\sin\delta = \sin\beta'\sin\beta - \sin\gamma'\sin\gamma.$$

But $$\sin\alpha' = \sin(\mathrm{A}+\delta'); \ \sin\beta' = \sin(\mathrm{C}+\gamma'),$$
$$\sin\delta = \sin(\mathrm{A}+\alpha), \quad \sin\gamma = \sin(\mathrm{C}+\beta);$$
$$\therefore \ \sin(\mathrm{A}+\delta')\sin\alpha - \sin\delta'\sin(\mathrm{A}+\alpha) = \sin(\mathrm{C}+\gamma')\sin\beta - \sin\gamma'\sin(\mathrm{C}+\beta);$$
$$\therefore \quad \sin\mathrm{A}\sin(\alpha-\delta') = \sin\mathrm{C}\sin(\beta-\gamma');$$
$$\text{and } \sin\mathrm{A} = \sin\mathrm{C}, \quad \therefore \ \sin(\alpha-\delta') = \sin(\beta-\gamma').$$

Hence either $\alpha - \delta' = \pi - (\beta - \gamma')$, and therefore $\alpha + \beta = \pi + \gamma' + \delta'$, which is impossible; or $\alpha - \delta' = \beta - \gamma'$ and $\therefore \ \alpha - \beta = \delta' - \gamma'$;

and we have also $$\alpha + \beta = \pi - \gamma' - \delta',$$
$$\therefore \ \alpha = \pi/2 - \gamma', \ \beta = \pi/2 - \delta'.$$

A geometrical investigation may also be given, in the following manner.

Let ABCD (fig. 31) be the given quadrilateral. Take U any point in DA; K the image of U in AB; L the image of K in BC; M the image of L in CD. Join MU, XL, WK, VU. Then it may easily be shown, that if ABCD is cyclic, UVWX has its adjacent side equally inclined to the sides of ABCD.

If U move along DA, the locus of K will be AR, the locus of L will be RS, the locus of M will be SM.

Now $\angle$SMU = $\angle$SLX = $\angle$VKA = $\angle$VUA = $\angle$XUD; $\therefore$ SM is parallel to DA.

Now let U′ be another point in DA; K′, L′, M′, the points corresponding to K, L, M. Then UU′ = KK′ = LL′ = MM′. Hence MM′U′U is a parallelogram; the direction of UM is invariable; and UM which is equal to the perimeter of the inscribed quadrilateral is of constant length. [It should be noted that if the quadrilateral UVWX be crossed, one of its sides has to be considered negative.]

Having shown that the directions of the sides of the inscribed quadrilateral are invariable, and that the primeter is constant, we may make use of a particular case to determine the directions of the sides and the length of the perimeter.

Let one of the vertices of the inscribed quadrilateral (fig. 32) coincide with D, so that PQ is equal to the primeter.

The triangles KCD and HAD are similar, and therefore the triangles KDH and CDA are similar; hence

$$\mathrm{KH:CA} = \mathrm{KD:DC} = \sin\mathrm{C}$$
$$\therefore\ \mathrm{KH} = \mathrm{CA}\sin\mathrm{C}$$
$$\therefore\ \mathrm{PQ} = 2\mathrm{KH} = 2\mathrm{CA}\sin\mathrm{C} = hk/\mathrm{R},$$

where h and k are the diagonals of the quadrilateral and R is the radius of the circumcircle.

Denoting DMH by a, as before, we get

$$\pi/2 - a = \angle \mathrm{Q} = \angle \mathrm{KHD} = \angle \mathrm{CAD};$$
$$\therefore\ a = \pi/2 - \mathrm{CAD};$$

that is, a is the complement of the angle subtended at the circumference by the side of ABCD opposite to that in which the vertex of a lies.

It might be thought that there must in every case be some minimum inscribed quadrilateral. But if the given quadrilateral is not cyclic, the minimum inscribed quadrilateral is one which has negative infinity for the length of its perimeter. If the inscribed quadrilateral be restricted to be non-crossed, the minimum one will not be determinable by a general method.

It will be seen how it is that some sides may have to be considered negative, by consideration of figs. 33 and 34, where A′ and B′ are the geometrical images of A and B. In fig. 33, AP + PQ + QB (= A′B′) is a minimum; while if we make the same construction in fig. 34 we only get AP + PQ + QB equal to A′B′ and a minimum if we consider PQ negative.

We may also determine the value of the perimeter without considering merely a limiting case.

In fig. 30, let $\mathrm{BP} = x$, $\mathrm{BQ} = y$, $\mathrm{DR} = z$, $\mathrm{DS} = w$; $\mathrm{PQ} = k$, $\mathrm{QR} = l$, $\mathrm{RS} = m$, $\mathrm{SP} = n$; $\angle \mathrm{BPQ} = a$, etc.

Then $\quad k = x\cos a + y\cos\beta, \qquad l = (b - y)\cos\beta + (c - z)\cos\gamma,$
$\qquad\quad m = z\cos\gamma + w\cos\delta, \qquad n = (d - w)\cos\delta + (a - x)\cos a.$

Hence, if p denote the perimeter,

$$\begin{aligned} p &= k + l + m + n \\ &= a\cos a + b\cos\beta + c\cos\gamma + d\cos\delta \\ &= a\sin\gamma' + b\sin\delta' + c\sin a' + d\sin\beta' \\ &= 2\mathrm{R}[\sin a'\sin\gamma' + \sin\beta'\sin\delta' + \sin\gamma'\sin a' + \sin\delta'\sin\beta'] \\ &= 4\mathrm{R}[\sin a'\sin\gamma'\sin\beta'\sin\delta'] \\ &= (ac + bd)/\mathrm{R} = hk/\mathrm{R}. \end{aligned}$$

General Case.

In a similar way it may be shown that in any polygon of an odd number of sides, there may be inscribed a polygon of the same number of sides whose perimeter is a minimum; and that this inscribed polygon is uniquely determined. The directions of its sides and the positions of its vertices may in fact easily be calculated as in the case of the triangle.

In the case of a polygon with an even number of sides no polygon of minimum perimeter can be inscribed, unless a relation holds among the angles of the given polygon, namely, $A - B + C - D + \ldots$ must be zero. If this relation holds, an infinite number of polygons of minimum perimeter may be inscribed. Any two of these inscribed polygons have equal perimeters and have corresponding sides parallel.

Particular Case.

The particular case of the regular polygon is of some interest.

The polygon of minimum perimeter inscribed in a regular odd polygon is a regular polygon having its vertices at the middle points of the sides of the given polygon.

Thus, in the case of the pentagon (fig. 35), putting $\angle APT = \angle BPQ = \alpha$, etc., we get, successively $\alpha = \gamma = \epsilon = \beta$; and therefore all the triangles, APT, BQP, etc., are isosceles. Also $PB = RD = DS = TA = AP$; and thus the vertices of PQRST bisect the sides of ABCDE.

If now we consider a regular hexagon (fig. 36), we do not get $\alpha = \beta$, by the reasoning that gave us this result in the case of the pentagon; but only $\alpha = \gamma = \epsilon$; and $\beta = \delta = \zeta$.

But if we suppose $\alpha > \beta$, we have

$$BQ > BP,\ CQ > CR,\ DS > DR,$$
$$ES > ET,\ FU > FT,\ AU > AP;$$

whence, adding, we have $BC + DE + FA > CD + EF + AB$; which is not the case; and therefore $\alpha = \beta$.

And in this case P is not necessarily the middle point of AB; for if we make $AP = CQ = CR = ES = ET = AU$, the hexagon PQRSTU will satisfy the conditions of the problem.

It may easily be shown that the perimeter of PQRSTU is constant; it is in fact $\sqrt{3}/2$ times that of ABCDE.