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THE BERGMAN PROJECTION ON WEIGHTED NORM SPACES

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1. Introduction. Quite recently Bekollé and Bonami [**1**] have characterized the weighted measures λ on the unit disk Δ for which the Bergman projection is bounded on $L_p(\Delta : \lambda)$, $1 < p < \infty$. Our methods in [**4**] can be applied to even extend their result by replacing the unit disk with multiply connected domains. This is done via a rather interesting identity between the Bergman kernel and its "adjoint" [**2**]. As a corollary of our result we obtain a generalization of a result due to Shikhvatov [**7**].

Let D be a bounded plane domain and let λ be a positive locally integrable function in D. λ is said to belong to $M_p(D)$ $(1 < p < \infty)$ if it satisfies the Muckenhoupt condition:

$$\operatorname*{Sup}_V \left[\frac{1}{|V|}\int_V \lambda(z)d\sigma(z)\right]\left[\frac{1}{|V|}\int_V \lambda(z)^{-1/p-1}d\sigma(z)\right]^{p-1} < \infty,$$

where the supremum is taken over all sectors $V \subset D$, $d\sigma$ is the area Lebesgue measure and $|V| = \sigma(V)$.

We denote by $L_p(D : \lambda)$ the space of all measurable functions f in D for which

$$||f||_{L_p(D:\lambda)} = \left\{\int_D |f(z)|^p\lambda(z)d\sigma(z)\right\}^{1/p} < \infty, \quad 0 < p < \infty,$$

and we write $L_p(D)$ for $L_p(D : 1)$. We also write $||f||_p$ for $||f||_{L_p(D)}$. We shall always assume that $1 < p < \infty$ and that $q = p/(p-1)$. From the $M_p(D)$-definition follows that if $\lambda \in M_p(D)$ then $\lambda \in L_1(D)$ and $\lambda^{-1} \in L_{q/p}(D)$. Moreover, in this case

$$||\lambda||_1||\lambda^{-1}||_{q/p} \leqq C_p|D|^p,$$

where $C_p(D)$ is a positive constant which can be taken as the supremum in the definition of $\lambda \in M_p(D)$.

We consider the Hilbert transform

$$(T_D(f)(\zeta) = \frac{1}{\pi}\int_D \frac{1}{(z-\zeta)^2}f(z)d\sigma(z)$$

and the Riesz transform

$$(R_Df)(\zeta) = \frac{1}{2\pi}\int_D \frac{\overline{z-\zeta}}{|z-\zeta|^3}f(z)d\sigma(z)$$

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where the integrals are taken in the principal value sense. It is well known that these operators are bounded on $L_p(D)$ $1 < p < \infty$. Moreover, if $f_{\bar{z}} \in L_p(D)$ then $f_z = -R_D^2 f_{\bar{z}}$ and therefore

$$\|f_z\|_p \leqq A_p \|f_{\bar{z}}\|_p.$$

Here $f_z = \partial f/\partial z$ and $f_{\bar{z}} = \partial f/\partial \bar{z}$.

We shall be using the following proposition (cf. [**5**]):

PROPOSITION 1. *T_D is bounded on $L_p(D : \lambda)$ if and only if $\lambda \in M_p(D)$.*

Let $G = G_D(z, \zeta)$ be the Green's function of D. Here

$$G(z, \zeta) = H(z, \zeta) - \log |z - \zeta|,$$

where $H = H(z, \zeta)$ is symmetric and harmonic in $(z, \zeta) \in D \times D$. The Bergman kernel is given by

$$K_D(z, \bar{\zeta}) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \bar{\zeta}} \tag{1.1}$$

and its "adjoint" [**2**] is

$$L_D(z, \zeta) = -\frac{2}{\pi} \frac{\partial^2 G}{\partial z \partial \zeta}. \tag{1.2}$$

Consequently,

$$L_D(z, \zeta) = \frac{1}{\pi} \frac{1}{(z - \zeta)^2} - l_D(z, \zeta),$$

where

$$l_D(z, \zeta) = \frac{2}{\pi} \frac{\partial^2 H}{\partial z \partial \zeta},$$

is symmetric and holomorphic in $(z, \zeta) \in D \times D$. We note that $l_D(z, \zeta) \equiv 0$ when D is a disk and that $l_D(z, \zeta)$ is holomorphic in $(z, \zeta) \in \bar{D} \times \bar{D}$ when ∂D is analytic (cf. [**2**, p. 211]). If ϕ is a conformal mapping of D onto Ω then

$$G_D(z, \zeta) = G_\Omega(\phi(z), \phi(\zeta))$$

and therefore

$$K_D(z, \bar{\zeta}) = K_\Omega(\phi(z), \overline{\phi(\zeta)}) \phi'(z) \overline{\phi'(\zeta)}, \tag{1.3}$$

and

$$L_D(z, \zeta) = L_\Omega(\phi(z), \phi(\zeta)) \phi'(z) \phi'(\zeta).$$

We consider the "Bergman–Schiffer transforms"

$$(Q_D f)(\zeta) = \int_D \overline{L_D(z, \zeta)} f(z) d\sigma(z) \tag{1.4}$$

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and

$$(S_D f)(\zeta) = \int_D \overline{l_D(z, \zeta)} f(z) d\sigma(z)$$

where the first integral is taken in the principal value sense. It follows that

(1.5) $T_D = Q_D + S_D$.

The (formal) adjoint of Q_D is given by

$$(\bar{Q}_D f)(\zeta) = \int_D L(z, \zeta) f(z) d\sigma(z)$$

and the "Bergman projection" is

$$(1.6) \quad (P_D f)(\zeta) = \int_D \overline{K_D(z, \bar{\zeta})} f(z) d\sigma(z).$$

2. Identities amongst operators. In this section we shall assume that D is a bounded domain with ∂D being analytic. It will be clear from the sequel that this assumption could be weakened considerably. For example, it will be sufficient, after some modification of the argument, to assume that ∂D is of class C^1 with a Dini continuous normal (cf. [**4**]).

Since ∂D is analytic it follows that $l_D(z, \zeta)$ is holomorphic for $(z, \zeta) \in \bar{D} \times \bar{D}$ and therefore S_D is a bounded operator on $L_p(D)$, $1 \leqq p \leqq \infty$. Consequently, in view of (1.5), Q_D is a bounded operator on $L_p(D)$, $1 < p < \infty$.

Let $C_c^\infty(D)$ be the class of $C^\infty(D)$ functions with compact support inside D and let $H(D)$ designate the class of holomorphic functions in D. The following theorem is crucial to our work. It holds also for domains which are not so smooth (cf. [**4**]). For the special case that $p = 2$ and ∂D is analytic it was also proved by Block [**3**] by using different methods.

THEOREM 1. *$I - P_D = \bar{Q}_D Q_D$ on $L_p(D)$, $1 < p < \infty$. Here I is the identity operator on $L_p(D)$.*

Proof. Let $f \in L_p(D)$ and write

$$(2.1) \quad g(\zeta) = 2\pi^{-1} \int_D G_{\bar{z}}(z, \zeta) f(z) d\sigma(z).$$

From classical results of potential theory it is well known that g_ζ and $g_{\bar{\zeta}}$ exist a.e. in D, and they are given by

$$(2.2) \quad g_\zeta(\zeta) = f(\zeta) + 2\pi^{-1} \int_D H_{\bar{z}\zeta}(z, \zeta) f(z) d\sigma(z)$$

and

$$(2.3) \quad g_{\bar{\zeta}}(\zeta) = 2\pi^{-1} \int_D G_{z\zeta}(z, \zeta) f(z) d\sigma(z).$$

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Using (1.2) and (1.4), we can express (2.3) as

(2.4) $g_{\bar{\zeta}}(\zeta) = -(Q_D f)(\zeta)$.

Further, $H_{\bar{z}\zeta} = G_{\bar{z}\zeta}$ and so, using (1.1), (1.6) and (2.2),

(2.5) $g_\zeta(\zeta) = (I - P_D)f(\zeta)$.

From (2.4) it follows that $g_{\bar{\zeta}} \in L_p(D)$ and therefore $g_\zeta = -R_D{}^2 g_{\bar{\zeta}}$. This together with (2.4) and (2.5) shows that

(2.6) $I - P_D = R_D{}^2 Q_D$

and therefore $I - P_D$ is continuous on $L_P(D)$. Because of the continuity of the operators $I - P_D$ and $\bar{Q}_D Q_D$, and the density of $C_c{}^\infty(D)$ in $L_p(D)$, it suffices to prove the theorem for $C_c{}^\infty(D)$. Let now f be in $C_c{}^\infty(D)$. Since $P_D f \in H(D)$ we have from (2.5) that $g_{\zeta\bar{\zeta}} = f_{\bar{\zeta}}$. By Green's formula

(2.7) $$\int_D G_{\bar{z}} f d\sigma(z) = -\int_D G f_{\bar{z}} d\sigma(z) = -\int_D G g_{z\bar{z}} d\sigma(z) = \int_D G_{\bar{z}} g_z d\sigma(z).$$

We have used the fact that $g_{\bar{z}} = -Q_D f$, in accordance with (2.4), is in $C_c{}^\infty(D)$ because $f \in C_c{}^\infty(D)$. The above integrals are, of course, taken in the principal value sense. Therefore, using (2.1), (2.5) and (2.7),

$$(I - P_D)f = g_z = -\bar{Q}_D(g_{\bar{z}}) = -\bar{Q}_D(-Q_D f) = \bar{Q}_D Q_D f$$

and the assertion follows.

We have actually shown the following operators identities on $L_p(D)$ $(1 < p < \infty)$, as (2.6) above shows:

$$I - P_D = R_D{}^2 Q_D = \bar{Q}_D Q_D.$$

3. The Bergman projection. We again assume that ∂D is analytic and as before we note the possibility of weakening this assumption. We fix $1 < p < \infty$ and $q = p/(p-1)$. Our main theorem is:

THEOREM 2. (i) *If $\lambda \in M_p(D)$, then Q_D is a bounded operator on $L_p(D : \lambda)$.*
(ii) *If Q_D is bounded on $L_p(D : \lambda)$ and $\lambda, \lambda^{-q/p} \in L_1(D)$, then $\lambda \in M_p(D)$.*
(iii) *If D is a disk and Q_D is bounded on $L_p(D : \lambda)$, then $\lambda \in M_p(D)$.*

Proof. (i) Since $\lambda \in M_p(D)$ we have $\|\lambda\|_1 \|\lambda^{-1}\|_{q/p} \leqq C_p |D|^p$. Therefore

$$\begin{aligned}
\|S_D f\|^p_{L_p(D:\lambda)} &= \int_D |(S_D f)(\zeta)|^p \lambda(\zeta) d\sigma(\zeta) \\
&= \int_D \left| \int_D \overline{l_D(z, \zeta)} f(z) d\sigma(z) \right|^p \lambda(\zeta) d\sigma(\zeta) \\
&\leqq A^p \|\lambda\|_1 \left[\int_D |f(z)| d\sigma(z) \right]^p \leqq A^p \|\lambda\|_1 \left[\int_D |f(z)|^p \lambda(z) d\sigma(z) \right] \\
&\times \left[\int_D \lambda(z)^{-q/p} d\sigma(z) \right]^{p/q} = A^p \|\lambda\|_1 \|\lambda^{-1}\|_{q/p} \|f\|^p_{L_p(D:\lambda)} \\
&\leqq A^p C_p |D|^p \|f\|^p_{L\ (D:\lambda)}.
\end{aligned}$$

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Consequently, S_D is bounded on $L_p(D : \lambda)$. According to (1.5), $Q_D = T_D - S_D$ and therefore, by Proposition 1, Q_D is bounded on $L_p(D : \lambda)$.

(ii) Since $\lambda, \lambda^{-q/p} \in L_1(D)$ it follows, as in (i), that S_D is bounded on $L_p(D : \lambda)$. Also, since Q_D is bounded on $L_p(D : \lambda)$ it follows that $T_D = Q_D + S_D$ has this property too. Proposition 1 shows, therefore, that $\lambda \in M_p(D)$.

(iii) If D is a disk, then $l_D(z, \zeta) \equiv 0$. Consequently, $Q_D = T_D$ and so again, by Proposition 1, $\lambda \in M_p(D)$.

This leads to the following result about the Bergman projection P_D:

THEOREM 3. (i) *If* $\lambda \in M_p(D)$, *then* P_D *is bounded operator on* $L_p(D : \lambda)$.
(ii) P_D *is bounded on* $L_p(D)$ *if and only if* Q_D *is.*

Proof. This follows from Theorems 1 and 2, (1.5) and the fact that $S_D = T_D P_D$.

This theorem complements the result of [**1**].

4. Applications. We now derive some consequences from Theorem 3. We shall allow the domain D not to be smooth. However, we shall assume that D is bounded by n non-degenerate boundary components $C_1, \ldots, C_n$ where, say, C_1 is the outer boundary. Then D can be conformally mapped onto a domain Ω which is bounded by n closed analytic curves. Let $\phi : D \to \Omega$ be such a mapping. Then $\phi = \phi_n \circ \phi_{n-1} \circ \ldots \circ \phi_1$, where each factor ϕ_j is a conformal mapping of a simply connected domain D_j. For example, $\omega_1 = \phi_1(z)$ is conformal on the simply connected domain D_1 which is bounded by C_1 and contains D, and $\phi_1(D_1)$ is the unit disk. $\omega_j = \phi_j(\omega_{j-1})$ $(2 \leqq j \leqq n)$ is conformal on the simply connected domain D_j which is bounded by $\phi_{j-1} \circ \phi_{j-2} \circ \ldots \circ \phi_1(C_j)$ and contains $\phi_{j-1} \circ \phi_{j-2} \circ \ldots \circ \phi_1(D)$; $\phi_j(D_j)$ is the exterior of the unit disk. See [**4**] for additional details. We write $\psi = \phi^{-1}$.

We define

$$t(D) = \operatorname{Sup}\{r \in \mathbf{R} \cup \{\infty\} : ||\phi'||_r < \infty\}.$$

This definition is clearly independent of the particular choice of the analytic domain $\Omega = \phi(D)$ and it is also obvious that $t_D \equiv t(D) \geqq 2$. It can be shown [**4**] that in fact $t_D > 2$. We can, therefore, define the interval

$$I(D) = \begin{cases} \left[\dfrac{t_D}{t_D - 1}, t_D\right] ; \|\phi'\|_{t_D} < \infty, \\ \left(\dfrac{t_D}{t_D - 1}, t_D\right) ; \|\phi'\|_{t_D} = \infty. \end{cases}$$

We also write

$$J(D) = I(D) - \{1, \infty\}.$$

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For a fixed $p \in J(D)$ we let $q = p/(p-1)$ (and thus $q \in J(D)$). D is said to belong to class W_p if ϕ' satisfies

$$\text{(4.1)} \quad \operatorname*{Sup}_{U} \left(\frac{1}{\|\phi'\|_{2:U}^{2}} \cdot \|\phi'\|_{p:U} \|\phi'\|_{q:U} \right) < \infty,$$

where the supremum is taken over all sectors $U \subset D$ and

$$\|f\|_{k:U} = \left[\int_U |f(z)|^k d\sigma(z) \right]^{1/k}.$$

Obviously, this definition is independent of the particular choice of the analytic domain $\Omega = \phi(D)$. Further, always $D \in W_2$ and $D \in W_p$ if and only if $D \in W_q$.

The following result was also obtained in [**4**].

THEOREM 4. *Let $p \in J(D)$. Then P_D is a bounded operator on $L_p(D)$ if and only if $D \in W_p$.*

Proof. For $z, \zeta \in D$ we write $\omega = \phi(z)$, $\tau = \phi(\zeta)$ with $\omega, \tau \in \Omega$. Also, for $f \in L_p(D)$ we let $g = (f \circ \psi) \cdot \psi'$. Using (1.3) and (1.6), we have

$$(P_D f)(\zeta) = \phi'(\zeta) \int_\Omega \overline{K_\Omega(\omega, \bar{\tau})} g(\omega) d\sigma(\omega) = \phi'(\zeta)(P_\Omega g)(\tau).$$

Therefore,

$$\|P_D f\|_p^p = \int_D |\phi'(\zeta)|^p |(P_\Omega g)(\tau)|^p d\sigma(\zeta)$$

$$= \int_\Omega |(P_\Omega g)(\tau)|^p |\psi'(\tau)|^{2-p} d\sigma(\tau).$$

Hence the $L_p(D)$ boundedness of P_D is equivalent to the inequality

$$\int_\Omega |(P_\Omega g)(\omega)|^p |\psi'(\omega)|^{2-p} d\sigma(\omega) \leqq A_p \|f\|_p^p.$$

However,

$$\|f\|_p^p = \int_\Omega |g(\omega)|^p |\psi'(\omega)|^{2-p} d\sigma(\omega).$$

Therefore, the above inequality is equivalent to the boundedness of P_Ω on $L_p(\Omega : \lambda)$ with $\lambda + |\psi'|^{2-p}$. By Theorems 2 and 3, since $p \in J(D)$, this is equivalent to $\lambda \in M_p(\Omega)$ which exactly means $D \in W_p$. This concludes the proof.

If D is simply connected and ∂D is of class C^1 with a Dini continuous normal then it follows from a theorem of Warschawski (see [**6**, p. 298]) that there exist positive constants a and b so that $0 < a < |\phi'(z)| < b < \infty$, $z \in D$. This is also true in the more general case when D is

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multiply connected by appealing to the above mentioned factorization of ϕ. Therefore, for Dini smooth domain D, $t_D = \infty$, $I(D) = [1, \infty]$ and $J(D) = (1, \infty)$. Further, $D \in W_p$ for any $p \in J(D) = (1, \infty)$.

Assume now that ∂D is Dini smooth except at one point $c \in \partial D$. At the point c the boundary makes an angle with aperture π/α, $1/2 \leqq \alpha < \infty$. We denote the class of such domains by M_α.

Let $D \in M_\alpha$. We may assume that $c \in C_1$. It is then well-known that

$$a_1|z - c|^{\alpha-1} \leqq |\phi_1'(z)| \leqq a_2|z - c|^{\alpha-1} \quad \text{for all } z \in D_1.$$

Since $C_2, \ldots, C_n$ are Dini smooth it follows from the above factorization of ϕ that

$$a|z - c|^{\alpha-1} \leqq |\phi'(z)| \leqq b|z - c|^{\alpha-1}; \quad a, b \in (0, \infty), z \in D. \tag{4.2}$$

In this case, condition (4.1) is equivalent to the condition that

$$\frac{1}{\|\phi'\|_{2:U}^2} \|\phi'\|_{p:U}\|\phi'\|_{q:U} \leqq C$$

for the sector

$$U = D(\rho : \beta) = \{z \in D : z - c = re^{i\theta}, 0 \leqq r \leqq \rho, -\beta/2 \leqq \theta \leqq \beta/2\},$$

and, that the constant C is independent of ρ and β. Now,

$$\int_U |z - c|^{(\alpha-1)p} d\sigma(z) = \beta \int_0^\rho r^{(\alpha-1)p+1} dr = \frac{\beta}{(\alpha - 1)p + 2} \rho^{(\alpha-1)p+2},$$

only when $(\alpha - 1)p + 2 > 0$. This of course is true for all $p > 1$ if $\alpha > 1$. When $1/2 \leqq \alpha < 1$ we must have $p < 2/(1 - \alpha)$. Therefore, $J(D) = (1, \infty)$ if $D \in M_\alpha$ with $\alpha \geqq 1$, and $J(D) = (2/(1 + \alpha), 2/(1 - \alpha))$ if $D \in M_\alpha$ with $1/2 \leqq \alpha < 1$. Also, using (4.1) and (4.2) we obtain

$$\left(\frac{a}{b}\right)^2 \frac{\alpha}{g(\alpha, p, q)} \leqq \frac{1}{\|\phi'\|_{2:U}^2} \cdot \|\phi'\|_{p:U}\|\phi'\|_{q:U} \leqq \left(\frac{b}{a}\right)^2 \frac{\alpha}{g(\alpha, p, q)},$$

with $g(\alpha, p, q) = [(\alpha - 1)p + 2]^{1/p}[(\alpha - 1)q + 2]^{1/q}$. For domains D of class M_α therefore, we have that $D \in W_p$ whenever $p \in J(D)$. Consequently using Theorem 4, we obtain:

THEOREM 5. *Let $D \in M_\alpha$. If $\alpha \geqq 1$ then P_D is bounded on $L_p(D)$ for all $p \in (1, \infty)$. If $1/2 \leqq \alpha < 1$ then P_D is bounded on $L_p(D)$ for all $p \in (2/(1 + \alpha), 2/(1 - \alpha))$.*

A special case of this theorem was also obtained by Shikhvatov [**7**] by using different methods.

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