CERTAIN CLASSES OF IDEALS IN POLYNOMIAL RINGS

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1. Introduction. The purpose of this paper is to establish some results, of a somewhat miscellaneous nature, concerning certain classes of ideals in polynomial rings. Although the results will be formulated for a ring $R[x]$ of polynomials in one indeterminate over a given ring R, they can be easily extended to polynomial rings in any finite number of indeterminates. We shall be concerned both with right ideals and two-sided ideals, but the word *ideal* used alone will henceforth mean two-sided ideal.

A right ideal $\mathfrak{q}$ in the ring R is said to be a *semi-prime right ideal* if $\mathfrak{a}^2 \subseteq \mathfrak{q}$ implies that $\mathfrak{a} \subseteq \mathfrak{q}$, where $\mathfrak{a}$ is any right ideal of R. Of course, a semi-prime right ideal which is an ideal will be called a *semi-prime ideal.*

Each class of ideals which we shall consider is a sub-class of the class of semi-prime right ideals. Clearly the class of semi-prime ideals (right ideals) of R is closed under arbitrary intersection. The unique minimal semi-prime ideal of R is the *lower radical* of R as defined by Baer **(2)**. If the lower radical of R is zero, we call R a *semi-prime ring*. Thus a ring R is semi-prime if and only if the zero ideal is a semi-prime ideal of R. It is known **(1, 10)** that if N is the lower radical of R, then the lower radical of $R[x]$ is $N[x]$. New proofs of this fact will appear in §3 and §4. In particular, the polynomial ring $R[x]$ is semi-prime if and only if R is semi-prime.

Johnson **(7)** has introduced the concept of *prime right ideal* in a semi-prime ring, and has shown that these ideals play an important role in the structure theory of such rings. The definition will be found in §5. However, we may now state the various classes of ideals which will be considered in this paper. If R is an arbitrary ring, we shall study in R and in $R[x]$:

(1) semi-prime right ideals,

(2) semi-prime ideals, and

(3) prime ideals.

Also, for the case in which R (and therefore also $R[x]$) is semi-prime, we shall consider

(4) prime right ideals, and

(5) annihilating ideals.

All of these are special instances of semi-prime right ideals.

The following terminology will be convenient. Let σ be a property of ideals (right ideals) defined for ideals in R and in $R[x]$. For example, σ may be the property defining any one of the above five classes of ideals. An ideal (right

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ideal) with property σ may be called a σ-ideal. If σ is such that $\mathfrak{a}$ a σ-ideal in R implies that $\mathfrak{a}[x]$ is a σ-ideal in $R[x]$, we shall find it convenient to say that the "going up" theorem holds for σ-ideals. If A a σ-ideal in $R[x]$ implies that $A \cap R$ is a σ-ideal in R, we shall say that the "going down" theorem holds for σ-ideals. It will be shown that both of these theorems hold for σ-ideals, where σ-ideals are any one of the five classes of ideals mentioned above. This implies a certain closure property which will be obtained in §2.

Let A be a right ideal in $R[x]$ and $\mathfrak{a}$ the right ideal in R generated by all coefficients of elements of A. If $\mathfrak{q}$ is a semi-prime right ideal of R, we shall show in Theorem 4 that $(\mathfrak{q}[x]:A) = (\mathfrak{q}:\mathfrak{a})[x]$. As a special case of this result it follows that if R is a semi-prime ring, the only annihilating ideals in $R[x]$ are those ideals of the form $\mathfrak{a}[x]$, where $\mathfrak{a}$ is an annihilating ideal in R. As a consequence of this fact, we can say something about the structure of $R[x]$ as it is induced by the structure of R. This is based on results of Johnson **(7)** and will be discussed in §6. In the final section we make a few remarks about prime right ideals in $R[x]$, where R is a *division ring*, and give an example that may be of some interest.

2. A general closure property. Before considering any particular class of ideals, we prove a result in a fairly general setting.

Let σ be a property of ideals (right ideals) defined in the ring R and also in the polynomial ring $R[x]$. An ideal (right ideal) with property σ may be called a σ-ideal. As an example, σ-ideals may be the semi-prime right ideals. If R is itself a σ-ideal, the property σ may be used to define a closure operation on the set of all ideals (right ideals) of R. If $\mathfrak{a}$ is an ideal (right ideal) in R, the *closure* of $\mathfrak{a}$ relative to σ, which we may denote by $\mathfrak{a}_\sigma$, is the intersection of all σ-ideals which contain $\mathfrak{a}$. Of course, the *closed ideals* (relative to σ) are those ideals that are intersections of σ-ideals.

We shall prove the following result:

THEOREM 1. *Let σ be a property of ideals (right ideals) defined in R and in $R[x]$ such that the following are true:*

(a) *R is a σ-ideal,*

(b) *If A is an ideal (right ideal) and B a σ-ideal in $R[x]$ such that $AR[x] \subseteq$ B, then $A \subseteq B$,*

(c) *The "going up" and "going down" theorems hold for σ-ideals.*
Then for each ideal (right ideal) $\mathfrak{a}$ in R, $(\mathfrak{a}[x])_\sigma = \mathfrak{a}_\sigma[x]$.

Clearly (a) and (c) imply that $R[x]$ is a σ-ideal in $R[x]$. Moreover, it follows easily from (c) that the "going up" and "going down" theorems hold also for *closed* ideals.

Since $\mathfrak{a} \subseteq \mathfrak{a}_\sigma$, we have $\mathfrak{a}[x] \subseteq \mathfrak{a}_\sigma[x]$. But by the "going up" theorem for closed ideals, $\mathfrak{a}_\sigma[x]$ is a closed ideal in $R[x]$ and hence $(\mathfrak{a}[x])_\sigma \subseteq \mathfrak{a}_\sigma[x]$. To obtain inclusion in the other direction, suppose that $\mathfrak{a}[x] \subseteq C$, where C is any closed ideal in $R[x]$. Then $C \cap R$ is a closed ideal in R by the "going

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down" theorem, and since $\mathfrak{a} \subseteq C \cap R$ we have $\mathfrak{a}_\sigma \subseteq C \cap R$. It follows that $\mathfrak{a}_\sigma[x]R[x] \subseteq C$ and (b) implies that $\mathfrak{a}_\sigma[x] \subseteq C$. Applying this result to the case in which $C = (\mathfrak{a}[x])_\sigma$, we have $\mathfrak{a}_\sigma[x] \subseteq (\mathfrak{a}[x])_\sigma$ and the proof is completed.

If σ-ideals are any one of the various classes of ideals to be studied in this paper, it will be obvious that (a) and (b) hold. Accordingly, in order to apply this theorem it will only be necessary to establish the "going up" and "going down" theorems.

3. The prime radical of a polynominal ring. In this section we give a simple application of the preceding theorem to the case in which σ-ideal means prime ideal. We need therefore to establish the following result:

LEMMA 1. *The "going up" and "going down" theorems hold for prime ideals.*

The "going up" theorem follows easily from the fact that if $\mathfrak{p}$ is an ideal in R, then $R[x]/\mathfrak{p}[x] \cong (R/\mathfrak{p})[x]$, and $(R/\mathfrak{p})[x]$ is a prime ring (that is, the zero ideal is prime) if and only if $R/\mathfrak{p}$ is a prime ring.

The "going down" theorem has been established in **(10)** but for completeness we reproduce the proof here. If P is a prime ideal in $R[x]$, we wish to prove that $P \cap R$ is a prime ideal in R. By Theorem 1 of **(9)** we only need to show that if a and b are elements of R such that $aRb \subseteq P \cap R$, then $a \in P \cap R$ or $b \in P \cap R$. If $aRb \subseteq P \cap R$, then each element of $aR[x]bR[x]$ is a sum of terms belonging to P and thus $aR[x]bR[x] \subseteq P$. Since P is a prime ideal in $R[x]$ it follows that $a \in P$ or $b \in P$, and hence $a \in P \cap R$ or $b \in P \cap R$, as required.

By the *prime radical* of the ring R we shall mean the radical as defined in **(9)**, that is, the intersection of all prime ideals in R. It is known **(8; 12)** that the prime radical coincides with the *lower radical* of Baer **(2)**. If σ-ideal means prime ideal, then the prime radical of R is the closure of the zero ideal in the terminology of the preceding section. A simple application of Theorem 1 then yields the following result:

THEOREM 2. *If K is the prime radical of the ring R, the prime radical of the polynomial ring $R[x]$ is $K[x]$.*

Different proofs of this result are to be found in **(1)** and **(10)**.

4. Semi-prime right ideals. Before passing to a consideration of polynomial rings we make a few preliminary remarks about semi-prime right ideals in general.

A right ideal $\mathfrak{q}$ in the ring R is said to be *semi-prime* if $\mathfrak{a}^2 \subseteq \mathfrak{q}$ implies that $\mathfrak{a} \subseteq \mathfrak{q}$, where $\mathfrak{a}$ is any right ideal in R. Henceforth we shall find it convenient to denote the sets of right ideals, ideals, semi-prime right ideals, and semi-prime ideals of any ring R by $\mathfrak{J}_r(R)$, $\mathfrak{J}(R)$, $\mathfrak{S}_r(R)$, and $\mathfrak{S}(R)$, respectively.

(A) *If $\mathfrak{q} \in \mathfrak{J}_r(R)$, then $\mathfrak{q} \in \mathfrak{S}_r(R)$ if and only if $a \in R$ such that $aRa \subseteq \mathfrak{q}$ implies that $a \in \mathfrak{q}$.*

To show this, suppose that $\mathfrak{q} \in \mathfrak{S}_r(R)$ and that $aRa \subseteq \mathfrak{q}$. Then $(aR)^2 \subseteq \mathfrak{q}$ and this implies that $aR \subseteq \mathfrak{q}$. If $(a)_r$ denotes the right ideal of R generated by a, we now have $[(a)_r]^2 \subseteq \mathfrak{q}$ and hence $a \in \mathfrak{q}$. Conversely, let $\mathfrak{q} \in \mathfrak{J}_r(R)$ such that $aRa \subseteq \mathfrak{q}$ implies that $a \in \mathfrak{q}$, and let $\mathfrak{a}$ be a right ideal with $\mathfrak{a}^2 \subseteq \mathfrak{q}$. If $a \in \mathfrak{a}$, then $aRa \subseteq \mathfrak{a}^2 \subseteq \mathfrak{q}$ and $a \in \mathfrak{q}$. Hence $\mathfrak{a} \subseteq \mathfrak{q}$ and $\mathfrak{q} \in \mathfrak{S}_r(R)$.

(B) *If* $\mathfrak{q} \in \mathfrak{S}_r(R)$ *and* $a, b \in R$ *such that* $aRb \subseteq \mathfrak{q}$, *then* $ab \in \mathfrak{q}$.

Clearly $aRb \subseteq \mathfrak{q}$ implies that $(abR)^2 \subseteq \mathfrak{q}$. It follows that $abR \subseteq \mathfrak{q}$, and the desired result follows from (A).

We may remark in passing that Herz **(3)** has shown that if R is commutative, the lattice $\mathfrak{S}(R)$ is distributive. More recently, Tominaga **(13)** has established the same result for an arbitrary ring R.

We shall next prove the following useful lemma.

LEMMA 2. *Let* $\mathfrak{q}$ *be a semi-prime right ideal in the arbitrary ring* R, *and let* f *and* g *be elements of the polynomial ring* $R[x]$ *such that* $fRg \subseteq \mathfrak{q}[x]$. *Then* $aRb \subseteq \mathfrak{q}$, *where* a *is any coefficient of* f *and* b *is any coefficient of* g.

This is obvious if g is of degree zero, and we proceed to use induction on the degree of g. Let us set

$$f = a_0x^n + \ldots + a_n, \qquad (a_0 \neq 0),$$

and

$$g = b_0x^m + \ldots + b_m, \qquad (b_0 \neq 0).$$

Hence we have

$$(a_0x^n + \ldots + a_n)R(b_0x^m + \ldots + b_m) \subseteq \mathfrak{q}[x]. \tag{1}$$

By considering the coefficients of $x^{n+m}, \ldots, x^m$, we obtain the following in which r is an arbitrary element of R:

$$\begin{aligned} &a_0rb_0 \in \mathfrak{q}, \\ &(a_0rb_1 + a_1rb_0) \in \mathfrak{q}, \\ &(a_0rb_2 + a_1rb_1 + a_2rb_0) \in \mathfrak{q}, \\ &\qquad \ldots \\ &(\qquad \ldots + a_nrb_0) \in \mathfrak{q}. \end{aligned} \tag{2}$$

If we multiply the second of these relations by sb_0 on the right, where s is an arbitrary element of R, we get

$$(a_0rb_1sb_0 + a_1rb_0sb_0) \in \mathfrak{q}, \tag{3}$$

and the first term on the left side of (3) is in $\mathfrak{q}$ by the first of relations (2). Hence $a_1Rb_0Rb_0 \subseteq \mathfrak{q}$, and it follows that $(a_1Rb_0R)^2 \subseteq \mathfrak{q}$. Since $\mathfrak{q}$ is semi-prime, this implies that $a_1Rb_0R \subseteq \mathfrak{q}$ and then property (A) shows that $a_1Rb_0 \subseteq \mathfrak{q}$.

Now if we multiply the third of relations (2) by sb_0 on the right, the first two terms are in $\mathfrak{q}$ and it follows that $a_2rb_0sb_0 \in \mathfrak{q}$. As above, this implies that $a_2Rb_0 \subseteq \mathfrak{q}$. Obviously, this can be continued to yield

$$a_iRb_0 \subseteq \mathfrak{q} \ (i = 0,1, \ldots, n).$$

Thus $fRb_0 \subseteq \mathfrak{q}[x]$ and hence from (2) we have

$$fR(b_1x^{m-1} + \ldots + b_m) \subseteq \mathfrak{q}[x],$$

and the induction hypothesis shows that $aRb \subseteq \mathfrak{q}$, where a and b are any coefficients of f and g, respectively.

We shall now prove the following theorem.

THEOREM 3. *The "going up" and "going down" theorems hold for semi-prime right ideals.*

Suppose, first, that $\mathfrak{q} \in \mathfrak{S}_r(R)$ and let us show that $\mathfrak{q}[x] \in \mathfrak{S}_r(R[x])$. By (A), we only need to show that if $f \in R[x]$ such that $fR[x]f \subseteq \mathfrak{q}[x]$, then $f \in \mathfrak{q}[x]$. In particular, $fRf \subseteq \mathfrak{q}[x]$ and Lemma 2 shows that $aRb \subseteq \mathfrak{q}$, where a and b are any coefficients of f. Since also $ab \in \mathfrak{q}$ by property (B) it follows that $\mathfrak{a}^2 \subseteq \mathfrak{q}$, where $\mathfrak{a}$ is the right ideal in R generated by the coefficients of f. Hence $\mathfrak{a} \subseteq \mathfrak{q}$ and $f \in \mathfrak{q}[x]$.

To prove the "going down" theorem, suppose that $Q \in \mathfrak{S}_r(R[x])$ and let us show that $\mathfrak{q} = Q \cap R \in \mathfrak{S}_r(R)$. Clearly $(\mathfrak{q}\,[x])^2 \subseteq Q$, and therefore $\mathfrak{q}[x] \subseteq Q$. If $a \in R$ such that $aRa \subseteq \mathfrak{q}$, it follows that

$$(aR[x])^2 \subseteq \mathfrak{q}[x] \subseteq Q$$

and hence $aR[x] \subseteq Q$. However, by (A), we then have $a \in Q$ so that $a \in \mathfrak{q}$, completing the proof.

Remark. Although we have stated and proved this result for semi-prime right ideals, it is clear that it holds also for semi-prime *ideals.* In this case, the proof of the "going up" theorem need not depend on Lemma 2 but can be easily established as follows. Let $fR[x]f \subseteq \mathfrak{q}[x]$, where f is of degree n, and let a be the leading coefficient of f. It follows that $aRa \subseteq \mathfrak{q}$ and hence $a \in \mathfrak{q}$. Therefore $ax^n \in \mathfrak{q}[x]$ and we have

$$(f - ax^n)R[x]\,(f - ax^n) \subseteq \mathfrak{q}[x].$$

Since $f - ax^n$ is zero or has degree less than n, it is clear how to complete the proof by induction on n.

From Theorems 1 and 3 we now obtain at once the following result.

COROLLARY. *If σ-ideals are taken to be the semi-prime right ideals and $\mathfrak{a}$ is any right ideal in R, then* $(\mathfrak{a}[x])_\sigma = \mathfrak{a}_\sigma[x]$.

We may also point out that if σ-ideals are taken to be the semi-prime *ideals*, the lower radical of R is just $(0)_\sigma$ and it follows that if N is the lower radical of R, then the lower radical of $R[x]$ is $N[x]$. Since the lower radical coincides with the prime radical, this is still another proof of Theorem 2.

If $\mathfrak{a}, \mathfrak{b} \in \mathfrak{J}_r(R)$, we shall use the familiar notation

$$(\mathfrak{b}:\mathfrak{a}) = \{r;\, r \in R,\, \mathfrak{a}r \subseteq \mathfrak{b}\}.$$

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In case $\mathfrak{b} = 0$, we shall write $\mathfrak{a}^r$ in place of $((0):\mathfrak{a})$. Hence $\mathfrak{a}^r$ is the ideal of right annihilators of the right ideal $\mathfrak{a}$. We next prove the following theorem.

THEOREM 4. *Let $A \in \mathfrak{J}_r(R[x])$ and $\mathfrak{q} \in \mathfrak{S}_r(R)$. Then if $\mathfrak{a}$ is the right ideal in R generated by all coefficients of elements of A, we have $(\mathfrak{q}[x]:A) = (\mathfrak{q}:\mathfrak{a})[x]$.*

Let f be any element of $(\mathfrak{q}:\mathfrak{a})[x]$, and c any coefficient of f. Then $\mathfrak{a}c \subseteq \mathfrak{q}$, which implies that $Ac \subseteq \mathfrak{q}[x]$ and hence that $Af \subseteq \mathfrak{q}[x]$. It follows that $(\mathfrak{q}:\mathfrak{a})[x] \subseteq (\mathfrak{q}[x]:A)$. This part is trivial and makes no use of the hypothesis that q is semi-prime.

Now assume that $f \in (\mathfrak{q}[x]:A)$. If a is any coefficient of an element of A and c is any coefficient of f, Lemma 2 assures us that $aRc \subseteq \mathfrak{q}$. Since this implies that $ac \in \mathfrak{q}$, we have $\mathfrak{a}c \subseteq \mathfrak{q}$ or $c \in (\mathfrak{q}:\mathfrak{a})$. Thus $f \in (\mathfrak{q}:\mathfrak{a})[x]$, and hence $(\mathfrak{q}[x]:A) \subseteq (\mathfrak{q}:\mathfrak{a})[x]$. The proof is therefore completed.

We recall that a semi-prime *ring* is a ring in which the zero ideal is semi-prime. The following result is then obtained from the theorem just proved by specifying q to be the zero ideal.

COROLLARY. *If A is a right ideal in the semi-prime ring $R[x]$ and $\mathfrak{a}$ is the right ideal in R generated by all coefficients of elements of A, then $A^r = \mathfrak{a}^r[x]$.*

Thus, if R is a semi-prime ring, $Ag = 0$ if and only if each coefficient of g is a right annihilator of all coefficients of elements of A. This is not true if R is an arbitrary ring, as will be apparent from an example to be given below, but it is known **(11)** that $A^r \neq 0$ if and only if $\mathfrak{a}^r \neq 0$.

THEOREM 5. *Suppose that $\mathfrak{q}[x] \subseteq A$, where $\mathfrak{q} \in \mathfrak{S}_r(R)$ and $A \in \mathfrak{J}_r(R[x])$, R being an arbitrary ring. If $\mathfrak{a}$ is the right ideal in R generated by all coefficients of elements of A and $\mathfrak{b}$ is the right ideal in R generated by the leading coefficients of elements of A, then $(\mathfrak{q}:\mathfrak{a}) = (\mathfrak{q}:\mathfrak{b})$.*

Since $\mathfrak{b} \subseteq \mathfrak{a}$, it is clear that $(\mathfrak{q}:\mathfrak{a}) \subseteq (\mathfrak{q}:\mathfrak{b})$. Let $c \in (\mathfrak{q}:\mathfrak{b})$, and let us show that $c \in (\mathfrak{q}:\mathfrak{a})$. That is, if $f \in A$, we wish to show that

$$fc \in \mathfrak{q}[x]. \tag{4}$$

This is true if f is of degree zero, for then $f \in \mathfrak{b}$. Let us therefore set

$$f = a_0x^n + \ldots + a_n \qquad (a_0 \neq 0) \tag{5}$$

and assume (4) for all elements of A of degree less than n. Clearly $a_0 \in \mathfrak{b}$ and $a_0R \subseteq \mathfrak{b}$, so that $a_0Rc \subseteq \mathfrak{q}$ and therefore $a_0c \in \mathfrak{q}$. Hence $a_0cx^n \in \mathfrak{q}[x] \subseteq A$, and it follows that

$$(a_1x^{n-1} + \ldots + a_n)c \subseteq A.$$

By use of the induction hypothesis we can conclude that $a_icRa_ic \subseteq \mathfrak{q}$, and hence finally that $a_ic \in \mathfrak{q}$ $(i = 0, 1, \ldots, n)$. It follows that $fc \in \mathfrak{q}[x]$, and the proof is completed.

In case R is a semi-prime ring, we can choose $\mathfrak{q}$ to be the zero ideal and obtain the following corollary.

COROLLARY. *Let $A \in \mathfrak{J}_r(R)$, where R is a semi-prime ring, and let $\mathfrak{a}$ and $\mathfrak{b}$ be as defined in the preceding theorem. Then $\mathfrak{a}^r = \mathfrak{b}^r$.*

As an example to show that the conclusions of the preceding corollary, and also of this one, are not true for arbitrary rings, let R be the ring $I[t]/(4, t^2)$, where I is the ring of integers and t an indeterminate. In $R[x]$, let $A = (tx + 2)$. Then it is obvious that $(tx + 2) \subseteq A^r$, whereas $t \notin \mathfrak{a}^r$ since $2t \neq 0$. Moreover, it can be verified by a detailed but straight-forward calculation that t annihilates the leading coefficient of each element of A, and hence that $t \in \mathfrak{b}^r$. In particular, then, we have $\mathfrak{a}^r \neq \mathfrak{b}^r$.

5. Prime right ideals in semi-prime rings. Throughout this section and the next we shall consider semi-prime rings only.

A right ideal $\mathfrak{p}$ in a semi-prime ring R is said to be a *prime right ideal* if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$, where $\mathfrak{a}$, $\mathfrak{b} \in \mathfrak{J}_r(R)$ with $\mathfrak{b}^r = 0$, implies that $\mathfrak{a} \subseteq \mathfrak{p}$.

Johnson **(7)** has introduced this definition and has shown that the prime right ideals play an important role in the structure theory of semi-prime rings. We may emphasize that the concept of prime right ideal has meaning only for semi-prime rings.

By Theorem 2 we know that $R[x]$ is a semi-prime ring if and only if R is semi-prime. It is known **(7**, p. 378**)** that a prime right ideal is semi-prime, and therefore some of our previous results are applicable to prime right ideals. We may denote the set of prime right ideals of R by $\mathfrak{P}_r(R)$, and hence $\mathfrak{P}_r(R) \subseteq \mathfrak{S}_r(R)$ for any semi-prime ring R.

We now prove the following theorem.

THEOREM 6. *The "going up" and "going down" theorems hold for prime right ideals.*

Suppose that $\mathfrak{p} \in \mathfrak{P}_r(R)$ and that $AB \subseteq \mathfrak{p}[x]$, where A, $B \in \mathfrak{J}_r(R[x])$ with $B^r = 0$. If $\mathfrak{a}$ and $\mathfrak{b}$ are the respective right ideals in R generated by the coefficients of elements of A and of B, it follows from Lemma 2 that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$. Moreover, $\mathfrak{b}^r = 0$ and it follows that $\mathfrak{a} \subseteq \mathfrak{p}$ and therefore that $A \subseteq \mathfrak{p}[x]$. This proves the "going up" theorem.

Now let $P \in \mathfrak{P}_r(R[x])$, and set $\mathfrak{p} = P \cap R$. Suppose that $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$, where $\mathfrak{a}, \mathfrak{b} \in \mathfrak{J}_r(R)$ with $\mathfrak{b}^r = 0$. Then $\mathfrak{a}[x]\mathfrak{b}[x] \subseteq \mathfrak{p}[x]$ and the Corollary to Theorem 4 implies that $(\mathfrak{b}[x])^r = 0$. But $\mathfrak{p}[x]R[x] \subseteq P$, and this implies that $\mathfrak{p}[x] \subseteq P$. Hence we have $\mathfrak{a}[x]\mathfrak{b}[x] \subseteq P$ with $(\mathfrak{b}[x])^r = 0$, and hence $\mathfrak{a}[x] \subseteq P$. It follows that $\mathfrak{a} \subseteq \mathfrak{p}$, and this establishes the "going down" theorem.

Now let σ-ideal mean prime right ideal. Evidently the set of prime right ideals is closed under arbitrary intersection and if $\mathfrak{a} \in \mathfrak{J}_r(R)$, $\mathfrak{a}_\sigma$ is the unique least prime right ideal which contains $\mathfrak{a}$. In the terminology of **(7)**, $\mathfrak{a}_\sigma$ is the *prime cover* of $\mathfrak{a}$. In view of Theorem 6, we may apply Theorem 1 to show that if $\mathfrak{p}$ is the prime cover of $\mathfrak{a}$ in R, then $\mathfrak{p}[x]$ is the prime cover of $\mathfrak{a}[x]$ in $R[x]$.

If $\mathfrak{a} \in \mathfrak{J}_r(R)$, Theorem 6 shows that $\mathfrak{a}[x] \in \mathfrak{P}_r(R[x])$ if and only if $\mathfrak{a} \in \mathfrak{P}_r(R)$. Of course, this says nothing about the prime right ideals of $R[x]$ which are not of the form $\mathfrak{a}[x]$. An example to be given in §7 will show that there may exist many prime right ideals not of this form.

6. Annihilating ideals in semi-prime rings. If T is a set of elements of the semi-prime ring R, we may set $T^r = \{a;\ a \in R,\ Ta = 0\}$ and $T^l = \{a;\ a \in R,\ aT = 0\}$. The right ideal $\mathfrak{a}$ in R is an *annihilating right ideal* if $\mathfrak{a} = T^r$ for some set T. Annihilating left ideals are defined in an analogous way, and an ideal which is both an annihilating right ideal and an annihilating left ideal will be called an *annihilating ideal*. We may denote the set of all annihilating ideals in R by $\mathfrak{A}(R)$. It is known **(7**, p. 376**)** that if $\mathfrak{a} \in \mathfrak{J}(R)$, then $\mathfrak{a} \in \mathfrak{A}(R)$ if and only if $\mathfrak{a} = \mathfrak{a}^{rl}$, where $\mathfrak{a}^{rl}$ means $(\mathfrak{a}^r)^l$. Moreover, it can be shown **(7**, p. 379**)** that $\mathfrak{A}(R) = \mathfrak{J}(R) \cap \mathfrak{P}_r(R)$, and clearly $\mathfrak{A}(R)$ is closed under arbitary intersection.

We now prove the following theorem.

THEOREM 7. *If R is a semi-prime ring, the annihilating ideals of $R[x]$ are precisely those ideals of the form $\mathfrak{a}[x]$, where $\mathfrak{a}$ is an annihilating ideal of R.*

This follows readily from the Corollary to Theorem 4 and the corresponding result for left ideals. If A is any ideal in $R[x]$ and $\mathfrak{a}$ is the ideal in R generated by all coefficients of elements of A, then $A^{rl} = \mathfrak{a}^{rl}[x]$. If, then, $A \in \mathfrak{A}(R[x])$ so that $A = A^{rl}$, it follows that $A = \mathfrak{a}^{rl}[x]$ and therefore $\mathfrak{a}^{rl} = \mathfrak{a}$. Hence $\mathfrak{a} \in A(R)$ and $A = \mathfrak{a}[x]$.

Conversely, if $\mathfrak{a} \in \mathfrak{A}(R)$ and we set $A = \mathfrak{a}[x]$, it follows that

$$A^{rl} = \mathfrak{a}^{rl}[x] = \mathfrak{a}[x] = A,$$

and $A \in \mathfrak{A}(R[x])$.

We may call $\mathfrak{A}(R)$ an *algebra* under the operations $\{\subseteq, \cap\}$. A minimal nonzero element of $\mathfrak{A}(R)$ will be called an *atom*. Since, by the preceding theorem, the algebras $\mathfrak{A}(R)$ and $\mathfrak{A}(R[x])$ are isomorphic, it follows that $\mathfrak{A}(R[x])$ has atoms if and only if $\mathfrak{A}(R)$ has atoms. We shall now make a few remarks about some relations between the structures of R and of $R[x]$ as determined by the isomorphism of $\mathfrak{A}(R)$ and $\mathfrak{A}(R[x])$. Throughout the rest of this section *we assume that the algebra $\mathfrak{A}(R)$ has atoms.* Our remarks are based on work of Johnson **(7)** whose structure theory for semi-prime rings involves a study of the prime right ideals and is more penetrating than any analysis depending only on the annihilating ideals.

If the atoms of $\mathfrak{A}(R)$ are $\mathfrak{b}_\alpha$, where α ranges over some index set, we shall call the ring sum of all these atoms the *support* of R and denote it by S. This is related to what Johnson calls the *base* of R, but we use a different term in order to avoid any confusion. It follows by the method of proof of Theorem 4.1 of **(7)** that $S = \cup_\alpha \mathfrak{b}_\alpha$, where $\cup\alpha$ indicates the finite direct sum of the

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$\mathfrak{b}_\alpha$. Since the atoms of $R[x]$ are $\mathfrak{b}_\alpha[x]$, we know that $S[x]$ is then the support of the ring $R[x]$. Of course, this is the reason that the concept is of significance for our purposes.

It is known **(6)** that with a semi-prime ring R there is associated a unique semi-prime ring $N(R)$ such that (i) R is an ideal in $N(R)$, (ii) $R^r = 0$ in $N(R)$, and (iii) any ring T which contains R as an ideal and in which $R^r = 0$ is a subring of $N(R)$. Since $S = \bigcup_\alpha \mathfrak{b}_\alpha$, it follows **(7**, p. 387**)** that

$$N(S) = \textstyle\sum_\alpha N(\mathfrak{b}_\alpha),$$

where $\sum_\alpha$ refers to the full direct sum.

Now $S \cap S^r = 0$ and therefore the sum of S and S^r is a direct sum $S \oplus S^r$. Moreover, $(S \oplus S^r)^r = 0$ and since $N(S \oplus S^r) = N(S) \oplus N(S^r)$, it follows that

$$S \oplus S^r \subseteq R \subseteq N(S) \oplus N(S^r). \tag{6}$$

We now investigate the question as to whether the relations (6) imply anything of the same sort about $R[x]$. From (6) it follows at once that

$$S[x] \oplus S^r[x] \subseteq R[x] \subseteq (N(S))[x] \oplus (N(S^r))[x]. \tag{7}$$

Moreover, we know that $S[x]$ is the support of $R[x]$, and the Corollary to Theorem 4 implies that $S^r[x] = (S[x])^r$. That is, the left side of (7) has the same relation to $R[x]$ that the left side of (6) has to R. Since S has no right annihilator in $N(S)$, S[x] has no right annihilator in $(N(S))[x]$, and hence $(N(S))[x] \subseteq N(S[x])$. Similarly, $(N(S^r))[x] \subseteq N(S^r[x])$, and thus

$$(N(S))[x] \oplus (N(S^r))[x] \subseteq N(S[x]) \oplus (N(S^r))[x]. \tag{8}$$

Moreover, $R[x]$ has no right annihilator in the left side of (8), so the second inclusion in (7) is apparently a little better than that obtained by simply applying relations (6) to the ring $R[x]$.

7. An example. We have established as Theorem 6 the "going up" and "going down" theorems for prime right ideals in semi-prime rings. The principal purpose of this section is to show by means of an example that there may be many prime right ideals in $R[x]$ not obtained from prime right ideals in R by the "going up" process.

The right ideal $\mathfrak{p}$ in a *prime* ring R is a prime right ideal if and only if $aRb \subseteq \mathfrak{p}$, where $a, b \in R$ with $b \neq 0$, implies that $a \in \mathfrak{p}$. Throughout this section R will be a *division* ring, and $R[x]$ is then a prime ring.

Let $c \in R$ and let us set $P = (x - c)R[x]$. It is not difficult to see that P is a semi-prime right ideal. However, if c is algebraic over the center of R, P is not a prime right ideal. For in this case there exists a nonzero polynomial $g(x)$ with coefficients in the center of R such that $g(c) = 0$, and $g(x)$ is therefore both right and left divisible by $x - c$. It follows that

$$1 \cdot R[x]g(x) = g(x)R[x] \subseteq P,$$

whereas $1 \notin P$.

From these remarks it follows that if R is a division ring and $R[x]$ has a prime right ideal of the form $(x - c)R[x]$, then c can not be algebraic over the center of R. We now give an example showing that there do exist prime right ideals of this form.

Henceforth, R will be the division ring of Hilbert (**4**, pp. 103ff). For our purpose it is sufficient to know that R contains the field of rational numbers in its center and that it contains elements s and t such that $ts = 2st$. From this, it follows by induction that if α and β are nonnegative integers, then

$$s^{-\beta}\, t^{\alpha}\, s^{\beta} = 2^{\alpha\beta}\, t^{\alpha}. \tag{9}$$

We shall show that in $R[x]$ the right ideal $P = (x - t)R[x]$ is a prime right ideal. Accordingly, we assume that

$$g(x)R[x]h(x) \subseteq P \qquad h(x) \neq 0, \tag{10}$$

and shall show that $g(x) \in P$. By the division algorithm, we may write

$$g(x) = (x - t)q(x) + c,$$

where $c \in R$. From (10) it follows that

$$cR[x]h(x) \subseteq P. \tag{11}$$

The proof will be completed by showing that $c = 0$ and hence that $g(x) \in P$.

Let us assume that $c \neq 0$ and seek a contradiction. In particular, (11) then implies that $Rh(x) \subseteq P$. Thus, for every b in R, $bh(x)$ is left divisible by $x - t$. If $b \neq 0$ and $bh(x) = (x - t)F(x)$, it follows that

$$h(x) = b^{-1}(x - t)F(x) = (x - b^{-1}tb)b^{-1}F(x),$$

and $h(x)$ is left divisible by $x - b^{-1}tb$ for every nonzero $b \in R$. We proceed to show that this is impossible. Let

$$h(x) = a_0x^n + a_1x^{n-1} + \ldots + a_n \qquad (a_0 \neq 0).$$

If $d \in R$, we may write

$$h_L(d) = d^na_0 + d^{n-1}a_1 + \ldots + a_n,$$

and the factor theorem states that $h_L(d) = 0$ if and only if $h(x)$ is left divisible by $x - d$. Accordingly, we have that $h_L(b^{-1}tb) = 0$ for every nonzero b in R. In particular, $h_L(s^{-\beta}ts^{\beta}) = 0$ $(\beta = 0, 1, \ldots, n)$. Using (9), we then have the following system of equations:

$$\begin{aligned}
t^na_0 &+ t^{n-1}a_1 &&+ \ldots + a_n = 0,\\
2^nt^na_0 &+ 2^{n-1}t^{n-1}a_1 &&+ \ldots + a_n = 0,\\
2^{2n}t^na_0 &+ 2^{2(n-1)}t^{n-1}a_1 &&+ \ldots + a_n = 0,\\
\ldots & &&\ldots\\
2^{n.n}t^na_0 &+ 2^{n(n-1)}t^{n-1}a_1 &&+ \ldots + a_n = 0.
\end{aligned}$$

However, since the determinant

$$\begin{vmatrix} 1 & 1 & . & . & . & 1 & 1 \\ 2^n & 2^{n-1} & . & . & . & 2 & 1 \\ 2^{2n} & 2^{2(n-1)} & & . & . & 2^2 & 1 \\ . & . & . & . & . & . & . \\ 2^{n.n} & 2^{n(n-1)} & & . & . & 2^n & 1 \end{vmatrix}$$

is different from zero, it follows that

$$t^n a_0 = t^{n-1} a_1 = \ldots = a^n = 0,$$

and hence $a_i = 0$ $(i = 0, 1, \ldots, n)$. This implies that $h(x) = 0$ and we have the desired contradiction. It follows that $c = 0$ and hence that $g(x) \in P$, completing the proof that P is a prime right ideal in $R[x]$.

Since $R[x]$ is an integral domain and has a prime right ideal other than (0) and $R[x]$, it is known **(5)** that $R[x]$ does not have minimal prime right ideals. In the present example this also follows from the following statement whose proof we omit. If c_i $(i = 1, 2, \ldots, k)$ are distinct elements of the center of R, then

$$(x - c_1 t)(x - c_2 t) \ldots (x - c_k t) R[x]$$

is a prime right ideal in $R[x]$, and thus there exist prime right ideals which are generated by polynomials of arbitrarily high degree.

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