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Analytic Variation of p-adic Abelian Integrals

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Abstract. In *Ann. of Math.* **121** (1985), 111–168, Coleman defines p-adic Abelian integrals on curves. Given a family of curves X/S, a differential ω and two sections s and t, one can define a function λ_ω on S by $\lambda_\omega(P) = \int_{s(P)}^{t(P)} \omega_P$. In this paper, we prove that λ_ω is locally analytic on S.

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1. Introduction

Let p be an odd prime number. Let K be a finite extension of $\mathbb{Q}_p$, and let R be its ring of integers. Let k be the residue field of R. Let S be a smooth affine curve over R. Let $X \xrightarrow{\pi} S$ be a family of curves over S; in other words, suppose that X/S is proper and smooth of relative dimension one. Let s and t be sections of this family, and set $D = s(S) \cup t(S)$.

Given a family of relative differentials of the second kind ω on X/S whose polar divisor does not meet the images of s and t, we would like to study the p-adic integrals (as defined by Coleman in [1]) of ω from s to t on the fibers of the family X/S. One possible definition of these integrals is the following. For each $P \in S$, the fiber X_P above P is a smooth curve, $s(P)$ and $t(P)$ are points of X_P and ω_P is a differential of the second kind on X_P, so we can use the construction of [1] to define $\lambda_\omega(P)$ by $\lambda_\omega(P) = \int_{s(P)}^{t(P)} \omega_P$.

We view this as giving a function λ_ω on S. Some interesting arithmetical properties of the family X/S can be phrased in terms of λ_ω. For example, the results of [1] imply

THEOREM 1.1. *The divisor class of* $(s(P)) - (t(P))$ *in* X_P *is torsion if and only if* $\lambda_\omega(P) = 0$ *for all* $\omega \in \Gamma(S, \pi_* \Omega^1_{X/S})$.

However, given the above definition, there is no reason to expect that λ_ω has any good properties at all, since it is a priori only a set-theoretic function. Zarhin gives an alternative construction of p-adic Abelian integrals in [4], but this constuction does not lend itself to studying families. In this paper,we establish that λ_ω is in fact *locally analytic* on S. By 'locally analytic,' we mean that λ_ω is given by a convergent power series on each residue class of S.

2. Restriction to Residue Classes

To make our results precise, we must formalize the notion of restricting to a residue class of S. The completion of S along a k-valued point P_0 is isomorphic to Spec $R[[T]]$ (for some non-canonical choice of a local parameter T). We view base change by the map Spec $R[[T]] \to S$ as restricting to the residue class of P_0. A function on S can be pulled back to a function of T on the residue class.

The integrals λ_ω will have denominators of p and hence will not be in the ring $R[[T]]$, so we introduce $K\{\{T\}\}$, the ring of power series which are convergent in the open disk of radius 1.More precisely, $K\{\{T\}\}$ consists of series $\sum_{i=0}^{n} a_i T^i$ such that $\lim_{i\to\infty} |a_i| r^i = 0$ for every real number $0 \leqslant r < 1$. One may regard $K\{\{T\}\}$ as the ring of rigid analytic functions on an open unit ball.

Our main result can now be stated:

THEOREM 2.1. *Let X/S be a family of curves and ω a family of differentials as in the introduction. On any residue class with local parameter T, the integral λ_ω, viewed as a function of T, is an element of $K\{\{T\}\}$.*

The proof of this theorem will be divided into two cases. First, we will give a proof based on crystalline cohomology for the residue classes where the sections do not meet mod p. Then we will give a fairly elementary proof for residue classes where the two sections s and t are congruent mod p.

3. Disjoint Sections

To prove the analyticity of integrals on residue classes where the two sections do not meet, we will use the language of crystalline cohomolgy. We will follow the notation for F-crystals used in [3].

We wish to integrate differentials of the second kind on X/S. However, the differential of a function that vanishes on D should integrate to zero. Thus we may view the objects we are integrating as differentials of the second kind modulo differentials of functions that are zero on D, *i.e.* as classes from $H^1_{\mathrm{DR}}(X/S, D)$. An integral should assign a function on the base to each such class. The problem of integration therefore amounts to finding a section σ of the dual of $H^1_{\mathrm{DR}}(X/S, D)$, namely $H^1_{\mathrm{DR}}((X \setminus D)/S)$, such that $\langle \omega, \sigma \rangle = \lambda_\omega$.

The cohomology modules $H^1_{\mathrm{DR}}((X \setminus D)/S)$ and $H^1_{\mathrm{DR}}(X/S, D)$ are F-crystals on S (see [2]). Briefly, this means that they are S-modules with an integrable, convergent connection and an action of Frobenius; see [3] for more details. We will show that the following properties determine a locally analytic section σ and also characterize Coleman's integrals:

(1) $d\langle \omega, \sigma \rangle = \langle \nabla\omega, \sigma \rangle$.
(2) $\langle dG, \sigma \rangle = t^*G - s^*G$ for G a function regular on D.

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(3) $\Phi\sigma = p\sigma$, where Φ denotes the Frobenius endomorphism.

3.1. CONSTRUCTION OF INTEGRALS

The section σ will be a locally analytic (Coleman uses the term "flabby") section of the cohomology sheaf; in other words, σ will be given locally as a section of the pullback of the cohomology to each residue class, with no relation required between residue classes. We have the following:

THEOREM 3.1. *For each k-valued point P_0 of S such that s and t do not meet above P_0, there is a unique section σ of $H^1_{\mathrm{DR}}((X \setminus D)/S)$ restricted to the residue class above P_0 satisfying the following conditions:*

(1) $d\langle\omega, \sigma\rangle = \langle\nabla\omega, \sigma\rangle$ *for all* $\omega \in H^1_{\mathrm{DR}}(X/S, D)$.
(2) $\langle dG, \sigma\rangle = t^*G - s^*G$ *for G a function on X regular on D.*
(3) $\Phi\sigma = p\sigma$, *where Φ denotes the Frobenius endomorphism of* $H^1_{\mathrm{DR}}((X \setminus D)/S)$.

Proof. Choose a local parameter T for the residue class above P_0. We will use the notation $-\otimes K\{\{T\}\}$ to denote the pullback of an S-module to the residue class above P_0. Let H be $H^1_{\mathrm{DR}}((X \setminus D)/S) \otimes K\{\{T\}\}$. We seek a section σ of H.

First, the pairing on cohomology is compatible with the connections, which means that for any sections ω and σ,

$$\langle\omega, \nabla\sigma\rangle + \langle\nabla\omega, \sigma\rangle = d\langle\omega, \sigma\rangle.$$

Condition 1 in the statement of the theorem may then be understood as requiring that $\langle\omega, \nabla\sigma\rangle = 0$ for all ω, i.e. as requiring that σ be horizontal. Therefore, we must find a vector with the desired properties in the finite-dimensional K-vector space of horizontal sections of H.

For the remainder of the proof, we restrict to the residue class above P_0. There is a horizontal exact sequence

$$0 \to H^1_{\mathrm{DR}}(X/S) \otimes K\{\{T\}\} \to H \xrightarrow{\mathrm{Res}} H^0_{\mathrm{DR}}(D)_0 \otimes K\{\{T\}\} \to 0,$$

where $H^0_{\mathrm{DR}}(D)_0$ denotes the degree 0 part of $H^0_{\mathrm{DR}}(D)$ (the residues of a differential form must sum to 0). Using Proposition 3.1.2 of [3], we obtain a corresponding exact sequence of horizontal sections

$$0 \to (H^1_{\mathrm{DR}}(X/S) \otimes K\{\{T\}\})^{\nabla} \to H^{\nabla} \to (H^0_{\mathrm{DR}}(D)_0 \otimes K\{\{T\}\})^{\nabla} \to 0.$$

This is an exact sequence of K-vector spaces. However, since all three vector spaces arise as spaces of horizontal sections of F-crystals, they are all equipped with a σ-linear endomorphism (where σ denotes the Frobenius automorphism of K) which we will call the action of Frobenius and write as Φ. A priori this endomorphism depends on a choice of lift of Frobenius, but convergence shows that every lift in fact gives the same endomorphism. Let n be the integer such that σ^n is the identity on K. Then Φ^n is a K-linear endomorphism of all the vector spaces in the exact sequence above, and the maps of the exact sequence respect this map.

$(H^0_{\mathrm{DR}}(D)_0 \otimes K\{\{T\}\})^\nabla$ is a one-dimensional K-vector space where Φ^n acts as multiplication by p^n. $(H^1_{\mathrm{DR}}(X/S) \otimes K\{\{T\}\})^\nabla$ is a $2g$-dimensional K-vector space where Φ^n acts with eigenvalues of complex absolute value $p^{n/2}$ (by comparison with crystalline cohomology and the Riemann hypothesis). Since these eigenvalues have different complex absolute value, this extension of vector spaces splits naturally in a unique Φ^n-invariant way.

Condition 2 specifies the image of σ in $(H^0_{\mathrm{DR}}(D)_0 \otimes K\{\{T\}\})^\nabla$, namely that it should have residue $+1$ on t and residue -1 on s. Condition 3 gives the action of Φ, which means that σ must actually be the unique preimage of these residues coming from the Frobenius-invariant splitting (and upon which Φ^n acts as multiplication by p^n).

3.2. COMPARISON WITH COLEMAN'S INTEGRALS

We would now like to show that the integrals constructed in the proof of Theorem 3.1 agree with the integrals constructed in [1]. As in the introduction, define a function λ_ω on S by $\lambda_\omega(P) = \int_{s(P)}^{t(P)} \omega_P$, where the integral is to be interpreted as in [1]. We now prove the following:

THEOREM 3.2. *Let P be a point of S. If σ is the locally analytic section of $H^1_{\mathrm{DR}}((X \setminus D)/S)$ constructed in Theorem 3.1 and ω is a section of $H^1_{\mathrm{DR}}(X/S, D)$, let σ_P and ω_P denote their pullbacks to the fiber above P. Then $\langle \omega_P, \sigma_P \rangle = \lambda_\omega(P)$.*

Proof. Let P_0 be the reduction of P. Since everything is at most locally analytic, we will restrict to the residue class above P_0. In particular, for the remainder of this proof, we will write S and X for the restriction of these objects to the residue class of P_0.

Let n be the positive integer such that P_0 is fixed by the nth power of Frobenius. There is a commutative diagram of affinoids

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, hook] & S \arrow[d] \\ S \arrow[r] & P \end{tikzcd}$$

The reduction of this diagram commutes with $\tilde{\phi}$, the nth power of the Frobenius map

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on S_0:

$$\begin{tikzcd} P_0 \arrow[r, hook] \arrow[d, hook] & S_0 \arrow[d] \\ S_0 \arrow[r] \arrow[ur, hook, "\tilde{\phi}"] & P_0 \end{tikzcd}$$

Therefore Theorem 1.1 of [1] implies that there is a lift ϕ of the nth power of Frobenius to S such that P is the Teichmuller point of ϕ above P_0.

Because ϕ fixes P, the induced endomorphisms Φ of $H^1_{\mathrm{DR}}(X/S, D)$ and $H^1_{\mathrm{DR}}((X \setminus D)/S)$ restrict to K-linear endomorphisms of the stalks of these sheaves above P (which are K-vector spaces). σ_P is characterized as the unique element of $H^1_{\mathrm{DR}}((X \setminus D)/S)$ which has residue $+1$ at $t(P)$ and -1 at $s(P)$ and is an eigenvector of Φ with eigenvalue p^n.

Let $\lambda(P)$ be the element of $H^1_{\mathrm{DR}}((X \setminus D)/S)$ (the dual of $H^1_{\mathrm{DR}}(X/S, D)_P$) determined by $\langle \omega, \lambda(P) \rangle = \lambda_\omega(P)$. If we show that $\lambda(P)$ satisfies the same conditions on residues and the action of Frobenius as σ_P, then this will imply $\lambda(P)$ is equal to σ_P. First, one easily sees that the condition of "the fundamental theorem of calculus" (Proposition 2.4 of [1]) for $\lambda(P)$ is equivalent to the condition on the residues of σ_P.

To check the second condition, we must determine the action of Frobenius on $\lambda(P)$. Let m be the positive integer such that the nmth power of Frobenius fixes the reductions $s(P_0)$ and $t(P_0)$. Again by Theorem 1.1 of [1], this time considering the diagram

$$\begin{tikzcd} s(P) \cup t(P) \arrow[r, hook] \arrow[d, hook] & X_P \arrow[d] \\ X_P \arrow[r] & P \end{tikzcd}$$

there is a lift ϕ_1 of the nmth power of Frobenius to X_P fixing $s(P)$ and $t(P)$, inducing an endomorphism Φ_1 of F-crystals. Then for any differential of the second kind ω on X_P, the change of variables formula for p-adic integrals (Theorem 2.7 of [1]) implies that

$$\int_{s(P)}^{t(P)} \omega = \int_{\phi_1(s(P))}^{\phi_1(t(P))} \omega = \int_{s(P)}^{t(P)} \Phi_1 \omega,$$

or in different notation, $\langle \omega, \lambda(P) \rangle = \langle \Phi_1 \omega, \lambda(P) \rangle$.

The duality pairing on cohomology maps into H^2_{DR}, which is then identified with the ground field via the trace, and Frobenius acts as multiplication by p on the one-dimensional space H^2_{DR}. Φ_1 is invertible. Define an endomorphism V as $p^{nm}\Phi_1^{-1}$. By functoriality, for any ω and τ,

$$\langle \Phi_1 V\omega, \Phi_1 \tau \rangle = \Phi_1 \langle V\omega, \tau \rangle = p^{nm} \langle V\omega, \tau \rangle.$$

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However, we also have

$$\langle \Phi_1 V\omega, \Phi_1\tau\rangle = \langle p^{nm}\omega, \Phi_1\tau\rangle = p^{nm}\langle \omega, \Phi_1\tau\rangle.$$

So we obtain the fact that $\langle V\omega, \tau\rangle = \langle \omega, \Phi_1\tau\rangle$ (informally, "the adjoint of Frobenius is Verschiebung").

Combining the above, we obtain that for any ω,

$$\begin{aligned}\langle \omega, \Phi_1\lambda(P)\rangle &= \langle V\omega, \lambda(P)\rangle \\ &= \langle \Phi_1 V\omega, \lambda(P)\rangle \\ &= \langle p^{nm}\omega, \lambda(P)\rangle \\ &= \langle \omega, p^{nm}\lambda(P)\rangle.\end{aligned}$$

Therefore $\Phi_1\lambda(P) = p^{nm}\lambda(P)$, so Frobenius does act with the same eigenvalue on $\lambda(P)$ as on σ_P. This completes the proof that $\lambda(P) = \sigma_P$.

4. Kernel of Reduction

Note that the two sections s and t of the family X/S give one section u of the Jacobian J (a family of Abelian varieties) of X/S by setting $u(P)$ equal to the divisor class of$(t(P)) - (s(P))$. Then

$$\lambda_\omega(P) = \int_{s(P)}^{t(P)} \omega_P = \int_0^{u(P)} \omega_P,$$

where the first integration is performed on X_P and the second integration is performed on J_P.

To complete the proof of Theorem 2.1, we must show that λ_ω varies analytically on the residue class of a k-valued point P_0 of S such that $s(P_0) = t(P_0)$. Then on the residue class of P_0, the image of u lies in the kernel of reduction on J. As described in [1], the integral can be thought of as a formal logarithm on the Jacobian. By computing this directly, we will show that λ_ω is locally analytic on the residue class of P_0. Because we are working in the kernel of reduction, we merely need to carry out the standard construction of formal logarithms with $R[[T]]$ and $K\{\{T\}\}$ replacing R and K.

We can choose local coordinates $(U_1, \dots, U_g)$ for the kernel of reduction of J so that the identity element of each fiber of J is at $(0, \dots, 0)$ (where g is the genus of X). Since the polar divisor of ω does not meet the images of s and t, ω can be given locally as

$$\sum_{i=1}^{g} C_i(T, U_1, \dots, U_g)dU_i,$$

where each $C_i \in R[[T, U_1, \dots, U_g]]$. The section u is given locally by $U_i = u_i(T)$ for

$u_i \in R[[T]]$. In fact, since the image of u is contained in the kernel of reduction, each u_i is contained in the ideal generated by p and T.

To compute the integral λ_ω, we first formally integrate each C_i with respect to U_i, which introduces denominators of p, but only of a very mild type: a term containing $U_i^{p^n}$ introduces a denominator of p^n to the integral, so the coefficients will grow at a small enough rate for the final result to lie in $K\{\{T\}\}$. To obtain the final result, we substitute u_i for U_i in the formal integral. Since every u_i is contained in the ideal $(p, T)R[[T]]$, this substitution is well-defined and is a member of $K\{\{T\}\}$. This completes the proof of our main result.

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