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HOMOMORPHISM-COMPACT SPACES

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1. Introduction. In 1979 Edgar asked for a characterization of those completely regular Hausdorff topological spaces X which have the property that any Boolean σ-homomorphism from the Baire σ-field of X into the measure algebra of an arbitrary complete probability space can be realized by a measurable point-mapping. Those spaces X will be called homomorphism-compact or, for short, H-compact hereafter. It is well-known that compact spaces are H-compact (cf. [**4**], p. 637, Proposition 3.4). We will show that the same is true for strongly measure compact spaces. On the other hand H-compact spaces are easily seen to be real-compact. Since the notions of measure-compactness and lifting-compactness (cf. [**3**]) also lie between strong measure-compactness and real-compactness it is natural to investigate the relations among these notions. Here the results are mainly negative (cf. Sections 4 and 6). Concerning the structural properties of H-compactness not very much can be said so far (cf. Section 7): it is, for instance, unknown whether the product of two H-compact spaces is again H-compact.

For our considerations the notion of H-compact measures turns out to be particularly useful:

A finite Baire measure μ on a completely regular space X is said to be *H-compact* if any μ-measure preserving σ-homomorphism from the Baire σ-field of X to the measure algebra of a complete finite measure space can be realized by a measurable point-mapping.

There are some nice characterizations of H-compact measures: A measure μ on X is H-compact if and only if (X, μ) is injective in the category of measure spaces, if and only if μ is inner regular with respect to the closed sets of a weaker compact topology on X. For metrizable spaces X H-compactness of a measure μ is equivalent to the tightness of μ (cf. Section 3). Moreover H-compact measures on metrizable spaces have the disintegration property, while, on the other hand, a measure, which lives on a measure compact space and has the disintegration property, is H-compact.

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The results proved in this note leave many natural questions unanswered. We therefore include a number of open problems whose solution would give some further insight into the structure of H-compact spaces.

2. Notation and terminology. Throughout this paper X will be a completely regular Hausdorff space, $C_b(X)$ the algebra of all bounded real-valued functions on X, $\mathscr{B}(X)$ the σ-field of Baire subsets of X (i.e., the smallest σ-field with respect to which every $h \in C_b(X)$ is measurable) and $P(X)$ the set of all σ-additive probability measures on $\mathscr{B}(X)$. For $h \in C_b(X)$ we denote by $\tilde{h}$ the canonical extension of h to the Stone-Čech compactification of X. For $\mu \in P(X)$ we denote by $\tilde{\mu}$ the Baire measure on βX induced via

$$\int_X h d\mu = \int_X \tilde{h} d\tilde{\mu} \quad (h \in C_b(X))$$

and by $\tilde{\tilde{\mu}}$ the Radon measure extending $\tilde{\mu}$ to all Borel sets of βX.

Let $(\Omega, \mathfrak{A}, \nu)$ be a probability space. Then $\mathfrak{A}_\nu$ denotes the completion of $\mathfrak{A}$ w. r. t. ν while $\mathfrak{A}/_\nu$ stands for the measure algebra of $(\Omega, \mathfrak{A}, \nu)$, i.e., the quotient algebra of $\mathfrak{A}$ w. r. t. the σ-ideal of ν-nullsets. The measure induced by ν on $\mathfrak{A}/_\nu$ is again denoted by ν.

By ν^* (resp. ν_*) we denote the outer (resp. inner) measure on $\mathfrak{P}(\Omega)$ corresponding to ν. For $\Omega' \subset \Omega$ we write $\mathfrak{A} \cap \Omega'$ for the σ-field $\{A \cap \Omega' \mid A \in \mathfrak{A}\}$ on Ω'.

For the notions not defined in this paper we refer the reader to [**3**], [**12**], and [**14**].

3. H-compact measures. In this section the notion of H-compact measure is introduced and studied in some detail.

Definition. Let X be a completely regular space and $\mu \in P(X)$. Then μ is called *H-compact* if and only if for every probability space $(\Omega, \mathfrak{A}, \nu)$ and any Boolean σ-homomorphism $\Phi: \mathscr{B}(X) \to \mathfrak{A}/_\nu$ with $\nu \circ \Phi = \mu$ there exists an $\mathfrak{A}_\nu - \mathscr{B}(X)$-measurable map $\varphi: \Omega \to X$ such that for all $B \in \mathscr{B}(X)$ we have $\varphi^{-1}(B) \in \Phi(B)$.

Due to the following proposition there is a large class of H-compact measures:

3.1. PROPOSITION. *Every tight measure is H-compact.*

Proof. The proof consists in a slight modification of the arguments used by Edgar [**4**], p. 637/638 and Graf [**8**], p. 68/69.

For our further considerations we need the following definitions:

Definitions. (i) Let $(\Omega, \mathfrak{A}, \nu)$ and $(\Omega', \mathfrak{A}', \nu')$ be measure spaces. A map j: $\Omega \to \Omega'$ is called an *embedding* if and only if j is one-to-one, $\mathfrak{A} - \mathfrak{A}'$-measurable and satisfies

$$j(\nu) = \nu' \quad \text{and} \quad \{j^{-1}(A)|A \in \mathfrak{A}'\} = \mathfrak{A}.$$

(ii) Let $(\Omega, \mathfrak{A}, \nu)$ and $(\Omega', \mathfrak{A}', \nu')$ be measure spaces and let j: $\Omega \to \Omega'$ be an embedding. An $\mathfrak{A}'_{\nu'} - \mathfrak{A}$-measurable map r: $\Omega' \to \Omega$ is called a *retraction* for j if and only if $r \circ j = \mathrm{id}_\Omega$.

(iii) A measure space $(\Omega, \mathfrak{A}, \nu)$ is called an *absolute retract* if and only if every embedding of $(\Omega, \mathfrak{A}, \nu)$ into another measure space admits a retraction.

(iv) A measure space $(\Omega, \mathfrak{A}, \nu)$ is called *injective* if and only if for every two measure spaces $(\Omega_1, \mathfrak{A}_1, \nu_1)$ and $(\Omega_2, \mathfrak{A}_2, \nu_2)$, every embedding j: $\Omega_1 \to \Omega_2$, and every $\mathfrak{A}_1 - \mathfrak{A}$-measurable map φ: $\Omega_1 \to \Omega$ with $\varphi(\nu_1) = \nu$ there exists an $(\mathfrak{A}_2)_{\nu_2} - \mathfrak{A}$-measurable map $\bar{\varphi}$: $\Omega_2 \to \Omega$ with $\bar{\varphi} \circ j = \varphi$.

(v) Given a measure space $(\Omega, \mathfrak{A}, \nu)$ and a collection $\mathscr{H}$ of subsets of X, ν is called *inner regular* w. r. t. $\mathscr{H}$ if and only if, for every $A \in \mathfrak{A}$, we have

$$\nu(A) = \sup\{\nu^*(K): K \in \mathscr{H}, K \subset A\}.$$

Our next theorem gives a characterization of H-compact measures, thereby relating the notions introduced above.

3.2. THEOREM. *For a completely regular space X and a measure $\mu \in P(X)$ the following properties are equivalent:*

(i) *μ is H-compact.*

(ii) *$(X, \mathscr{B}(X), \mu)$ is injective.*

(iii) *$(X, \mathscr{B}(X), \mu)$ is an absolute retract.*

(iv) *The canonical embedding of $(X, \mathscr{B}(X), \mu)$ into $(\beta X, \mathscr{B}(\beta X), \tilde{\mu})$ admits a retraction.*

(v) *There exists a $(\mathscr{B}(\beta X))_{\tilde{\mu}} - \mathscr{B}(X)$-measurable map φ: $\beta X \to X$ such that*

$$\tilde{\mu}(\varphi^{-1}(B) \,\Delta\, B) = 0 \quad \textit{for all } B \in \mathscr{B}(\beta X).$$

(vi) *There exists a compact topology τ_μ on X which is weaker than the original topology of X and such that μ is inner regular with respect to the collection of all τ_μ-closed subsets of X which are at the same time zero sets of the original topology.*

Proof. (i) $\Rightarrow$ (ii): Let $\mu \in P(X)$ be H-compact, $(\Omega, \mathfrak{A}, \nu)$ and $(\Omega', \mathfrak{A}', \nu')$ measure spaces, $j: \Omega \to \Omega'$ an embedding and $\varphi: \Omega \to X$ $\mathfrak{A} - \mathscr{B}(X)$-measurable with $\varphi(\nu) = \mu$. Define $\Phi: \mathscr{B}(X) \to \mathfrak{A}'/_{\nu'}$ by $\Phi(B) = [A]_{\nu'}$, where $A \in \mathfrak{A}'$ is any set with $j^{-1}(A) = \varphi^{-1}(B)$ and $[A]_{\nu'}$ denotes the residual class of A in $\mathfrak{A}'/_{\nu'}$. Then Φ is a well-defined Boolean σ-homomorphism with $\nu' \circ \Phi = \mu$. Since μ is H-compact there exists an $\mathfrak{A}'_{\nu'} - \mathscr{B}(X)$-measurable map $\varphi': \Omega' \to X$ such that $(\varphi')^{-1}(B) \in \Phi(B)$ for all $B \in \mathscr{B}(X)$. Define $\bar{\varphi}: \Omega' \to X$ by

$$\bar{\varphi}(\omega) = \begin{cases} \varphi(j^{-1}(\omega)), & \omega \in j(\Omega) \\ \varphi'(\omega), & \omega \in \Omega' \setminus j(\Omega). \end{cases}$$

It remains to show that $\bar{\varphi}$ is $\mathfrak{A}'_{\nu'} - \mathscr{B}(X)$-measurable. Let $B \in \mathscr{B}(X)$ be arbitrary and $A \in \mathfrak{A}'$ with $j^{-1}(A) = \varphi^{-1}(B)$. Then we have

$$A \cap (\varphi')^{-1}(B) \subset \bar{\varphi}^{-1}(B) \subset A \cup (\varphi')^{-1}(B)$$

and, according to the definition of Φ,

$$\nu'(A \,\Delta\, (\varphi')^{-1}(B)) = 0.$$

Together this implies

$$\bar{\varphi}^{-1}(B) \in \mathfrak{A}'_{\nu'}.$$

(ii) $\Rightarrow$ (iii): This implication follows by a standard argument of category theory.

(iii) $\Rightarrow$ (iv) is true by specialization.

(iv) $\Rightarrow$ (v): Let $j: X \to \beta X$ be the canonical embedding and $\varphi: \beta X \to X$ $\mathscr{B}(\beta X)_{\tilde{\mu}} - \mathscr{B}(X)$-measurable with $\varphi \circ j = \mathrm{id}_X$. Let $B \in \mathscr{B}(\beta X)$ be given. Because

$$j^{-1}(\varphi^{-1}(j^{-1}(B))) = j^{-1}(B)$$

we have

$$\begin{aligned} \tilde{\mu}(B \,\Delta\, \varphi^{-1}(B)) &= \tilde{\mu}(B \,\Delta\, \varphi^{-1}(j^{-1}(B))) \\ &= \mu(j^{-1}(B) \,\Delta\, j^{-1} \varphi^{-1}(j^{-1}(B))) = 0. \end{aligned}$$

(v) $\Rightarrow$ (vi): Let $\varphi: \beta X \to X$ be as in (v). Define

$$\tau_\mu := \{U \subset X \mid \exists\, V \subset \beta X: V \text{ open}, V \cap X = U \text{ and } \varphi^{-1}(V) \subset V\}.$$

Then τ_μ is obviously a topology on X which is weaker than the original topology of X. Let $(U_i)_{i \in I}$ be a covering of X by τ_μ-open sets. For $i \in I$ let $V_i \subset \beta X$ be open with

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$$U_i = V_i \cap X \quad \text{and} \quad \varphi^{-1}(V_i) \subset V_i.$$

Then $(V_i)_{i \in I}$ is an open covering of βX and, therefore, has a finite subfamily covering βX. The corresponding U_i obviously cover X. Hence (X, τ_μ) is compact. Let

$$\mathscr{H} = \{F \subset X \mid F \quad \tau_\mu\text{-closed and } F \text{ zero set}\}.$$

We will show that μ is inner regular with respect to $\mathscr{H}$. To this end let $B \in \mathscr{B}(X)$ and $\epsilon > 0$ be arbitrary. Let $\widetilde{B} \in \mathscr{B}(\beta X)$ be such that $\widetilde{B} \cap X = B$. Since $\widetilde{\mu}$ is a Baire measure on a compact space there exists a closed G_δ-set $F_1 \subset \widetilde{B}$ with

$$\widetilde{\mu}(\widetilde{B} \setminus F_1) < \epsilon/2.$$

Since

$$\widetilde{\mu}(F_1 \,\Delta\, \varphi^{-1}(F_1)) = 0$$

there exists a closed G_δ-set $F_2 \subset F_1 \cap \varphi^{-1}(F_1)$ with

$$\widetilde{\mu}(F_1 \setminus F_2) < \epsilon/4.$$

Continuing in the same way we construct a sequence $(F_n)_{n \in \mathbf{N}}$ of closed G_δ-sets with

$$F_{n+1} \subset F_n \cap \varphi^{-1}(F_n) \text{ and } \widetilde{\mu}(F_n \setminus F_{n+1}) < \epsilon/2^{n+1} \quad \text{for all } n \in \mathbf{N}.$$

Define

$$F := \bigcap_{n \in \mathbf{N}} F_n.$$

Then F is a closed G_δ-set and, therefore, $F \cap X$ is a zero set. We have

$$F = \bigcap_{n \in \mathbf{N}} F_n \subset \bigcap_{n \in \mathbf{N}} F_{n+1} \cap \varphi^{-1}(F_{n+1}) \subset \varphi^{-1}(F).$$

Hence $F \cap X$ is τ_μ-closed. Moreover

$$\mu(B \setminus (F \cap X)) = \widetilde{\mu}(\widetilde{B} \setminus F) \leqq \widetilde{\mu}(\widetilde{B} \setminus F_1) + \widetilde{\mu}(F_1 \setminus F_2) + \ldots < \epsilon.$$

Thus μ is inner regular w. r. t. $\mathscr{H}$ and (vi) is proved.

(vi) $\Rightarrow$ (i): Let τ_μ be a topology with the properties stated in (vi). Let $\mathscr{H}$ be defined as above. Let $(\Omega, \mathfrak{A}, \nu)$ be a probability space, $\Phi: \mathscr{B}(X) \to \mathfrak{A}/_\nu$ a σ-homomorphism with $\mu = \nu \circ \Phi$, and $\Theta: \mathfrak{A}/_\nu \to \mathfrak{A}_\nu$ a lifting (cf. [**9**], p. 36). For $\omega \in \Omega$ define

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$$\mathscr{H}_\omega = \{K \in \mathscr{H} | \omega \in \Theta(\Phi(K))\}.$$

Then $\mathscr{H}_\omega$ is non-empty, stable under finite intersections, and does not contain the empty set. Since (X, τ_μ) is compact this implies $\cap \mathscr{H}_\omega \neq \emptyset$.

Define $\varphi: \Omega \to X$ by $\varphi(\omega) \in \cap \mathscr{H}_\omega$. For $K \in \mathscr{H}$ and $\omega \in \Omega$

$$\varphi(\omega) \notin K \text{ implies } K \notin \mathscr{H}_\omega$$

and, hence, $\omega \notin \Theta(\Phi(K))$. We therefore deduce

$$\Theta(\Phi(K)) \subset \varphi^{-1}(K).$$

Now let $B \in \mathscr{B}_\mu(X)$ be arbitrary. Then there exist increasing sequences $(K_n)_{n \in \mathbf{N}}$ and $(L_n)_{n \in \mathbf{N}}$ in $\mathscr{H}$ such that

$$\bigcup_{n \in \mathbf{N}} K_n \subset B \quad \text{and} \quad \mu(B \setminus \bigcup_{n \in \mathbf{N}} K_n) = 0;$$

and

$$\bigcup_{n \in \mathbf{N}} L_n \subset X \setminus B \quad \text{and} \quad \mu((X \setminus B) \setminus \bigcup_{n \in \mathbf{N}} L_n) = 0.$$

This leads to

$$\begin{aligned} \bigcup_n \Theta(\Phi(K_n)) \subset \varphi^{-1}(\bigcup_n K_n) \subset \varphi^{-1}(B) = \Omega \setminus \varphi^{-1}(X \setminus B) \\ \subset \Omega \setminus \varphi^{-1}(\bigcup_n L_n) \subset \Omega \setminus \bigcup_n \Theta(\Phi(L_n)). \end{aligned}$$

Since

$$\Phi(\cup K_n) = \Phi(B) = \Phi(X \setminus \bigcup_n L_n),$$

$$[\bigcup_n \Theta(\Phi(K_n))]_\nu = \Phi(\bigcup_n K_n)$$

and

$$[\Omega \setminus \bigcup_n \Theta(\Phi(L_n)]_\nu = \Phi(X \setminus \bigcup_n \mathrm{L_n})$$

we derive

$$\varphi^{-1}(B) \in \mathfrak{A}_\nu$$

and

$$[\varphi^{-1}(B)]_\nu = \Phi(B).$$

Remarks. a) The topology τ_μ in (vi) can be chosen in such a way that all compact subsets of X are τ_μ-closed.

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b) The idea of constructing a topology τ_μ with the properties stated in (vi) originates in an analysis of Pachl's considerations concerning disintegration of measures (see [**13**], pp. 159-161).

3.3. THEOREM. *Let X be metrizable and $\mu \in P(X)$. Then μ is H-compact if and only if μ is tight.*

Proof. The "if" part of the theorem is a special case of Proposition 3.1. To prove the "only if" part let $\mu \in P(X)$ be H-compact. According to Theorem 3.2 (v) there exists a $\mathscr{B}(\beta X)_{\tilde{\mu}} - \mathscr{B}(X)$-measurable map $\varphi: \beta X \rightarrow X$ with

$$\tilde{\mu}(B \,\Delta\, \varphi^{-1}(B)) = 0 \quad \text{for all } B \in \mathscr{B}(X).$$

By a theorem of Fremlin [**7**] φ is Lusin measurable and, therefore, $\mu = \varphi(\tilde{\mu})$ is tight.

4. *H*-compact spaces. This section contains partial answers to Edgar's question stated in the introduction. His problem led us to the following

Definition. A completely regular Hausdorff space is called *H-compact* if and only if every measure $\mu \in P(X)$ is H-compact.

Our next theorem is an immediate corollary of Proposition 3.1.

4.1. THEOREM. *Every strongly measure-compact space is H-compact.*

4.2 THEOREM. *Every H-compact space X is real-compact.*

Proof. Let $\mu \in P(X)$ be $\{0, 1\}$-valued. Let Ω be a singleton, $\mathfrak{A} = \mathfrak{P}(\Omega)$, ν the unique probability measure on $\mathfrak{A}$ and $\Phi: \mathscr{B}(X) \rightarrow \mathfrak{A}/_\nu = \mathfrak{A}$ defined by

$$\Phi(B) = \begin{cases} \Omega, & \mu(B) = 1 \\ \emptyset, & \mu(B) = 0. \end{cases}$$

Then Φ is a σ-homomorphism with $\nu \circ \Phi = \mu$. Since μ is H-compact there exists a map $\varphi: \Omega \rightarrow X$ with $\varphi^{-1}(B) = \Phi(B)$ for $B \in \mathscr{B}(X)$. Thus, for $x_o \in \varphi(\Omega)$ and $B \in \mathscr{B}(X)$, we have

$$x_o \in B \Leftrightarrow \mu(B) = 1,$$

which leads to $\mu = \epsilon_{x_o}$. Hence X is real-compact.

As an immediate consequence of Theorem 3.3 we have

4.3. THEOREM. *A metrizable space is H-compact if and only if it is strongly measure compact.*

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The following examples show that neither of the notions of H-compactness and measure-compactness implies the other. Thus the converses of Theorems 4.1 and 4.2 are, in general, false.

4.4. *Examples*. a) X H-compact does not imply X measure compact. Let

$$X = \{0\} \times [0, 1] \cup \left\{\left(\frac{1}{n}, \frac{k}{n^2}\right) \mid k, n \in \mathbf{N}\right\}.$$

For $x \in X \setminus \{0\} \times [0, 1]$ let $\{x\}$ be a neighborhood of x, and for each $(0, y) \in \{0\} \times [0, 1]$ define a neighborhood base $(B_n(y))_{n \in \mathbf{N}}$ by

$$B_n(y) = \{(u, v) \in X \mid u \leqq 1/n \text{ and } |v-y| \leqq u\}.$$

This defines a completely regular topology τ on X which is finer than the topology inherited from $\mathbf{R}^2$ but possesses the same Baire sets. Since X with the euclidean topology is compact it follows from Theorem 3.1 that (X, τ) is H-compact. However, (X, τ) is not measure compact ([**1**], p. 681).

b) X measure compact does not imply X H-compact.

Let X be a non-Lebesgue measurable subset of $[0, 1]$. Then X is separable and metrizable and hence measure-compact. But X is not universally (Radon) measurable and, therefore, not strongly measure-compact ([**15**]). Since for metrizable spaces H-compactness and strong measure-compactness are equivalent properties, this proves our claim.

5. H-compactness and disintegration. The close formal relationship between the realization of homomorphisms and the disintegration of measures suggested the investigations concerning disintegration and H-compactness carried out in this section.

Definition. Let X be completely regular and $\mu \in P(X)$.

a) μ is said to have the *disintegration property* (DP) provided that for any probability space $(\Omega, \mathfrak{A}, \nu)$ and for any $\mathscr{B}(X) - \mathfrak{A}$-measurable map $q: X \to \Omega$ with $\nu = q(\mu)$ there exists a disintegration of μ w. r. t. q, i.e., there is a family $(\mu_\omega)_{\omega \in \Omega}$ in $P(X)$ with the following properties:

(i) $\forall\, B \in \mathscr{B}(X): \Omega \to \mathbf{R};\ \omega \to \mu_\omega(B)$ is $\mathfrak{A}_\nu$-measurable

(ii) $\forall\, B \in \mathscr{B}(X)\ \forall A \in \mathfrak{A}: \mu(B \cap q^{-1}(A)) = \int_A \mu_\omega(B)\, d\nu(\omega).$

b) μ is said to have the *integral representation property* (IRP) provided that for every probability space $(\Omega, \mathfrak{A}, \nu)$ and for every positive linear operator

$$T: C_b(X) \to \mathrm{L}^{\infty}(\Omega, \mathfrak{A}, \nu)$$

with $T1 = 1$ and

$$\int Tfd\nu = \int fd\mu \quad \text{for all } f \in C_b(X)$$

there exists a family $(\mu_\omega)_{\omega \in \Omega}$ in $P(X)$ such that $\Omega \to \mathbf{R}$; $\omega \to \int f\, d\mu_\omega$ is $\mathfrak{A}_\nu$-measurable and is a representative of the residual class Tf in $L^\infty (\Omega, \mathfrak{A}, \nu)$.

5.1. PROPOSITION. *If* $\mu \in P(X)$ *has the* IRP *then it has the* DP.

Proof. Let $(\Omega, \mathfrak{A}, \nu)$ be a probability space and q: $X \to \Omega$ a $\mathscr{B}(X)$ – $\mathfrak{A}$-measurable map with $q(\mu) = \nu$. For $h \in C_b(X)$ define ν_h: $\mathfrak{A} \to \mathbf{R}$ by

$$\nu_h(A) = \int_{q^{-1}(A)} h\, d\mu.$$

Then ν_h is a signed measure on $\mathfrak{A}$ with

$$-\|h\|_\infty\, \nu \leqq \nu_h \leqq \|h\|_\infty\, \nu.$$

Hence the Radon-Nikodym theorem implies the existence of a unique $f_h \in L^\infty(\Omega, \mathfrak{A}, \nu)$ with $\nu_h(A) = \int_A f_h\, d\nu$ for all $A \in \mathfrak{A}$. The map

$$T: C_b(X) \to L^\infty (\Omega, \mathfrak{A}, \nu);\ h \to f_h,$$

is obviously a positive linear operator with $T1 = 1$ and

$$\int Thd\nu = \int hd\mu \quad \text{for all } h \in C_b(X).$$

Since μ has the IRP, there exists a family $(\mu_\omega)_{\omega \in \Omega}$ in $P(X)$ such that $\omega \to \int h\, d\mu_\omega$ represents Th for all $h \in C_b(X)$. By standard arguments one sees that $(\mu_\omega)_{\omega \in \Omega}$ is a disintegration of μ w. r. t. q.

Problem. Does the converse of the above theorem also hold?

That "nice" measures satisfy the IRP is the content of the following proposition:

5.2. PROPOSITION. *Every tight measure has the* IRP, *hence the* DP.

Proof. Let $\mu \in P(X)$ be tight, $(\Omega, \mathfrak{A}, \nu)$ a probability space, and

$$T: C_b(X) \to L^\infty(\Omega, \mathfrak{A}, \nu)$$

a positive linear operator with $T1 = 1$ and

$$\int Thd\nu = \int hd\nu \quad \text{for all } h \in C_b(X).$$

Let Θ: $L^\infty(\Omega, \mathfrak{A}, \nu) \to M^\infty(\Omega, \mathfrak{A}, \nu)$ be a lifting (cf. [**9**]). Define ρ_ω: $C_b(X) \to \mathbf{R}$ by

$$\rho_\omega(f) = \Theta(Tf)(\omega).$$

Then ρ_ω is a positive linear functional.

Since μ is tight there exists an increasing sequence $(K_n)_{n\in\mathbf{N}}$ of compact subsets of X with

$$\mu_*(X \setminus \bigcup_{n\in\mathbf{N}} K_n) = 0.$$

We claim that there exists a ν-nullset N with

$$(1) \quad \inf_{n\in\mathbf{N}} \Theta\,(\sup\{Tf \mid f \in C_b(X),\ 0 \leqq f \leqq 1_{X\setminus K_n}\})(\omega) = 0$$

for all $\omega \in \Omega \setminus N$. Since

$$\int \Theta(\sup\{Tf \mid f \in C_b(X),\ 0 \leqq f \leqq 1_{X\setminus K_n}\})(\omega)\, d\nu(\omega) = \mu_*(X \setminus K_n)$$

we deduce

$$0 = \inf_{n\in\mathbf{N}} \int \Theta\,(\sup\{Tf \mid f \in C_b(X),\ 0 \leqq f \leqq 1_{X\setminus K_n}\})(\omega) d\nu(\omega)$$

$$= \int \inf_{n\in\mathbf{N}} \Theta\,(\sup\{Tf \mid f \in C_b(X),\ 0 \leqq f \leqq 1_{X\setminus K_n}\})(\omega)\, d\nu(\omega),$$

and the claim follows.

It is easy to verify that for any $h \in C_b(X)$ and any closed subset $F \subset X$ the inequality

$$(2) \quad \Theta(Th) \leqq \sup_{x\in F} |h(x)| + \|h\|_\infty\, \Theta\,(\sup\{Tf \mid f \in C_b(X),\ 0 \leqq f \leqq 1_{X\setminus F}\})$$

holds.

We now show that for every $\omega \in \Omega \setminus N$ and any sequence $(h_k)_{k\in\mathbf{N}}$ in $C_b(X)$ with $h_k \downarrow 0$ we have $\rho_\omega(h_k) \downarrow 0$.

Let $\epsilon > 0$ be arbitrary. Then, according to (1), there exists an $n \in \mathbf{N}$ with

$$(3) \quad \epsilon/2 > \|h_1\|_\infty\, \Theta(\sup\{Tf \mid f \in C_b(X),\ 0 \leqq f \leqq 1_{X\setminus K_n}\})(\omega).$$

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By Dini's theorem $(h_k)_k$ converges to 0 uniformly on K_n, i.e., there is a $k_o \in \mathbf{N}$ with $h_{k|K_n} \leqq \epsilon/2$ for all $k \geqq k_o$. Hence (2) combined with (3) implies

$$Th_k \leqq \epsilon/2 + \epsilon/2 = \epsilon$$

for all $k \geqq k_o$, and, therefore,

$$\rho_\omega(h_k) = \Theta(Th_k)(\omega) \leqq \epsilon$$

for all $k \geqq k_o$. It follows that there is a Baire measure $\mu_\omega \in P(X)$ with

$$\rho_\omega(h) = \int h d\mu_\omega \quad \text{for all } h \in C_b(X).$$

Choosing $\mu_\omega \in P(X)$ arbitrary for $\omega \in N$ gives us the family $(\mu_\omega)_{\omega \in \Omega}$ we are looking for.

Remark. The proof shows that the theorem remains true if one assumes μ to be only pseudotight, i.e., inner regular w. r. t. the collection of pseudo compact subsets of X (cf. [**2**]).

We now relate H-compactness to the above notions.

5.3. THEOREM. *A measure $\mu \in P(X)$ is H-compact if and only if μ has a disintegration $(\mu_\omega)_{\omega \in \beta X}$ with respect to the canonical embedding j: $X \to \beta X$, such that every μ_ω is τ-smooth.*

Proof. " $\Rightarrow$ ": According to Theorem 3.2 (v) there exists a $(\mathscr{B}(\beta X))_{\tilde{\mu}} - \mathscr{B}(X)$-measurable map φ: $\beta X \to X$ with

$$\tilde{\mu}(B \,\Delta\, \varphi^{-1}(B)) = 0 \quad \text{for all } B \in \mathscr{B}(\beta X).$$

Then $\beta X \to P(X)$, $\omega \to \epsilon_{\varphi(\omega)}$ is obviously a disintegration of μ w. r. t. j with the required properties.

" $\Leftarrow$ ": Let $(\mu_\omega)_{\omega \in \beta X}$ be a disintegration of μ w. r. t. j such that all μ_ω are τ-smooth. Define

$$\tilde{\mathscr{F}} = \{F \subset \beta X \mid F \text{ closed } G_\delta\text{-set and } \mu_\omega(F \cap X) = 1 \text{ for all } \omega \in F\}.$$

Then $\mathscr{F} := \{F \cap X \mid F \in \tilde{\mathscr{F}}\}$ is a collection of zero sets in X, which contains $\emptyset$, X and is stable with respect to forming finite unions and intersections. We show that μ is inner regular w. r. t. $\mathscr{F}$. Let $\epsilon > 0$ and $B \in \mathscr{B}(X)$ be given and let $\tilde{B} \in \mathscr{B}(\beta X)$ be such that $B = X \cap \tilde{B}$. Then there exists a closed G_δ-set F_1 in βX with $F_1 \subset \tilde{B}$ and $\tilde{\mu}(\tilde{B} \setminus F_1) < \epsilon/2$. The set

$$H_1 := \{\omega \in \beta X \mid \mu_\omega(F_1 \cap X) = 1\}$$

is in $(\mathscr{B}(X))_{\tilde{\mu}}$ and for all $A \in \mathscr{B}(\beta X)$ we have

$$\tilde{\mu}(A \cap F_1) = \mu(A \cap F_1 \cap X) = \int_A \mu_\omega(F_1 \cap X)\, d\tilde{\mu}(\omega).$$

This implies

$$\tilde{\mu}(F_1 \,\Delta\, H_1) = 0.$$

There is a closed G_δ-subset F_2 of βX with

$$F_2 \subset F_1 \cap H_1 \quad \text{and} \quad \tilde{\mu}(F_1 \setminus F_2) < \epsilon/2^2.$$

Continuing in the same way we get a sequence $(F_n)_{n \in \mathbf{N}}$ of closed G_δ-subsets of βX with

$$F_{n+1} \subset \{\omega \in F_n : \mu_\omega(F_n \cap X) = 1\}$$

and

$$\tilde{\mu}(F_n \setminus F_{n+1}) < \epsilon/2^{n+1}.$$

Define $F = \cap_{n \in \mathbf{N}} F_n$. Then F is a closed G_δ-subset of βX and for $\omega \in F$ we have

$$\mu_\omega(F) = \inf_{n \in \mathbf{N}} \mu_\omega(F_n) = 1,$$

because ω lies in all F_{n+1}; hence $F \in \tilde{\mathscr{F}}$. Moreover we deduce

$$\mu(B \setminus (F \cap X)) = \tilde{\mu}(\tilde{B} \setminus F) = \tilde{\mu}(\tilde{B} \setminus F_1) + \tilde{\mu}(F_1 \setminus F_2) + \ldots \leqq \epsilon.$$

Thus μ is inner regular w. r. t. $\mathscr{F}$.

Now let τ be the topology on X generated by $\{X \setminus F \mid F \in \mathscr{F}\}$. Then τ is obviously weaker than the original topology of X. Let $(U_i)_{i \in I}$ be a τ-open covering of X. Then there exists a family $(F_j)_{j \in J}$ in $\tilde{\mathscr{F}}$ such that $X \setminus F_j \subset U_{i_j}$ for some $i_j \in I$ and

$$\bigcup_{j \in J} X \setminus F_j = X.$$

This implies

$$X \cap \bigcap_{j \in J} F_j = \emptyset.$$

Assume there is an $\omega \in \cap_{j \in J} F_j$. Then we have on the one hand

$$\mu_\omega\Big(\bigcap_{j \in J_o} F_j\Big) = 1 \quad \text{for all finite } J_o \subset J.$$

Since μ_ω is τ-smooth we get, on the other hand,

$$\lim_{\substack{J_o \subset J \\ J_o \text{ finite}}} \mu_\omega\Big(\bigcap_{j \in J_o} F_j\Big) = 0,$$

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a contradiction. Thus $\cap_{j \in J} F_j$ is empty and there exists, therefore, a finite $J_o \subset J$ with

$$\bigcap_{j \in J_o} F_j = \phi.$$

But then the corresponding U_{i_j} form a finite covering of X. Thus (X, τ) is compact and according to Theorem 3.2 (vi) μ is H-compact.

As a consequence of the above theorem we have:

5.4. THEOREM. *If X is a measure-compact space then every measure $\mu \in P(X)$ with the* DP *is H-compact.*

Problems. a) Does the converse of Theorem 5.4 hold?

b) Can the assumption of "measure-compactness" in the theorem be weakened to "real-compactness"?

As a corollary we get the following characterization of H-compact metrizable spaces:

5.5. COROLLARY. *Let X be metrizable. Then the following properties of X are equivalent*:

(i) *X is H-compact.*

(ii) *X is real-compact and has the* IRP.

(iii) *X is real-compact and has the* DP.

(iv) *X is strongly measure compact.*

Proof. The equivalence of (i) and (iv) is the statement of Theorem 4.3 and it, therefore, follows from Propositions 5.1 and 5.2 together with Theorem 4.2 that (i) implies (ii) and (ii) implies (iii). Since every metrizable real-compact space is measure compact (cf. [**11**], [**12**]) the implication (iii) $\Rightarrow$ (i) follows from Theorem 5.4.

Problem. Are the equivalences (i) to (iii) in the above theorem still true if one drops the assumption of metrizability for the space X?

Remark. It should be noted that H-compactness of a measure $\mu \in P(X)$ implies the following weak forms of the integral representation and the disintegration property:

(WIRP): There exists a σ-field $\mathfrak{B} \subset \mathscr{B}(X)$ with $\mathscr{B}(X) \subset \mathfrak{B}_\mu$ such that for all probability spaces $(\Omega, \mathfrak{A}, \nu)$ and all positive linear operators $T: C_b(X) \to L^\infty(\Omega, \mathfrak{A}, \nu)$ with $T1 = 1$ and

$$\int Tf d\nu = \int f d\mu \quad (f \in C_b(X))$$

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there exists a family $(\mu_\omega)_{\omega\in\Omega}$ of probability measures on $\mathfrak{B}$ such that, for each $f \in C_b(X)$, one has a μ-nullset $N \in \mathfrak{B}$ for which $f \cdot 1_{X\setminus N}$ is $\mathfrak{B}$-measurable and $\omega \to \int_{X\setminus N} f\, d\mu_\omega$ is a representative of Tf.

(WDP): There exists a σ-field $\mathfrak{B} \subset \mathscr{B}(X)$ with $\mathscr{B}(\mathrm{X}) \subset \mathfrak{B}_\mu$ such that for all probability spaces $(\Omega, \mathfrak{A}, \nu)$ and every $\mathscr{B}(X)$-$\mathfrak{A}$-measurable map $q: X \to \Omega$ with $\nu = q(\mu)$ there exists a family $(\mu_\omega)_{\omega\in\Omega}$ of probability measures on $\mathfrak{B}$ such that $\Omega \to \mathbf{R}$, $\omega \to \mu_\omega(B)$ is $\mathfrak{A}_\nu$-measurable and satisfies

$$\mu(B\cap q^{-1}(A)) = \int_A \mu_\omega(B)d\nu(\omega) \quad \text{for all } B \in \mathfrak{B},\ A \in \mathfrak{A}.$$

The proof of these facts uses techniques similar to those of [**13**]. Using these techniques it can also be seen that the WIRP and the WDP of measures are equivalent.

6. H-compactness and lifting-compactness. Let us first recall the definition of lifting-compactness for completely regular spaces:

Definition. A completely regular space X is called *lifting-compact* if for any complete probability space $(\Omega, \mathfrak{A}, \nu)$, an $\mathfrak{A}$-$\mathscr{B}(X)$-measurable map φ: $\Omega \to X$, and any lifting

$$\Theta: L^\infty(\Omega, \mathfrak{A}, \nu) \to M^\infty(\Omega, \mathfrak{A}, \nu)$$

there exists a set $\Omega_0 \in \mathfrak{A}$ with $\nu(\Omega_0) = 1$ and a $\mathfrak{A} \cap \Omega_0 - \mathscr{B}(X)$-measurable map $\Theta(\varphi)$: $\Omega_0 \to X$ such that

$$\Theta(h \circ \varphi)(\omega) = (h \circ \Theta'(\varphi))(\omega)$$

for all $h \in C_b(X)$ and $\omega \in \Omega_0$.

It is known that every strongly measure-compact space is lifting-compact and that every lifting-compact space is measure-compact (cf. [**3**]). In this section we study the relation between lifting-compactness and H-compactness. We have the following negative results:

6.1. *Examples.* a) X H-compact and measure-compact does not imply X lifting-compact.

Edgar and Talagrand ([**5**], p. 347, Example) construct a completely regular topology τ' on $X = [0, 1]$, whose Baire sets agree with the usual Baire sets of $[0, 1]$, and which is such that (X, τ') is measure compact but not lifting-compact. Since $[0, 1]$ with the usual topology is H-compact the same is, therefore, true for (X, τ').

b) X lifting-compact does not imply X H-compact.

Since every separable metric space is lifting-compact Example 4.4 b) provides the required counterexample.

One question which remains is whether H-compactness and lifting-compactness together imply strong measure-compactness. Our next result gives a partial answer to this question:

6.2. THEOREM. *Let X be a completely regular Hausdorff space such that for every τ-smooth measure $\mu \in P(X)$ the Borel measure $\tilde{\tilde{\mu}}$ on βX has an almost strong lifting* (*cf.* [**9**], p. 129). *Then X is strongly measure-compact if and only if X is H-compact and lifting-compact.*

Proof. It follows from Theorem 4.1 and [**3**], Corollary 6.1 that every strongly measure-compact space is H-compact and lifting-compact. Now suppose that X is H-compact and lifting-compact. Then according to [**3**], Corollary 6.1, X is measure compact. To prove that X is strongly measure-compact it is enough to show that every τ-smooth measure $\mu \in P(X)$ is tight. Let μ be such a measure. By Theorem 3.2 (v) there is a map φ: $\beta X \to X$ which is $\mathscr{B}(\beta X)_{\tilde{\tilde{\mu}}} - \mathscr{B}(X)$-measurable and satisfies

$$\tilde{\mu}(\varphi^{-1}(A) \,\Delta\, A) = \tilde{\tilde{\mu}}(\varphi^{-1}(A) \,\Delta\, A) = 0 \quad \text{for all } A \in \mathscr{B}(\beta X).$$

This implies $h \circ \varphi(\omega) = \tilde{h}(\omega)$ for $\tilde{\tilde{\mu}}$ − a. a. $\omega \in \beta X$ for all $h \in C_b(X)$. Let $\mathfrak{A}$ be the completion of the Borel field of βX w. r. t. $\tilde{\tilde{\mu}}$. According to our assumptions there is an almost strong lifting

$$\Theta\colon L^{\infty}(\beta X, \mathfrak{A}, \tilde{\tilde{\mu}}) \to M^{\infty}(\beta X, \mathfrak{A}, \tilde{\tilde{\mu}}).$$

Since X is lifting-compact there is a Borel set $Y \subset \beta X$ with $\tilde{\tilde{\mu}}(Y) = 1$ and a map $\Theta'(\beta)\colon Y \to X$ which is $\mathfrak{A} \cap Y - \mathscr{B}(X)$-measurable such that

$$\Theta(h \circ \varphi)(\omega) = (h \circ \Theta'(\varphi))(\omega)$$

for all $h \in C_b(X)$, $\omega \in Y$.

Since Θ is almost strong there exists a Borel set N in βX with $\tilde{\tilde{\mu}}(N) = 0$ and such that

$$\tilde{h}(\omega) = \Theta(\tilde{h})(\omega) = \Theta(h \circ \varphi)(\omega) = (h \circ \Theta'(\varphi))(\omega)$$

for all $\omega \in Y \setminus N$ and all $h \in C_b(X)$. This implies $Y \setminus N \subset X$ and hence X is $\tilde{\tilde{\mu}}$-measurable with $\tilde{\tilde{\mu}}(X) = 1$. Therefore μ is tight and the theorem is proved.

6.3. COROLLARY. *Assume* CH. *Let X be separable. Then X is strongly measure compact if and only if X is H-compact and lifting-compact.*

Proof. Since X is separable βX has a base for its topology of cardinal at most that of the continuum. A result of Mokobodzki and Fremlin (cf. [**6**],

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Theorem 9) implies that every Radon measure on βX admits an almost strong lifting. The corollary, therefore, follows from Theorem 6.2.

7. Structural properties of H-compactness of spaces. Here we would like to discuss stability properties of H-compactness with respect to a) subspaces; b) products; c) images (continuous or just measurable). The corresponding properties for measure-compactness, strong measure-compactness, and lifting-compactness are well known (cf. [**12**] and [**3**]).

a) The following result is an immediate consequence of Theorem 3.2 (iii) combined with the fact that every Baire subset of a space admits a Baire measurable retraction (see the remark following Proposition 7.4):

7.1. PROPOSITION. *If X is H-compact and Y is a Baire subset of X then Y is H-compact.*

Unlike measure-compactness, strong measure-compactness and realcompactness, H-compactness is not necessarily inherited by closed subspaces.

7.2. *Example.* Let $X = [0, 1]$ and let τ' be the topology on X constructed by Edgar and Talagrand [**5**] (cf. Example 6.1 a), i.e., there exists a set $M \subset X$ with outer Lebesgue measure 1 and inner Lebesgue measure 0, such that

$$\tau' = \{G \cup P \mid G \text{ open w. r. t. the ordinary topology of } [0, 1] \text{ and } P \subset M\}$$

and (X, τ') has the same Baire sets as $[0, 1]$ equipped with the ordinary topology. Thus (X, τ') is H-compact. But its τ'-closed subset $X \setminus M$ is metrizable and not strongly measure compact ([**15**]), hence not H-compact (Theorem 4.3).

b) The problem whether the product of any two H-compact spaces is H-compact remains open. But we have the following rather special result:

7.3. PROPOSITION. *Let X be H-compact and measure-compact and let Y be strongly measure-compact. Then $X \times Y$ is H-compact.*

Proof. Let $(\Omega, \mathfrak{A}, \nu)$ be a probability space, $\Phi:\mathscr{B}(X \times Y) \to \mathfrak{a}/_\nu$ a σ-homomorphism, and $\mu = \nu \circ \Phi$. Define Φ_X: $\mathscr{B}(X) \to \mathfrak{A}/_\nu$ by $\Phi_X(B) = \Phi(B \times Y)$ and $\Phi_Y:\mathscr{B}(Y) \to \mathfrak{A}/_\nu$ in an analogous way. Then Φ_X and Φ_Y are σ-homomorphisms. Since X and Y are H-compact there exist $\mathfrak{A}_\nu - \mathscr{B}(X)$ – (resp. $\mathfrak{A}_\nu - \mathscr{B}(Y)-$) measurable maps $\varphi_X:\Omega \to X$ and $\varphi_Y:\Omega \to Y$ inducing Φ_X and Φ_Y respectively. Define $\varphi:\Omega \to X \times Y$ by

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$$\varphi(\omega) = (\varphi_X(\omega), \varphi_Y(\omega)).$$

We claim that φ is $\mathfrak{A}_\nu - \mathscr{B}(X \times Y)$-measurable. Obviously φ is $\mathfrak{A}_\nu - \mathscr{B}(X) \otimes \mathscr{B}(Y)$-measurable and satisfies

$$\varphi^{-1}(B) \in \Phi(B) \quad \text{for all } B \in \mathscr{B}(X) \otimes \mathscr{B}(Y).$$

Therefore our claim is proved if we can show that, for every $A \in \mathscr{B}(X \times Y)$, there are $B, B' \in \mathscr{B}(X) \otimes \mathscr{B}(Y)$ with $B \subset A \subset B'$ and $\mu(B' \setminus B) = 0$. By standard arguments the proof of this statement reduces to proving that for every non-negative $f \in C_b(X \times Y)$ there are $\mathscr{B}(X) \otimes \mathscr{B}(Y)$-measurable functions $g, g': X \times Y \to \mathbf{R}$ such that

$$g \leqq f \leqq g' \quad \text{and} \quad \int (g' - g) d\mu = 0.$$

To show this let

$$\mathscr{H} := \{h_{|X \times Y} | \, h \in C(\beta X \times \beta Y)\}.$$

Then it follows from [**3**], Lemma 3.1 that for each $f \in C_b(X \times Y)$ and all $(x, y) \in X \times Y$ we have

$$f(x, y) = \sup\{h(x, y) \mid h \in \mathscr{H}, h \leqq f\}.$$

According to [**12**], Theorem 5.3, the space $X \times Y$ is measure compact. Hence μ is τ-smooth and we deduce

$$\int f d\mu = \sup\left\{\int \mathrm{N} \, h d\mu \mid h \in \mathscr{H}, h \leqq f\right\}.$$

Since each $h \in \mathscr{H}$ is $\mathscr{B}(X) \otimes \mathscr{B}(Y)$-measurable there exists, therefore, a $\mathscr{B}(X) \otimes \mathscr{B}(Y)$-measurable function $g: X \times Y \to \mathbf{R}$ with $g \leqq f$ and

$$\int (f - g) d\mu = 0.$$

In an analogous way one can find a $\mathscr{B}(X) \otimes \mathscr{B}(Y)$-measurable function g': $X \times Y \to \mathbf{R}$ with $f \leqq g'$ and

$$\int (g' - f) d\mu = 0.$$

This completes the proof.

c) Note that a continuous one-to-one image of an H-compact space need not be H-compact: Take any non-H-compact space X whose cardinal $\mathfrak{k}$ in non-measurable. Let D be a discrete space of cardinality $\mathfrak{k}$. Then D is obviously H-compact. Let f: $D \to X$ be any bijection. Then f is continuous but X is not H-compact.

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Thus, concerning preservation w. r. t. images, the following is, in a sense, the best result one can hope for.

7.4. PROPOSITION. *Let X and Y be completely regular spaces, such that there is a $\mathscr{B}(X) - \mathscr{B}(Y)$-measurable map p: $X \to Y$ and a $\mathscr{B}(Y) - \mathscr{B}(X)$-measurable map q: $Y \to X$ with $p \circ q = \mathrm{id}_Y$. If X is H-compact then Y is H-compact.*

Proof. Let $\mu \in P(Y)$ be given. According to Theorem 3.2 (iii) it is enough to show that $(Y, \mathscr{B}(Y), \mu)$ is an absolute retract. Let $(\Omega, \mathfrak{A}, \mu)$ be a probability space and j: $Y \to \Omega$ an embedding. By Theorem 3.2 (ii) $(X, \mathscr{B}(X), \mu)$ is injective. Hence there exists a $\mathfrak{A}_\nu - \mathscr{B}(X)$-measurable map $\bar{q}$: $\Omega \to X$ with $\bar{q} \circ j =$ q. Define $r: = p \circ \bar{q}$. Then r is $\mathfrak{A}_\nu - \mathscr{B}(Y)$-measurable and

$$r \circ j = p \circ \bar{q} \circ j = p \circ q = \mathrm{id}_Y.$$

This completes the proof of the proposition.

Remark. If Y is a non-empty Baire subset of X then there exist maps p, q with the properties stated in Propositon 7.4.: Let q be the canonical embedding and let p: $X \to Y$ be defined by

$$p(x) = \begin{cases} x, & x \in Y \\ x_o, & x \in X \setminus Y \end{cases}$$

where x_o is any fixed point in Y.

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