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The Ample Cone for a K3 Surface

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Abstract. In this paper, we give several pictorial fractal representations of the ample or Kähler cone for surfaces in a certain class of $K3$ surfaces. The class includes surfaces described by smooth $(2, 2, 2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ defined over a sufficiently large number field K that have a line parallel to one of the axes and have Picard number four. We relate the Hausdorff dimension of this fractal to the asymptotic growth of orbits of curves under the action of the surface's group of automorphisms. We experimentally estimate the Hausdorff dimension of the fractal to be $1.296 \pm .010$.

The ample cone or Kähler cone for a surface is a significant and often complicated geometric object. Though much is known about the ample cone, particularly for $K3$ surfaces, only a few non-trivial examples have been explicitly described. These include the ample cones with a finite number of sides (see [N1] for $n = 3$, and [N2, N3] for $n \geq 5$; the case $n = 4$ is attributed to Vinberg in an unpublished work [N1]), the ample cone for a class of $K3$ surfaces with $n = 3$ [Ba3], and the ample cones for several Kummer surfaces, which are $K3$ surfaces with $n = 20$ [V, K-K, Kon]. Though the complexity of the problem generally increases with n, the problem for $K3$ surfaces with maximal Picard number ($n = 20$) appear to be tractable because of the small size of the transcendental lattice.

In this paper, we introduce accurate pictorial representations of the ample cone and the associated fractal for surfaces within a class of $K3$ surfaces with Picard number $n = 4$ (see Figures 1, 3, 4, and 5). As far as the author is aware, the associated fractal has not been studied in any great depth for any ample cone for which the fractal has a non-integer dimension, except the one in [Ba3]. The fractal in that case is Cantor-like (it is a subset of $\mathbb{S}^1$), and rigorous bounds on its Hausdorff dimension are calculated in [Ba1]. The Hausdorff dimension of the fractal in this paper is estimated to be $1.296 \pm .010$.

Our second main result is to relate the Hausdorff dimension of the fractal to the growth of the height of curves for an orbit of curves on a surface in this class. Precisely, let V be a surface within our class of $K3$ surfaces, and let $\mathcal{A} = \mathrm{Aut}(V/K)$ be its group of automorphisms over a sufficiently large number field K. Let D be an ample divisor on V, and let C be a curve on V. Define

$$N_{\mathcal{A}(C)}(t, D) = \#\{C' \in \mathcal{A}(C) : C' \cdot D < t\}.$$

Here we have abused notation by letting C' also represent the divisor class that contains C'. The intersection $C' \cdot D$ should be thought of as a logarithmic height of

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C', and the quantity $N_{\mathcal{A}(C)}(t, D)$ should be thought of as an analog of $N_{\mathcal{A}(P)}(t, h_D)$, which is the number of points P' in an $\mathcal{A}$ orbit of a point P with Weil height $h_D(P')$ bounded by t. See [Ba1, Section 5] for a detailed comparison. Let

$$\alpha = \lim_{t\to\infty} \frac{\log(N_{\mathcal{A}(C)}(t, D))}{\log t}.$$

For curves C with $C \cdot C > 0$, this limit exists and is the Hausdorff dimension of the fractal associated with the ample cone. Our estimate for α is inferred from a plot of $\log(N_{\mathcal{A}(C)}(t, D))$ as a function of $\log t$.

It should be stressed that the dynamics studied in this paper are in $\operatorname{Pic}(V) \otimes \mathbb{R}$, and not on V itself. The rich subject of dynamics on $K3$ surfaces has been studied by Cantat [C] and McMullen [McM]. Both authors study the dynamics of an automorphism whose pullback to $\operatorname{Pic}(V)$ is hyperbolic. On $\operatorname{Pic}(V) \otimes \mathbb{R}$, the dynamics of the pullback is discrete and uninteresting, but on the $K3$ surface, the orbits of points under iterations of this map are in places dense and quite fascinating. See in particular [C, Figure 1] and [McM, Figure 2].

1 Background

Let V be an algebraic $K3$ surface defined over a number field K. That is, V is a surface with trivial canonical divisor and irregularity equal to zero. Its arithmetic genus ρ_a is 1. If C is a smooth curve on V, then the genus of C is given by the adjunction formula $2g - 2 = C \cdot C$. The Picard group $\operatorname{Pic}(V)$ is an even lattice of dimension $n \le 20$. Let $\mathcal{D} = \{D_1, \dots, D_n\}$ be a basis of $\operatorname{Pic}(V)$, so

$$\operatorname{Pic}(V) = \mathbb{Z}D_1 \oplus \mathbb{Z}D_2 \oplus \cdots \oplus \mathbb{Z}D_n.$$

Let $J = [D_i \cdot D_j]$ be the intersection matrix for V with respect to the basis $\mathcal{D}$. By the Hodge index theorem, the signature of J is $(1, n-1)$. That is, J has one positive eigenvalue and $n-1$ negative eigenvalues.

Let $\mathcal{A} = \operatorname{Aut}(V/K)$ be the group of automorphisms on V. For an automorphism $\sigma \in \mathcal{A}$, the pullback σ^* acts linearly on $\operatorname{Pic}(V)$, so can be represented by a matrix with integer entries. Since σ preserves intersections, we further have that σ^* is in

$$\mathcal{O} = \mathcal{O}(\mathbb{Z}) = \{T \in \mathrm{Sl}_n(\mathbb{Z}) : T^t J T = J\}.$$

For an ample divisor D, the hypersurface $\mathbf{x}^t J \mathbf{x} = D \cdot D$ is a hyperboloid of two sheets, one of which contains D. Let us distinguish this sheet with $\mathcal{H}$, and define

$$\mathcal{O}^+ = \mathcal{O}^+(\mathbb{Z}) = \{T \in \mathcal{O} : T(\mathcal{H}) = \mathcal{H}\}.$$

The surface $\mathcal{H}$ can be thought of as a model of $n-1$ dimensional hyperbolic geometry imbedded in a Lorentz space, where the Lorentz product is the negative of the intersection product. The distance $|AB|$ between two points A and B in this model $\mathcal{H}$ is given by

$$A^t J B = A \cdot B = \|A\|\|B\| \cosh(|AB|) = -D \cdot D \cosh(|AB|),$$

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where $\|A\| = \sqrt{-A \cdot A}$. With this metric, the group of isometries on $\mathcal{H}$ is $\mathcal{O}^+(\mathbb{R})$, where the definition of this group is the obvious analog of $\mathcal{O}^+(\mathbb{Z})$. For more details of this model, we refer the reader to Ratcliff [R].

Let $\mathcal{E}$ be the set of effective divisor classes in $\mathrm{Pic}(V)$. That is, $E \in \mathcal{E}$ if we can write $E = a_1C_1 + \cdots + a_mC_m$ with $a_i \geq 0$ and C_i a divisor class that can be represented with a curve on V.

A divisor D is ample if $D \cdot E > 0$ for all effective divisors E. The ample cone, also known as the Kähler cone, is the set

$$\mathcal{K} = \{C \in \mathrm{Pic}(V) \otimes \mathbb{R} : C \cdot C > 0,\ C \cdot E > 0 \text{ for all } E \in \mathcal{E}\}.$$

The ample cone is convex and is a subset of the light cone

$$\mathcal{L}^+ = \{\mathbf{x} \in \mathrm{Pic}(V) \otimes \mathbb{R} : \mathbf{x} \cdot \mathbf{x} > 0, \mathbf{x} \cdot D > 0\},$$

where D is any ample divisor. If an irreducible curve has negative self intersection on V, then it is a smooth rational curve and has self intersection -2. Such a curve is called a -2 curve. If V contains no -2 curves, then $\mathcal{K} = \mathcal{L}^+$ [Kov]. Let $\mathcal{E}_{-2}$ be the set of nodal classes, that is, the set of divisor classes of smooth rational curves. Then

$$\mathcal{K} = \{C \in \mathcal{L}^+ : C \cdot E > 0 \text{ for all } E \in \mathcal{E}_{-2}\}.$$

Furthermore, every plane $C \cdot x = 0$ with $C \in \mathcal{E}_{-2}$ is a face of $\mathcal{K}$, so there are no superfluous inequalities in the above description [St].

When the Picard number n is 4, an appropriate (Euclidean) cross section of the light cone is a sphere. The ample cone is bounded by hyperplanes, each of which cut out a circle on this sphere. This is one way of describing Figure 1; each circle in the pattern represents one of the hyperplanes that properly bound $\mathcal{K}$. There is, however, a more satisfying interpretation. Each bounding hyperplane of $\mathcal{K}$ intersects the hypersurface $\mathcal{H}$ in a (hyperbolic) plane. When viewed from a point $-P_0$ with P_0 on $\mathcal{H}$, we get the Poincaré sphere representation of $\mathcal{H}$, as is further explained in Section 4. Using this interpretation, each circle in Figure 1 represents a hyperbolic plane, and $\mathcal{K} \cap \mathcal{H}$ is the region in the Poincaré sphere bounded by all these planes. The advantage of this interpretation is that it does not depend on the choice of basis. The different pictures one gets correspond to different choices of P_0, so the object is the same; it is only our perspective that has changed.

It is clear that if $\sigma \in \mathcal{A}$, then $\sigma^*(\mathcal{K}) = \mathcal{K}$, so let us define

$$\mathcal{O}'' = \{T \in \mathcal{O}^+ : T\mathcal{K} = \mathcal{K}\}.$$

If there are any -2 curves on V, then there exists a large subset of $\mathcal{O}^+$ that cannot be in $\mathcal{O}''$. Suppose $C \cdot C = -2$ and C is effective. Let us define the *reflection through* C by

$$R_C D = D + (C \cdot D)C.$$

In the Lorentz space, this is a reflection through the hyperplane $C \cdot \mathbf{x} = 0$, or in the hyperbolic geometry $\mathcal{H}$, it is reflection through the (hyper)line formed by the

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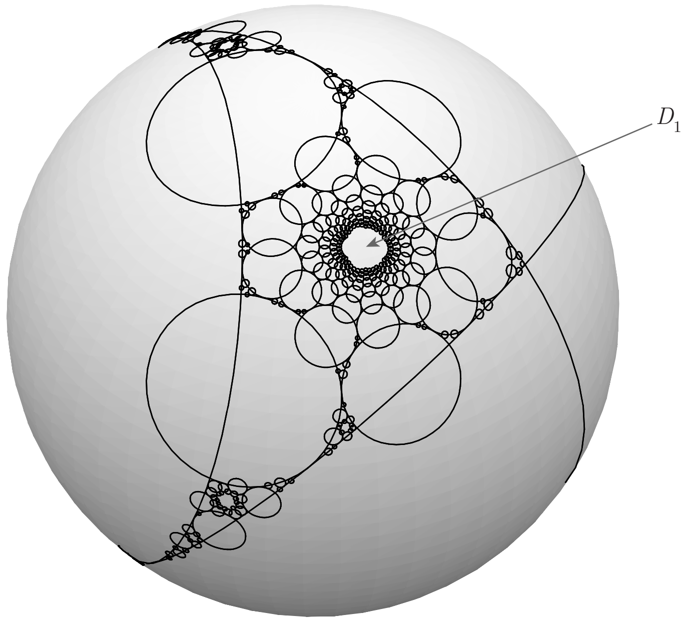

Figure 1: A representation of the ample cone. The point D_1 is an accumulation point of circles (one of infinitely many).

intersection of $\mathcal{H}$ with $C \cdot \mathbf{x} = 0$. Note that R_C is in $\mathcal{O}$, since it preserves intersections. But since $R_C C = -C$, we have that $R_C \notin \mathcal{O}''$.

Since the canonical divisor on V is trivial and the arithmetic genus is one, the Riemann–Roch formula on V is

$$l(D) + l(-D) \geq \tfrac{1}{2} D \cdot D + 2,$$

where $l(D)$ is one more than the dimension of the complete linear system for D. Hence, if $l(D) > 0$, then D is effective. In particular, if $C \cdot C = -2$, then either C or $-C$ is effective. Thus $R_C = R_{-C} \notin \mathcal{O}''$ for all divisors $C \in \operatorname{Pic}(V)$ with $C \cdot C = -2$. Let $\mathcal{O}'$ be the subgroup of $\mathcal{O}^+$ generated by the reflections through -2 curves. Note that $T R_C T^{-1} = R_{TC}$. Hence, $\mathcal{O}'$ is a normal subgroup of $\mathcal{O}^+$. In [PS-S], Pjateckiĭ-Šapiro and Šafarevič show that, for a sufficiently large number field K, the natural map

$$\Phi\colon \operatorname{Aut}(V/K) \to \mathcal{O}'', \quad \sigma \mapsto \sigma^*$$

has a finite kernel and co-kernel, and that $\mathcal{O}'' \cong \mathcal{O}^+/\mathcal{O}'$.

2 $(2,2,2)$ Forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

The results of this paper depend only on the Picard lattice $\operatorname{Pic}(V)$ and apply to any $K3$ surface with the intersection matrix J described below. Yet the result is, in the author's opinion, much more appealing when coupled with a class of surfaces that have the given Picard lattice. In this section, we describe such a class of $K3$ surfaces.

Let V be a surface described by a smooth $(2,2,2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Such a surface is the zero locus of a polynomial in three projective variables that is quadratic in each variable. Let us write

$$F(X,Y,Z) = X_0^2 F_0(Y,Z) + X_0 X_1 F_1(Y,Z) + X_1^2 F_2(Y,Z) = 0, \tag{2.1}$$

where $X = (X_0 : X_1) \in \mathbb{P}^1$, and the $F_i(Y,Z)$ are $(2,2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1$. Since smooth $(2,2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1$ are elliptic curves, V is fibered by elliptic curves in each of the three directions.

Let

$$p_1 : \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1, \quad (X,Y,Z) \mapsto X$$

be projection onto the X-axis, and similarly define p_2 and p_3. Let $D_i = p_i^{-1} H$ be the pullback of a point $H \in \mathbb{P}^1$ for $i = 1, 2, 3$.

2.1 The Generic Case

Generic surfaces in this class have been studied by Wang [W], Billard [Bi], and the author [Ba2]. Explicit examples (defined over $\mathbb{Q}$ and with full dimension in the moduli space) are given in [B-L]. If V is generic, then $\operatorname{Pic}(V) = D_1\mathbb{Z} + D_2\mathbb{Z} + D_3\mathbb{Z}$ so the Picard number is $n = 3$; the intersection matrix with respect to this basis is

$$\begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix};$$

and V contains no -2 curves, so its ample cone is the light cone.

An attractive feature of studying smooth $(2,2,2)$ forms in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is that we get several automorphisms "for free". Let us view (2.1) as a quadratic in X with two roots (say) X and X'. We define the map

$$\sigma_1 : V \to V \quad (X,Y,Z) \mapsto (X', Y, Z).$$

Explicitly,

$$X' = \begin{cases} (F_1(Y,Z)X_1 + F_0(Y,Z)X_0 : -F_0(Y,Z)X_1) & \text{if this is in } \mathbb{P}^1, \\ (F_1(Y,Z)X_0 + F_2(Y,Z)X_1 : -F_2(Y,Z)X_0) & \text{otherwise.} \end{cases}$$

In the generic case, the curves $F_i(Y,Z) = 0$ do not intersect, so σ_1 is defined everywhere. We similarly define σ_2 and σ_3. The action of σ_1 on $\operatorname{Pic}(V)$ is

$$\sigma_1^* = \begin{bmatrix} -1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix},$$

while σ_2^* and σ_3^* are symmetrically defined. The map σ_i^* is reflection across a line in $\mathbb{H}^2$ with endpoints D_j and D_k, where i, j, k is a permutation of 1, 2, and 3. Thus, the fundamental domain for $\langle \sigma_1^*, \sigma_2^*, \sigma_3^* \rangle$ is a triply asymptotic triangle, so by Pjateckiĭ-Šapiro and Šafarevič's result, $\langle \sigma_1, \sigma_2, \sigma_3 \rangle$ has finite index in $\operatorname{Aut}(V/K)$.

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2.2 A Class with Picard Number 4

If there exists a point $(Q,Q') \in \mathbb{P}^1 \times \mathbb{P}^1$ over $\bar{K}$ such that

$$F_0(Q,Q') = F_1(Q,Q') = F_2(Q,Q') = 0,$$

then (X,Q,Q') is a point on V for all $X \in \mathbb{P}^1$. That is, V contains a line parallel to the X-axis. Since this is a smooth curve of genus zero, it must be a -2 curve. If a surface is generic, modulo this condition, then it has Picard number $n = 4$. These are the $K3$ surfaces we study in this paper.

An example of such a $K3$ surface is the surface V given by $F(X,Y,Z) = 0$ where, in affine coordinates,

$$\begin{aligned} &F((x:1),(y:1),(z:1)) \\ &\quad = f(x,y,z) \\ &\quad = x^2(y^2+yz+z^2+z) + x(y^2z^2+y^2z+z) + (y^2z^2+y^2z+y+z) \end{aligned}$$

This surface includes the line $(X,(0:1),(0:1))$, so its Picard number is at least four. The equation $f(x,1/y,z-1)$ is equivalent modulo two to the polynomial that describes the surface Y_2 in [B-L], and so V has Picard number four. We can use this example to find infinitely many. Let

$$G(X,Y,Z) = X_0^2G_0(Y,Z) + X_0X_1G_1(Y,Z) + X_1^2G_2(Y,Z) = 0$$

be a $(2,2,2)$ form with rational coefficients with odd denominators. The curves $G_0(Y,Z) = 0$ and $G_1(Y,Z) = 0$ intersect somewhere, say at $Y = Q$ and $Z = Q'$. Let us suppose that Q and Q' are rational. Though we have imposed some constraints on G, those constraints are of codimension 0 in the moduli space of all $(2,2,2)$ forms. By making a change of basis, we may assume that $Q = Q' = 0$. After this change of basis, let us require that $G_2(Y,Z)$ have no constant term, so $G_2(0,0) = 0$. This is a one-dimensional constraint. The set of $(2,2,2)$ forms given by

$$F(X,Y,Z) + 2G(X,Y,Z) = 0$$

is of codimension one in the set of all $(2,2,2)$ forms, and all have Picard number 4. We have therefore constructed a set of $K3$ surfaces of full dimension in the moduli space of all $(2,2,2)$ forms with Picard number 4.

Let V be a surface in this class. Let us suppose that the line is parallel to the X-axis and denote the divisor class for this line with D_4. Since there is only one curve in this divisor class $(D_4 \cdot D_4 = -2)$, let us abuse notation slightly and represent the line with D_4 too. Note that the fibers in D_2 and D_3 over Q and Q' are each the union of the line D_4 and a conic.

Lemma 2.1 *The set* $\mathcal{D} = \{D_1, D_2, D_3, D_4\}$ *forms a basis of* $\operatorname{Pic}(V)$.

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Proof The set $\mathcal{D}$ is a linearly independent subset of $\mathrm{Pic}(V)$, so generates a sublattice of finite index in $\mathrm{Pic}(V)$. Let this index be m. It is easy to check that

$$D_i \cdot D_j = \begin{cases} 0 & \text{if } i = j \neq 4, \\ 2 & \text{if } i \neq j; i, j = 1, 2, \text{or } 3, \end{cases}$$

$$D_4 \cdot D_i = \begin{cases} 1 & \text{if } i = 1, \\ 0 & \text{if } i = 2,\ 3. \end{cases}$$

Thus, the intersection matrix J for the sublattice generated by $\mathcal{D}$ is

$$J = \begin{bmatrix} 0 & 2 & 2 & 1 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}.$$

Since m^2 divides $\det(J) = -28$, we know $m = 1$ or 2. If $m = 2$, then there exists a nonzero divisor $D' \in \mathrm{Pic}(V)$ that is a linear combination of the elements of $\mathcal{D}$ with coefficients 0 or $1/2$. Since $D' \in \mathrm{Pic}(V)$, we know $D' \cdot D'$ is even and $D' \cdot D_i \in \mathbb{Z}$. Of the fifteen combinations, only two satisfy these conditions, namely $D_2/2$, and $D_3/2$. Since D_2 and D_3 represent classes of elliptic curves, they cannot be reduced as the sum of two elliptic curves. ■

Remark 2.2 Since the lattice with basis $\{D_1, D_2/2, D_3, D_4\}$ is even, we know there must exist a $K3$ surface whose Picard lattice is this lattice [Mo].

3 The Group of Automorphisms

A surface V in our class of $K3$ surfaces includes the automorphisms σ_2 and σ_3, as defined in the generic case. The map σ_1 as described above in the generic case is defined in our case for all points on $V \setminus D_4$. We can extend this map to D_4 as follows. Without loss of generality, we may assume $Q_1 \neq 0$ and $Q_1' \neq 0$ (where $Q = (Q_0 : Q_1)$, $Q' = (Q_0' : Q_1')$). Let $q = Q_0/Q_1$, $q' = Q_0'/Q_1'$, and consider $(y, z) = (Y_0/Y_1, Z_0/Z_1)$ in a neighborhood of (q, q'). Let

$$M_1(z - q') = M_0(y - q),$$

and write F_0, F_1, and F_2 as functions of y and $M = (M_0 : M_1) \in \mathbb{P}^1$. Then,

$$F_i(q, M) = 0$$

for all M and all i, so $(y - q)$ divides $F_i(y, M)$ for all i. Define $G_i(y, M)$ so that

$$(y - q)G_i(y, M) = F_i(y, M).$$

Note that if $G_i(q, M) = 0$ for all i and some M, then V has a singularity. Thus, since V is smooth, the equation

$$X_0^2 G_0(y, M) + X_0 X_1 G_1(y, M) + X_1^2 G_2(y, M) = 0$$

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is a quadratic in X for all y in a neighborhood of q. It is defined on an open subset U of V containing D_4, and, as before, we may define a map σ_1' on U. The two maps σ_1 and σ_1' agree wherever they are both defined, and, piecing them together, we get an automorphism σ_1 of V.

The following describes how σ_i^* acts on $\operatorname{Pic}(V)$.

Theorem 3.1 *Let T_i be the matrix representation of σ_i^* in the basis $\mathcal{D}$. Then*

$$T_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Proof We calculate T_1 by considering the intersection numbers of the basis elements in $\mathcal{D}$ intersected with the elements in the image of $\mathcal{D}$ under σ_1^*. Since σ_1 leaves Y and Z fixed, we know $\sigma_1^* D_2 = D_2$, $\sigma_1^* D_3 = D_3$ and $\sigma_1^* D_4 = D_4$. Thus,

$$\begin{aligned} \sigma_1^* D_1 \cdot D_2 &= D_1 \cdot (\sigma_1^*)^{-1} D_2 = D_1 \cdot \sigma_1^* D_2 = D_1 \cdot D_2 = 2 \\ \sigma_1^* D_2 \cdot D_1 &= 2 \\ &\cdots \end{aligned}$$

Observe that σ_i^2 is the identity, a property that was made use of in the above. The only difficult calculation is $\sigma_1^* D_1 \cdot D_1 = 7$. Recall that D_1 is the divisor class of the curve $F(H, Y, Z) = 0$ for some fixed H. From (2.1) we have

$$H_0^2 F_0(Y, Z) + H_0 H_1 F_1(Y, Z) + H_1^2 F_2(Y, Z) = 0,$$

which is a $(2, 2)$ form in $\mathbb{P}^1 \times \mathbb{P}^1$ (if we think of H as a constant). The divisor D_1 intersects $\sigma_1^* D_1$ if H is a double root of this equation. That is, if

$$2H_0 F_0(Y, Z) + H_1 F_1(Y, Z) = 0,$$

which is another $(2, 2)$ form. These two curves intersect at eight points, but one solution is $(Y, Z) = (Q, Q')$, for which the above argument that H is a double point is not valid. Thus, $\sigma_1^* D_1 \cdot D_1 = 7$. We therefore get

$$T_1^t J = J T_1 = \begin{bmatrix} 7 & 2 & 2 & 1 \\ 2 & 0 & 2 & 0 \\ 2 & 2 & 0 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix},$$

from which we derive T_1.

For T_2, we note that $\sigma_2^* D_1 = D_1$ and $\sigma_2^* D_3 = D_3$, so the only difficult intersections to calculate are $\sigma_2^* D_2 \cdot D_2$, $\sigma_2^* D_2 \cdot D_4$ and $\sigma_2^* D_4 \cdot D_4$. We calculate $\sigma_2^* D_2 \cdot D_2$ in the same way that we calculated the intersection $\sigma_1^* D_1 \cdot D_1$ above, only this time no intersections are discarded, so $\sigma_2^* D_2 \cdot D_2 = 8$.

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We calculate $\sigma_2^* D_2 \cdot D_4$ by noting that the curve $F(X, Y, Q')$ is the union of the line D_4 and the conic $\sigma_2 D_4$. Thus $D_3 = D_4 + \sigma_2^* D_4$. Hence,

$$\sigma_2^* D_2 \cdot D_4 = \sigma_2^* D_2 \cdot D_3 - \sigma_2^* D_2 \cdot \sigma_2^* D_4 = 2 + 0 = 2.$$

We also use this observation to calculate $\sigma_2^* D_4 \cdot D_4$:

$$0 = D_3 \cdot D_3 = (D_4 + \sigma_2^* D_4) \cdot (D_4 + \sigma_2^* D_4) = -2 + 2\sigma_2^* D_4 \cdot D_4 - 2.$$

Thus,

$$T_2^t J = J T_2 = \begin{bmatrix} 0 & 2 & 2 & 1 \\ 2 & 8 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 1 & 2 & 0 & 2 \end{bmatrix},$$

from which we derive T_2. The matrix T_3 is found in a symmetric way. ∎

The elements T_1, T_2, and T_3 are all in $\mathcal{O}^+$ (as they should be) but the group $\langle T_1, T_2, T_3 \rangle$ is not all of $\mathcal{O}^+$. There is, of course, the reflection R_{D_4} through D_4 and the map S that switches D_2 and D_3:

$$R_{D_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}, \qquad S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

There is also another matrix

$$T_4 = \begin{bmatrix} 1 & 8 & 8 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 4 & 4 & 1 \end{bmatrix},$$

which was found by trial and error. Note that $T_3 = S T_2 S$.

Lemma 3.2 *The maps S and T_4 are in $\mathcal{O}''$.*

Proof Any map $T \in \mathcal{O}^+$ can be written as a product $T = T'T''$, where $T' \in \mathcal{O}'$ and $T'' \in \mathcal{O}''$ (by Pjateckiĭ-Šapiro and Šafarevič's result). The map T is in $\mathcal{O}''$ if and only if $T' = \mathrm{I}$. Note that $T'\mathcal{K} \cap \mathcal{K} = \varnothing$ for any $T' \in \mathcal{O}'$ that is not the identity. Thus, to show that $T \in \mathcal{O}''$, it is enough to find an ample D such that TD is also ample.

The divisor $D = D_1 + D_2 + D_3$ is ample, and $SD = D$, so $S \in \mathcal{O}''$.

Since the divisor D_1 represents an elliptic fibration, it is irreducible. Since D_4 represents a -2 curve, it is irreducible. Let $D' = 2D_1 + D_4$. The intersection of D' with its irreducible components are non-negative, so the intersection of D' with any effective divisor is non-negative. Hence, D' is in the closure of the ample cone. Since $D_1 \cdot D_1 = 0$, it is on the boundary of the ample cone. Thus, since the ample cone is convex, the divisor $D'' = D' + D_1 = 3D_1 + D_4$ is either in the ample cone or on the boundary of the ample cone. If it is on the boundary of the ample cone, then there exists a -2 curve whose intersection with D_1 and $2D_1 + D_4$ are both 0. It is not difficult to verify that no such -2 curve exists, so D'' is not on the boundary of the ample cone, and is therefore ample. Finally, $T_4 D'' = D''$, so $T_4 \in \mathcal{O}''$. ∎

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Remark 3.3 There exists an automorphism $\sigma_4 \in \operatorname{Aut}(V)$ such that $\sigma_4^* = T_4$. It can be described as follows. For a point $P \in V$, let E be the elliptic curve in the divisor class D_1 such that $P \in E$. Let O_E be the unique point of intersection between E and D_4. Define $\sigma_4(P) = -P$, where $-P$ is the additive inverse of P with respect to addition on the elliptic curve E with zero at O_E. Then σ_4 is an automorphism on V. For every E in the class D_1, σ_4 fixes E, and since $\sigma_4(O_E) = O_E$, it also fixes D_4. Thus D_1 and D_4 are both eigenvectors of σ_4^*. This is enough to narrow down the possibilities for σ_4^* to T_4, ST_4, S, and the identity. In [B-McK] we show $\sigma_4^* = T_4$.

This also shows that, for the class of $K3$ surfaces described in Section 2, we may take $K = \mathbb{Q}$. That is, $\mathbb{Q}$ is a sufficiently large number field.

4 Visualizing the Group $\mathcal{O}''$

As noted earlier, the hypersurface $\mathcal{H}$ is a three-dimensional hyperbolic space. In this section, we describe how to project $\mathcal{H}$ to the Poincaré ball and upper half plane models of $\mathbb{H}^3$, and through these models, we investigate the tilings generated by the actions of $\mathcal{O}^+$ and $\mathcal{O}''$ on $\mathcal{H}$.

The matrix J has three negative eigenvalues and one positive eigenvalue. Let us denote them with $-a_1^2$, $-a_2^2$, $-a_3^2$, and a_4^2. Let Q be the matrix that diagonalizes J so that $J = -Q^t A^t J_0 A Q$, where

$$A = \begin{bmatrix} a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & 0 & 0 & a_4 \end{bmatrix} \qquad \text{and} \qquad J_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

The surface $\mathcal{V}^+$ described by $\mathbf{y}^t J_0 \mathbf{y} = -1$ with $y_4 > 0$ is the usual Lorentz model of $\mathbb{H}^3$. The map $\mathbf{y} = \pm\lambda AQ\mathbf{x}$ sends $\mathcal{H}$ to $\mathcal{V}^+$, where $\lambda = (\sqrt{D \cdot D})^{-1}$. The stereographic projection of $\mathcal{V}^+$ to the hypersurface $y_4 = 0$ through the point $(0,0,0,-1)$ is an isomorphism of $\mathcal{V}^+$ to the Poincaré ball model of $\mathbb{H}^3$. This projection distinguishes a point, the point P_0 on $\mathcal{H}$ where $P_0 = \lambda^{-1}Q^{-1}A^{-1}(0,0,0,1)$. The point P_0 is the point “closest to the eye” and is mapped to the center of the Poincaré ball. By applying an isometry to $\mathcal{H}$ before projecting, we can make a choice for P_0. Throughout this paper, we make the choice $P_0 = D_1 + D_2 + D_3$, which is an ample divisor.

The Poincaré ball can be unwrapped to give the Poincaré upper half space model for $\mathbb{H}^3$. This map distinguishes a point P_∞ on $\partial\mathcal{H}$, the boundary of $\mathcal{H}$, which is sent to the point at infinity. The point P_∞ can be represented with a point on the light cone $\mathbf{x} \cdot \mathbf{x} = 0$. In the upper half space model (with $z > 0$), planes in $\mathbb{H}^3$ are modeled with planes and half spheres that are perpendicular to the plane $z = 0$. This model is often represented by its boundary, the plane $z = 0$ together with the point at infinity, on which planes in $\mathbb{H}^3$ appear as circles or lines. The rest of the figures in this paper are these types of representations of $\mathbb{H}^3$.

The map T_1 is a reflection through a plane in $\mathcal{H}$. That plane is the intersection of $\mathcal{H}$ with the hyperplane $(2,-2,-2,1) \cdot \mathbf{x} = 0$. When mapped to the plane $z = 0$, as described above with the point $P_\infty = D_1$, its boundary is a circle Γ_1, as shown in Figure 2. The plane includes the points (divisors) D_2 and D_3, which are on the light cone $\mathbf{x} \cdot \mathbf{x} = 0$, which represents $\partial\mathcal{H}$.

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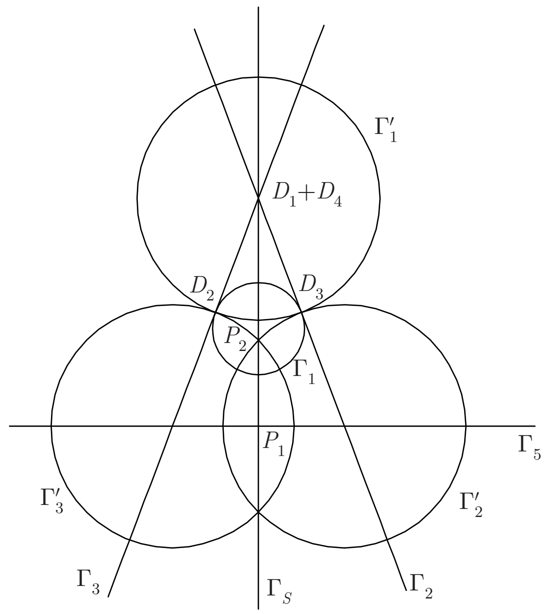

Figure 2: The fundamental domain for $\mathcal{G}(= \mathcal{O}^+)$: The region in $\mathbb{H}^3$ bounded by the planes above $\Gamma_2, \Gamma_5, \Gamma_S$, and above the hemispheres represented by Γ_1, Γ_1', and Γ_2'. The base angles θ of the isosceles triangle satisfy $\cos\theta = \frac{1}{2\sqrt{2}}$.

The map S is also a reflection through a plane in $\mathcal{H}$. That plane is the intersection of $\mathcal{H}$ with the hyperplane $(0, 1, -1, 0) \cdot \mathbf{x} = 0$, and is represented by the line Γ_S in Figure 2. It includes the points D_1, $D_1 + D_4$, and $P_1 = (1, 1, 1, -2)$.

The maps T_2, T_3, and T_4 are each rotations by π about parallel lines (parallel in $\mathcal{H}$), each with one endpoint at D_1 and the other endpoint at D_3, D_2, and $D_1 + D_4$, respectively. The map T_2 rotates everything on one side of the plane represented by Γ_2 to the other side. The map T_3 maps everything from on one side of Γ_3 to the other.

The map R_{D_4} is reflection through the plane represented by Γ_1'. In Figure 2, the planes Γ_2' and Γ_3' are the planes perpendicular to T_2D_4 and T_3D_4, respectively, so represent the reflections $T_3R_{D_4}T_3$ and $T_2R_{D_4}T_2$, respectively.

We also represent, with Γ_5, the plane through which $T_2T_4T_3S = T_2T_4ST_2$ reflects. We can now see enough to prove something.

Let $\mathcal{G} = \langle T_1, T_2, T_4, S, R_{D_4}\rangle$, and let $\mathcal{F}$ be the region in $\mathbb{H}^3$ bounded by the planes represented by Γ_2, Γ_5, and Γ_S in Figure 2, and above the hemispheres represented by Γ_1', Γ_2', and Γ_1.

Lemma 4.1 *The group $\mathcal{G}$ has finite index in $\mathcal{O}^+$.*

Proof The region $\mathcal{F}$ is a fundamental domain for $\mathcal{G}$. Since $\mathcal{G}$ is a subgroup of $\mathcal{O}^+$, a fundamental domain for $\mathcal{O}^+$ lies within $\mathcal{F}$. Since $\mathcal{F}$ has finite volume, the group $\mathcal{G}$ has finite index in $\mathcal{O}^+$. ■

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Since $\mathcal{G}' = \langle T_1, T_2, T_4, S\rangle$ is the largest subgroup of $\mathcal{G}$ that is in $\mathcal{O}''$, we get the following corollary.

Corollary 4.2 *The group $\mathcal{G}'$ has finite index in $\mathcal{O}''$.*

At the end of the next section, we will show that $\mathcal{G} = \mathcal{O}^+$ and $\mathcal{G}' = \mathcal{O}''$.

5 The Ample Cone

In this section, we explain how we generate Figure 1 and why it represents a cross section of the ample cone.

Lemma 5.1 *There are no -2 divisors C such that the plane $C \cdot \mathbf{x} = 0$ properly intersects $\mathcal{F}$.*

Proof Suppose C_1 and C_2 are two -2 divisors. Let us write $C_i = a_{i1}D_1 + \cdots + a_{i4}D_4$. From the calculation $C_i \cdot C_i = -2$, we find $a_{i1} \equiv 0 \pmod 2$, from which it follows that $C_1 \cdot C_2 \equiv 0 \pmod 2$. If the planes $C_1 \cdot \mathbf{x} = 0$ and $C_2 \cdot \mathbf{x} = 0$ intersect, then the angle θ of intersection is given by $-2\cos\theta = C_1 \cdot C_2$. Hence, the planes are either coincident (*i.e.,* $C_1 = C_2$), perpendicular ($C_1 \cdot C_2 = 0$), tangent ($C_1 \cdot C_2 = 2$), or do not intersect ($C_1 \cdot C_2 > 2$).

Now suppose there exists a -2 divisor C such that the plane $C \cdot \mathbf{x} = 0$ properly intersects the fundamental domain $\mathcal{F}$. If it goes through D_1, then it is either tangent to Γ_1', in which case it does not intersect Γ_3' perpendicularly, or it is perpendicular to Γ_1', in which case it must be Γ_2, since it must also be perpendicular to Γ_2'. But no normal to Γ_2 with integer coefficients has self intersection -2. Thus, the plane given by C cannot go through D_1, so it is represented by a circle. If D_2 or D_3 is inside this circle, then either $C \cdot D_1 < 0$, $C \cdot D_2 < 0$, or $C \cdot D_3 < 0$, which is a contradiction, since D_1, D_2, and D_3 represent elliptic curves. If the plane $C \cdot \mathbf{x} = 0$ does not intersect either Γ_1' or Γ_2', or if it intersects only one of the two, then it lies entirely in Γ_1, and so does not intersect $\mathcal{F}$. Finally, if it intersects both Γ_1' and Γ_2' perpendicularly, then solving we discover the coefficient a_4 of D_4 is even, from which we conclude $C \cdot C \equiv 0 \pmod 4$. Thus no such C exists. ■

Corollary 5.2 *The fundamental domain $\mathcal{F}$ is a subset of the closure of the ample cone.*

Proof We first note that $D = D_1 + D_2 + D_3$ is ample. It is not difficult to check that D is on the face of $\mathcal{F}$ given by the plane represented by Γ_S in Figure 2. Suppose there exists a divisor D' in $\mathcal{F}$ that is not ample, and not on either of the faces represented by Γ_1' and Γ_2'. Since $\mathcal{F}$ is convex, the line segment joining D and D' lies entirely within $\mathcal{F}$. This line segment must cross the boundary of the ample cone, so must intersect a plane given by $C \cdot \mathbf{x} = 0$ where $C \cdot C = -2$. By the previous lemma, no such C exists, so D' cannot exist. ■

Thus the set $\{x \in T\mathcal{F} : T \in \mathcal{G}'\}$, shown in Figure 3, is contained in the ample cone. Since the boundary of this set is the union of planes generated by -2 curves, this set is exactly the intersection of the ample cone with $\mathcal{H}$. Wrapped up into the Poincaré ball model, we get Figure 1. The same region is shown again in Figure 4 with D_3 at infinity.

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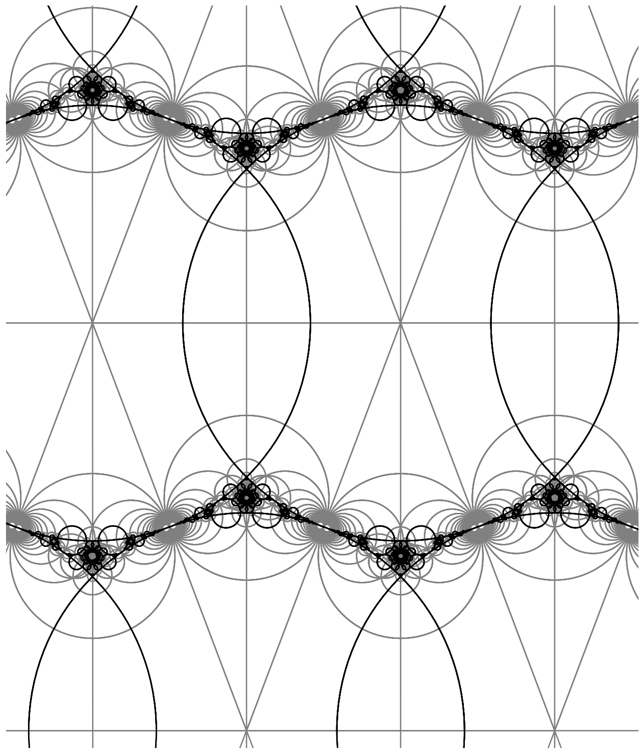

Figure 3: A hyperbolic cross section of the ample cone – the region in $\mathbb{H}^3$ above the hemispheres represented by all the heavy black circles. The set of heavy black circles is the image of Γ_1' under the action of $\mathcal{G}'$. The light lines represent the the action of $\mathcal{G}'$. The point at infinity is D_1.

We close this section with the following result, which we include for completeness.

Lemma 5.3 *The groups $\mathcal{G}$ and $\mathcal{G}'$ are, in fact, all of $\mathcal{O}^+$ and $\mathcal{O}''$, respectively.*

Proof We know $\mathcal{G} \subset \mathcal{O}^+$. Suppose $T \in \mathcal{O}^+$. There exists a $T' \in \mathcal{O}^+$ such that $T'TD_1 \in \mathcal{F}$. Since D_1 is a cuspidal point whose intersection with any -2 curve is non-zero, we know $T'TD_1 = D_1$. The region in Figure 2 bounded by Γ_2, Γ_5, and Γ_S tiles $\mathbb{H}^3$ under the action of $\langle T_2, T_4, S\rangle$. This is clear, since T_2 is rotation by π about the line D_1D_3; S is reflection through Γ_S, and $T_2T_4ST_2$ is a reflection through Γ_5. Thus there exists a $T'' \in \langle T_2, T_4, S\rangle$ such that $T''D_1 = D_1$, and $T''T'TD_3 \in \mathcal{F}$. Since D_3 is cuspidal and $D_3 \cdot D_4 = 0$, we know $T''T'TD_3 = D_3$. Let $\tau = T''T'T$. Then D_1 and D_3 are eigenvectors for τ. Let λ_1 and λ_3 be their associated eigenvalues, respectively.

Consider $\tau^k(cD_1 + cD_3) = \lambda_1^k cD_1 + \lambda_3^k cD_3$, where c is chosen so that $cD_1 + cD_3$ is on $\mathcal{H}$. Since τ is an isometry, these images are all on $\mathcal{H}$, and on the line given by the intersection of $\mathcal{H}$ with the plane spanned by D_1 and D_3. Since τ maps $\mathcal{H}$ to $\mathcal{H}$, neither eigenvalue can be negative. Thus, either both eigenvalues are 1, or we get

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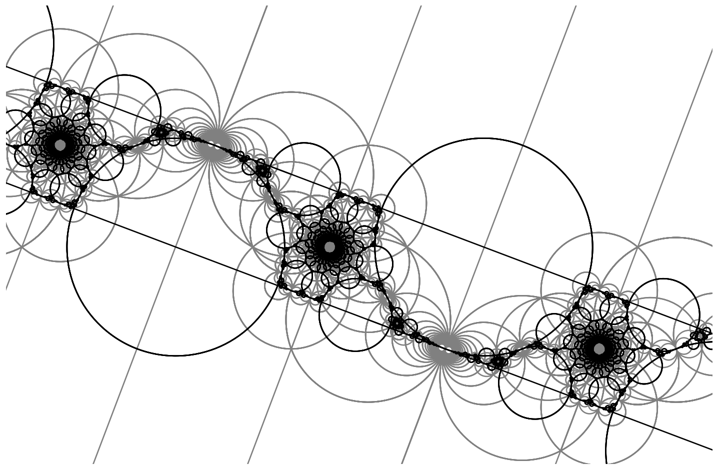

Figure 4: A different perspective of a hyperbolic cross section of the ample cone, the region between the two planes above the two dark parallel (Euclidean) lines and above the hemispheres represented by each dark circle. The point at infinity is D_3.

an infinite number of points in $\mathcal{F}$ under iterations of τ, which contradicts $\mathcal{G}$ having finite index in $\mathcal{O}^+$. Thus, both D_1 and D_3 are fixed by τ. Hence, either τ is a rotation with axis D_1D_3, or is reflection through a plane that includes the line D_1D_3.

If τ is a rotation, then it must send Γ_1' to either itself or Γ_2', so τ is either the identity, or rotation by π. The latter is T_2. In either case, τ, and hence T, is in $\mathcal{G}$.

If τ is reflection through a plane that includes D_1D_3, then by considering the possible image of Γ_1', we conclude τ is either reflection through the plane above Γ_2, or reflection through the plane perpendicular to the plane above Γ_2. Neither of these reflections are in $\mathcal{O}^+$, since both have fractional entries in their matrix representations. Thus, τ is neither of these.

Hence, $T \in \mathcal{G}$, so $\mathcal{O}^+ = \mathcal{G}$. Since $\mathcal{G}' \subset \mathcal{O}''$, and $\mathcal{G}'$ is the largest subgroup of $\mathcal{G}$ that does not contain any elements of $\mathcal{O}'$, we have $\mathcal{G}' = \mathcal{O}''$. ■

Note that the set $\mathcal{K} \cap \mathcal{H}$ is a fundamental domain for $\mathcal{O}'$, as expected. One can also surmise, from Figure 2 and a little thought, that the group $\mathcal{O}''$ is the free group on four elements T_1, T_2, T_4, and S, modulo the relations $T_i^2 = 1$, $S^2 = 1$, and $T_2T_4T_2ST_2ST_4ST_2S = 1$.

6 The Fractal

Let $\partial\mathcal{H}$ be the points at infinity in the usual compactification of $\mathcal{H}$. Figures 1, 3, and 4 suggest that there exists a fractal on $\partial\mathcal{H}$ associated with the ample cone for V.

Given a plane $\mathcal{P}$ in $\mathcal{H}$, let us write $x \overset{\mathcal{P}}{\sim} y$ if x and y are on the same side of $\mathcal{P}$, where

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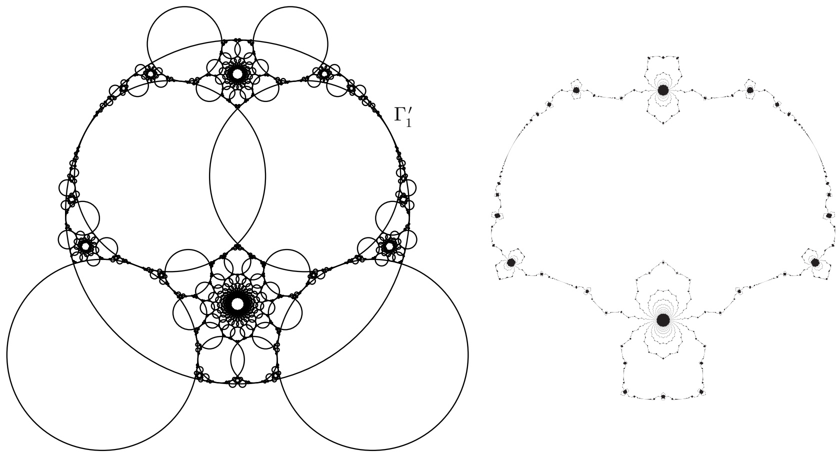

Figure 5: (a) The fractal associated with the ample cone, with the point $D_1 + D_2 + D_3 + 3D_4$ at infinity. (b) The associated fractal. The dark portions are regions with denser fractal dust.

x and y are any two points in the compactification of $\mathcal{H}$ not on $\mathcal{P}$. Let us define

$$\Lambda_{\text{res}} = \left\{ x \in \partial\mathcal{H} \;:\; \begin{array}{l} \text{for any plane } \mathcal{P} \in H \text{ that does not contain } x, \\ \text{there exists } y \in \mathcal{K} \text{ such that } x \overset{\mathcal{P}}{\sim} y \end{array} \right\}.$$

The plane $\mathcal{P}$ should be thought of as defining an open neighborhood of x on $\partial\mathcal{H}$. By results due to Kovács [Kov], Λ_{res} cannot include any proper open subset of $\partial\mathcal{H}$.

To better visualize this fractal Λ_{res}, let us consider Figure 5(a), which is another perspective of the hyperbolic cross section of $\mathcal{K}$, this time with the point $D_1 + D_2 + D_3 + 3D_4$ at infinity. The fractal is the set of points on $\partial\mathcal{H}$ inside the disc represented by Γ_1', and outside all other discs. This set is shown in Figure 5(b).

The set Λ_{res} is reminiscent of the residual set of the Apollonian packing (see [L-M-W]), and so ample cones of $K3$ surfaces generate a rich variation of the Apollonian packing that is quite different from the many variations considered in *Indra's Pearls* [M-S-W].

There is another less obvious fractal associated with the surface V, namely the *limit set* of $\mathcal{O}''$, which is

$$\Lambda(\mathcal{O}'') = \left\{ x \in \partial\mathcal{H} \;:\; \begin{array}{l} \text{for any plane } \mathcal{P} \in \mathcal{H} \text{ that does not contain } x, \\ \text{there exists } T \in \mathcal{O}'' \text{ such that } x \overset{\mathcal{P}}{\sim} T(x_0) \end{array} \right\}.$$

The point x_0 is a fixed point in $\mathcal{H}$. The limit set $\Lambda(O'')$, though, does not depend on the choice of x_0. We introduce the limit set because of the attention it has received in the literature.

Theorem 6.1 *The residual set Λ_{res} and the limit set $\Lambda(\mathcal{O}'')$ are equal.*

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We will prove this result with the help of the following rather interesting finiteness result.

Lemma 6.2 *Let y be a point in the usual compactification of $\mathcal{H}$. Let*

$$\mathcal{S} = \{C \in \mathcal{E}_{-2} : C \cdot y < 0\}.$$

Then $|\mathcal{S}| \leq 3$.

Proof Suppose there exist four elements, $C_1, \dots, C_4$ in $\mathcal{S}$. Suppose two planes $C_j \cdot x = 0$ and $C_k \cdot x = 0$ do not intersect. Then the set of points x such that $C_i \cdot x > 0$ for $i = j$ or k is described by just one of these inequalities. That is, one of these planes is not a face of $\mathcal{K}$, which contradicts the observation made by Sterk [St] mentioned in Section 1. Thus these planes intersect, and by the observation made in the proof of Lemma 5.1, are perpendicular. In the Poincaré upper half space model of $\mathcal{H}$, each plane is represented by a circle (with a suitable choice for the point at infinity). Fix one of these circles and think of it as representing a Poincaré disc model of $\mathbb{H}^2$. Since the other three circles intersect our distinguished circle perpendicularly, they represent lines in $\mathbb{H}^2$, and therefore form a triangle whose angle sum exceeds π, a contradiction. Thus, $|\mathcal{S}| \leq 3$. ■

Proof of Theorem 6.1 If we choose $x_0 \in \mathcal{K}$, then $\mathcal{O}''(x_0) \subset \mathcal{K}$, so $\Lambda(\mathcal{O}'') \subset \Lambda_{\text{res}}$.

Suppose now that $y \in \Lambda_{\text{res}}$. Let us fix a Poincaré upper half space model of $\mathcal{H}$ with y not at infinity, and let U be a neighborhood of y. We wish to show that there exists a $T \in \mathcal{O}''$ such that $T\Gamma_1' \subset U$. If such a T exists for every U, then y is an accumulation point for $\mathcal{O}''(x_0)$ for any x_0 on the plane $D_4 \cdot x = 0$, so $y \in \Lambda(\mathcal{O}'')$.

Suppose then that there does not exist a $T \in \mathcal{O}''$ such that $T\Gamma_1'$ is contained in U. Let r be the Euclidean radius of a disc centered at y contained in U (in our fixed Poincaré model). Let U_1 and U_2 be discs centered at y with (Euclidean) radii $r/2$ and $r/4$, respectively. If there exists a $T \in \mathcal{O}''$ such that the center of $T\Gamma_1'$ lies in U_1, then its radius must be less than $r/2$, since it cannot cover y. But then $T\Gamma_1'$ is contained in U, which contradicts our assumption. Consider the set of discs $T\Gamma_1'$ that intersect U_2. Each such disc must have radius greater than $r/4$, so must cover an area in U_1 of at least $1/3\pi(r/4)^2$. By the previous lemma, each point in U_1 is covered by at most three such discs, so there can be only a finite number of such discs that intersect U_2. Thus, y is an isolated point in Λ_{res}, so must be at the intersection of at least two of these discs. This means y is the image under some T of the point P_2 of intersection of Γ_2' and Γ_3' (see Figure 2). This point of intersection is $P_2 = D1 - \alpha D_2 - \alpha D_3 + (\alpha - 1)D_4$ where $\alpha = 1 + \sqrt{7}/2$, and is the eigenvector of $T_1T_2T_4T_3$ with eigenvalue $2 + 6\alpha$. Hence P_2 is in $\Lambda(O'')$, so y is in $\Lambda(O'')$ too. ■

7 Orbits of Curves

These fractals are related to the lattice point problem in hyperbolic geometry. Given a Kleinian group $\mathcal{G}$ of $\mathbb{H}^3$ (a discrete group of isometries of $\mathbb{H}^3$), and points $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{H}^3$, let $n(s, \mathbf{x}, \mathbf{y})$ be the number of points in the $\mathcal{G}$ orbit of $\mathbf{x}$ that are in the ball of radius s centered at $\mathbf{y}$. That is,

$$n(s, \mathbf{x}, \mathbf{y}) = \#\{\gamma\mathbf{x} : \gamma \in \mathcal{G}, d(\gamma\mathbf{x}, \mathbf{y}) < s\}.$$

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Here $d(*,*)$ is the hyperbolic metric. Determining the behavior of $n(s,\mathbf{x},\mathbf{y})$, particularly its asymptotics, is the lattice point problem. When $C\cdot C>0$, our quantity $N_{\mathcal{A}(C)}(t,D)$ is just a lattice point problem. In fact, with $\mathcal{G}=\mathcal{O}''$, we have

$$N_{\mathcal{A}(C)}(t,D)=kn\left(\cosh^{-1}\left(\frac{t}{\sqrt{C\cdot C}\sqrt{D\cdot D}}\right),\frac{C}{\sqrt{C\cdot C}},\frac{D}{\sqrt{D\cdot D}}\right),$$

where the constant k depends on the size of the kernel and cokernel of the natural map Φ from $\mathcal{A}$ to $\mathcal{O}''$.

For an arbitrary Kleinian group $\mathcal{G}$, we can also define the limit set

$$\Lambda(\mathcal{G})=\left\{x\in\partial\mathbb{H}^3\ :\ \begin{array}{l}\text{for any plane } \mathcal{P}\in\mathbb{H}^3 \text{ that does not contain } x,\\ \text{there exists } \gamma\in\mathcal{G} \text{ such that } x\overset{\mathcal{P}}{\sim}\gamma(x_0)\end{array}\right\},$$

for any fixed $x_0\in\mathbb{H}^3$.

Sullivan [Su] showed that if the group $\mathcal{G}$ is geometrically finite (i.e. the boundary of a fundamental domain is a finite number of planes), then the quantity

$$\alpha(\mathcal{G})=\limsup_{s\to\infty}\frac{\log n(s,\mathbf{x},\mathbf{y})}{s} \tag{7.1}$$

and $\mathrm{H.dim}(\Lambda(\mathcal{G}))$, the Hausdorff dimension of $\Lambda(\mathcal{G})$, are equal.

Lax and Phillips [L-P] related $n(s,\mathbf{x},\mathbf{y})$ to the minimal eigenvalue λ_0 of the Laplacian on the hyperbolic three-fold $\mathcal{M}=\mathcal{G}\backslash\mathbb{H}^3$. They showed that if $\mathcal{G}$ is geometrically finite, then

$$n(s,\mathbf{x},\mathbf{y})=k_1e^{\alpha(\mathcal{G})s}+O(e^{\beta(\mathcal{G})s}),$$

where the constant k_1 depends on $\mathcal{M}$,

$$\alpha(\mathcal{G})(2-\alpha(\mathcal{G}))=\lambda_0,$$

and $\beta(\mathcal{G})<\alpha(\mathcal{G})$. This shows that we may replace the limit supremum in (7.1) with the limit.

Combining these results, we get

$$\alpha=\mathrm{H.dim}(\Lambda(\mathcal{O}''))=\mathrm{H.dim}(\Lambda_{\mathrm{res}}).$$

8 The Dimension

In this section, we empirically estimate $\mathrm{H.dim}(\Lambda_{\mathrm{res}})$. We do this by estimating $\alpha(\mathcal{G})$ (see (7.1)) for $\mathcal{G}=\langle T_1,T_2,T_3,T_5\rangle$, where $T_5=T_2T_4T_3$. The map T_5 is a rotation by π about the line P_1D_1 (see Figure 2). The fundamental domain $\mathcal{F}_{\mathcal{G}}$ for $\mathcal{G}$ is the region bounded by the planes represented by Γ_2, Γ_3, and Γ_5, and above the hemisphere represented by Γ_1. Since $\mathcal{G}$ has index two in $\mathcal{O}''$, we know $\alpha(\mathcal{G})=\mathrm{H.dim}(\Lambda(\mathcal{G}))=\mathrm{H.dim}(\Lambda(\mathcal{O}''))=\mathrm{H.dim}(\Lambda_{\mathrm{res}})$.

To efficiently estimate $\alpha(\mathcal{G})$, we choose $\mathbf{y}$ carefully and identify a descent on $\mathcal{G}$-orbits. We choose $\mathbf{y}=(1,2,2,-1)$, which is in $\mathcal{K}$, so is ample. Let $\mathbf{y}_i$ be the

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s	$n(s, \mathbf{y}, \mathbf{y})$	$\log(n(s, \mathbf{y}, \mathbf{y}))$	s	$n(s, \mathbf{y}, \mathbf{y})$	$\log(n(s, \mathbf{y}, \mathbf{y}))$
6.5	2795	7.936	9.5	136429	11.824
7	5381	8.591	10	259600	12.467
7.5	10225	9.233	10.5	498266	13.119
8	19547	9.881	11	954663	13.769
8.5	37173	10.523	11.5	1820845	14.415
9	71435	11.177	12	3484566	15.064

Table 1: The number of points in a $\mathcal{G}$-orbit of $\mathbf{y} = (1, 2, 2, -1)$ that is a distance at most s away from $\mathbf{y}$.

reflection of $\mathbf{y}$ through the (hyperbolic) plane represented by Γ_i for $i = 1, 2, 3$, and 5. Then $T_i(\mathbf{y}) = \mathbf{y}_i$. That is, the line segment $\mathbf{y}\mathbf{y}_i$ has the property that it is perpendicular to the axes of rotation for each of T_2, T_3, and T_5. In particular, the set of points equidistant from both $\mathbf{y}$ and $\mathbf{y}_i$ is the plane represented by Γ_i. Because of this property, we know

$$d(T_i\mathbf{x}, \mathbf{y}) = d(\mathbf{x}, T_i\mathbf{y}) = d(\mathbf{x}, \mathbf{y}_i)$$

for $i = 1, 2, 3$, and 5, and if $\mathbf{x}$ and $\mathbf{y}$ are on opposite sides of the plane represented by Γ_i, then $\mathbf{x}$ and $\mathbf{y}_i$ are on the same side, so $d(T_i\mathbf{x}, \mathbf{y}) < d(\mathbf{x}, \mathbf{y})$. Note that for any $\mathbf{x} \in \mathcal{H}$ such that $\mathbf{x}$ is not in $\mathcal{F}_\mathcal{G}$, there exists at least one plane represented by Γ_i for $i = 1, 2, 3$, or 5 such that $\mathbf{x}$ and $\mathbf{y}$ are on opposite sides of this plane. Note too that if $\mathbf{x}$ has integer entries, then $T\mathbf{x} \cdot \mathbf{y}$ is a non-negative integer for all $T \in \mathcal{G}$, and $T_i\mathbf{x} \cdot \mathbf{y} < \mathbf{x} \cdot \mathbf{y}$, so descent must terminate. Thus, for any $\mathbf{x} \in \mathcal{H}$, there exists a finite sequence of points $\mathbf{x}_0, \ldots, \mathbf{x}_k$ such that $\mathbf{x}_0 = \mathbf{x}$, $\mathbf{x}_{j+1} = T_i\mathbf{x}_j$ for some i, $d(\mathbf{x}_{j+1}, \mathbf{y}) < d(\mathbf{x}_j, \mathbf{y})$, and $\mathbf{x}_k \in \mathcal{F}_\mathcal{G}$. This gives us a method of descent. Reversing this, we get an efficient algorithm for passing through all points in an orbit. We view the orbit as a tree, pruning branches that descend and pruning when we reach a node we have been at before. Implementing this algorithm gives us the data in Table 1, and a least squares fit to the data (for s vs. $\log(n(s, \mathbf{y}, \mathbf{y}))$ with $s > 7$) gives us the approximation $\alpha \approx 1.296 \pm .010$.

The error estimate is based on calculating the slope of secants to the curve s vs. $\log(n(s, \mathbf{y}, \mathbf{y}))$ for $s > 7$. All slopes lie within the given range. This latter technique for estimating Hausdorff dimension was tested in [P-S] for a couple of known examples and was found to be fairly accurate.

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References

[Ba1] A. Baragar, *Orbits of curves on certain K3 surfaces.* Compositio Math. **137**(2003), no. 2, 115–134. doi:10.1023/A:1023960725003

[Ba2] ———, *Rational points on K3 surfaces in* $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Math. Ann. **305**(1996), no. 3, 541–558. doi:10.1007/BF01444236

[Ba3] ———, *Fractals and eigenvalue of the Laplacian on certain noncompact surfaces.* Experiment. Math. **15**(2006), no. 1, 33–42.

https://doi.org/10.4153/CJM-2011-006-7 Published online by Cambridge University Press

[B-L] A. Baragar and R. van Luijk, *$K3$ surfaces with Picard number three and canonical heights.* Math. Comp. **76**(2007), no. 259, 1493–1498. doi:10.1090/S0025-5718-07-01962-X

[B-McK] A. Baragar and D. McKinnon, *$K3$ surfaces, rational curves, and rational points.* J. Number Theory **130**(2010), no. 7, 1470–1479. doi:10.1016/j.jnt.2010.02.014

[Bi] H. Billard, *Propriétés arithmétiques d'une famille de surfaces $K3$.* Compositio Math. **108**(1997), no. 3, 247–275. doi:10.1023/A:1000128300511

[C] S. Cantat, *Dynamique des automorphismes des surfaces $K3$.* Acta Math. **187**(2001), no. 1, 1–57. doi:10.1007/BF02392831

[L-M-W] J. C. Lagarias, C. L. Mallows, and A. R. Wilks, *Beyond the Descartes circle theorem.* Amer. Math. Monthly **109**(2002), no. 4, 338–361. doi:10.2307/2695498

[K-K] J. Keum and S. Kondō, *The automorphism groups of Kummer surfaces associated with the product of two elliptic curves.* Trans. Amer. Math. Soc. **353**(2001), no. 4, 1469–1487. doi:10.1090/S0002-9947-00-02631-3

[Kon] S. Kondō, *The automorphism group of a generic Jacobian Kummer surface.* J. Algebraic Geom. **7**(1998), no. 3, 589–609.

[Kov] S. Kovács, *The cone of curves of a $K3$ surface.* Math. Ann. **300**(1994), no. 4, 681–691. doi:10.1007/BF01450509

[L-P] P. D. Lax and R. S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces.* J. Funct. Anal. **46**(1982), no. 3, 280–350. doi:10.1016/0022-1236(82)90050-7

[McM] C. T. McMullen, *Dynamics on $K3$ surfaces: Salem numbers and Siegel disks.* J. Reine Angew. Math. **545**(2002), 201–233.

[M-S-W] D. Mumford, C. Series, and D. Wright, *Indra's pearls. The vision of Felix Klein.* Cambridge University Press, New York, 2002.

[Mo] D. R. Morrison, *On $K3$ surfaces with large Picard number.* Invent. Math. **75**(1984), no. 1, 105–121. doi:10.1007/BF01403093

[N1] V. V. Nikulin, *$K3$ surfaces with a finite number of automorphisms and a Picard group of rank three, Algebraic geometry and its applications.* (Russian) Trudy Mat. Inst. Steklov. **165**(1984), 119–142.

[N2] ———, *Quotient-groups of groups of automorphisms of hyperbolic forms of subgroups generated by 2-reflections.* (Russian) Dokl. Akad. Nauk SSSR **248**(1979), no. 6, 1307–1309.

[N3] ———, *Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications.* (Russian) In: Current problems in mathematics, 18, Akad. Nauk SSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981, pp. 3–114.

[P-S] R. Phillips and P. Sarnak, *The Laplacian for domains in hyperbolic space and limit sets of Kleinian groups.* Acta. Math. **155**(1985), no. 3–4, 173–241. doi:10.1007/BF02392542

[PS-S] I. I. Pjateckiĭ-Šapiro and I. R. Šafarevič, *Torelli's theorem for algebraic surfaces of type $K3$.* (Russian) Nauk SSSR Ser. Mat. **35** (1971), 530–572.

[R] J. S. Ratcliff, *Foundations of hyperbolic manifolds.* Springer-Verlag, New York, 1994.

[St] H. Sterk,*Finiteness results for algebraic $K3$ surfaces.* Math. Z. **189**(1985), no. 4, 507–513. doi:10.1007/BF01168156

[Su] D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups.* Acta Math. **153**(1984), no. 3–4, 259–277. doi:10.1007/BF02392379

[V] É. B. Vinberg, *The two most algebraic $K3$ surfaces.* Math. Ann. **265**(1983), no. 1, 1–21. doi:10.1007/BF01456933

[W] L. Wang, *Rational points and canonical heights on $K3$-surfaces in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$*. In: Recent developments in the inverse Galois problem (Seattle, WA, 1993), Contemp. Math., 186, American Mathematical Society, Providence, RI, 1995, pp. 273–289.

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