FOUNDATIONS OF THE THEORY OF DYNAMICAL SYSTEMS OF INFINITELY MANY DEGREES OF FREEDOM, II

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1. Introduction. The notion of quantum field remains at this time still rather elusive from a rigorous standpoint. In conventional physical theory such a field is defined in essentially the same way as in the original work of Heisenberg and Pauli **(1)** by a function $\phi(x, y, z, t)$ on space-time whose values are operators. It was recognized very early, however, by Bohr and Rosenfeld **(2)** that, even in the case of a free field, no physical meaning could be attached to the values of the field at a particular point—only the suitably smoothed averages over finite space-time regions had such a meaning. This physical result has a mathematical counterpart in the impossibility of formulating $\phi(x, y, z, t)$ as a bona fide operator for even the simplest fields (in any fashion satisfying the most elementary non-trivial theoretical desiderata), while on the other hand for suitable functions f, the integral $\int \phi(x, y, z, t) f(x, y, z, t)\, dxdydzdt$ could be so formulated. This mathematical development began with the work of Fock **(3)**, in which the field was treated in the conventional way without smoothing, but which gave a concrete representation for a free field that was capable of extension to a representation by bona fide operators, of the smoothed field operators, in the non-relativistic case, an observation that formed the basis for the independent work of Friedrichs **(4)** and Cook **(5)**. The latter gave in rigorous terms the basic mathematical theory of the situation. Additional complications arise in giving an effective relativistic treatment, but it is now established that the suitably smoothed averages of the standard relativistic free real fields may be formulated as bona fide self-adjoint operators in Hilbert space in the strict mathematical sense (see below).

There has not yet been analogous progress for the case of interacting fields, and in the work of Wightman **(6)**, for example, it has merely been postulated that field averages could be given meaning as operators. The expectation values of functions of these operators in the so-called physical vacuum state determine the observable consequences of the theory, and instead of attempting to specify the theory by partial differential equations one may rather attempt

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this through a more direct description of such vacuum expectation values. The efforts of Källén and Wightman **(7)** have been directed towards a determination of possible forms for the vacuum expectation values of simple products of smoothed field operators, and in the case of the triple product certain highly refined analytical information has been obtained from the postulates of Lorentz-invariance, microcausality (= "local commutativity"), and positivity of the energy. The finiteness of the vacuum expectation values of such products may, however, be questioned, and in fact in conventional unrenormalized field theory they appear as infinite. In addition, even in the hypothetical case that these values are all finite, they do not necessarily fix the theory, that is, the vacuum expectation values of *all* (smooth) functions of the field operators.

The situation is in a way rather similar to, although vastly more complex than, that with regard to the specification of a probability distribution by its moments. The moments need not be finite; and even when they are finite they do not necessarily determine the distribution (cf. for example **(8)**). The argument that such expectation values must be finite because they have simple physical interpretations **(9A)** is quite parallel to the argument that the second moment of the distribution on the line with element of probability $\pi^{-1}(\alpha^2 + x^2)^{-1}dx$ must be finite because it measures the physical parameter α representing the dispersion of the distribution; speaking loosely in the manner of conventional physical theory, this second moment is easily seen to be proportional to α by an "infinite constant."

A rather natural way to attempt to remedy this situation is to pursue the analogue in the field-theoretic case for the characteristic function in the theory of probability distributions. This is always finite, is known to determine the distribution, and moreover is capable of being characterized intrinsically. The present paper obtains such an analogue in connection with a general (non-pathological) state of a linear field. An interacting field on a particular space-like surface can be transformed into a linear field (by taking it in the so-called interaction representation, as described for example in **(9B)**), whereupon its vacuum state transforms into an (analytically rather inaccessible) state of the linear field. The present results thereby have implications for the vacuum state of an interacting field. It would in certain respects be more useful to be able to treat the interacting fields directly, but the mathematically ambiguous character of such fields at present seems to make it out of the question to give any *rigorous* treatment of the matter, and in addition there is the apparent lack of any *formal* characterization for the generating functions $E[e^{i\int\phi f}]$ (where E is the vacuum state expectation functional, ϕ is an interacting field, and f a general smoothing function) that would form the analogue for the "Heisenberg" fields of the present functional for a linear field.

At any rate, we show here (rigorously) that a *regular state of a linear field* (of arbitrary unitary transformation properties) *can be characterized by a functional on the corresponding classical wave functions*. That is to say, for example, that any regular state of the *quantized* hermitian Klein-Gordon field satisfying the equation

$$\square \phi = m^2\phi$$

is determined by a functional defined on the manifold of all *classical* real-valued normalizable solutions of this equation. The generating functionals that arise in this way are characterized intrinsically, and it is shown how the state may be recovered from the functional. The attainment of these results requires the suitable grouping together of a fairly wide array of scientific developments, but the proofs do not involve any individual points of great technical difficulty.

The present generating functional thus appears as considerably more economical, and mathematically distinctly more viable, than the characterization of a state through the expectation values of products of field operators. It has, however, a rather less direct connection with conventional practice in so-called renormalization theory than the product approach.

In the present paper only Bose-Einstein fields are treated, but the same methods can be adapted to the case of Fermi-Dirac fields.

2. The general linear boson field. The conventional treatments of linear field theory start from specific sets of linear partial differential equations, and arrive at formal operator-valued functions satisfying the same partial differential equations and certain non-trivial commutation relations, after a procedure that varies somewhat from equation to equation. The treatment for the photon case is in particular rather parallel to that for the scalar meson case, but involves additional technical complications, which are somewhat space-consuming and significantly complicate the notation. In addition these treatments have no immediate extension to systems that may be defined not by partial differential equations in ordinary space-time, but in a more general space-time manifold; or which are covariant not with respect to the Lorentz group, but with respect to a more general one. A further difficulty is a fundamental lack of uniqueness—for any given linear quantum field, there exist infinitely many others, satisfying the same commutation relations and partial differential equations, but no two of which are connected by a unitary transformation (cf. **(10)**).

It is therefore relevant that there is available a perfectly general, rigorous, and quite mechanical procedure for linear quantization, whenever the states of the classical system being considered form a complex Hilbert space (or in fact, somewhat more generally). The commutation relations in particular are fixed once the structure of this Hilbert space is specified, and no further examination of the field equations is required. This unique mathematical structure may appropriately be called the "general linear boson field"; its use makes it possible to deal with the commutation relations for extensive classes of fields without the burden of complicated singular functions in the formalism, or the need to utilize generalized functions such as Schwartz' distributions in order to rigorize parts of the analysis. The quantization of a photon field is in particular, for example, reduced to the *classical* (that is, unquantized) problem of showing that the real normalizable solutions of Maxwell's equations in a vacuum form a complex Hilbert space in a unique Lorentz-invariant manner.

More generally, the normalizable classical solutions of a relativistic linear field equation form the complex Hilbert space $\mathfrak{H}$ that is basic in the following for the treatment of the corresponding field. To present this development in the most elementary fashion, consider for example the real solutions of the Klein-Gordon equation

$$\Box\, \phi = m^2\phi.$$

This is to be interpreted as a heuristic equation, for the relevant solutions are not necessarily conventional functions, but generalized ones. For a rigorous treatment it is simplest to take the Fourier transform Φ of ϕ as basic. In such terms $\mathfrak{H}$ consists of all complex-valued Φ on the hyperboloid $k^2 = m^2$ (here k is the vector with components (k_0, k_1, k_2, k_3) and k^2 denotes the Lorentz squared-length, $k^2 = k_0^2 - k_1^2 - k_2^2 - k_3^2$), such that $\Phi(-k) = \overline{\Phi}(k)$ (corresponding to the reality of the field) and with

$$|\Phi|^2 = \int|\Phi(k)|^2 d\chi(k) < \infty,$$

where $d\chi(k) = |k_0|^{-1}dk_1dk_2dk_3$, and is characterized as the unique regular measure on the hyperboloid (within a constant factor) that is Lorentz-invariant. The physical significance of the finiteness condition is more apparent if one deals, equivalently, with positive frequency rather than real solutions of the Klein-Gordon equation (cf. §3 of **(11)**), for these are conventionally interpretable as single-particle wave functions, and the normalizability corresponds to the existence in physical principle of an *individual* free particle (a non-normalizable wave function such as a plane wave having a somewhat ambiguous interpretation as a *beam* of particles).

The description of the corresponding linear quantum field involves basically the formulation and labelling of the field observables, and in particular the specification of the commutation relations of the field variables. Conventionally this is achieved in a heuristic fashion, by postulating the existence of an essentially unique (that is, unique within unitary equivalence, when irreducibility is present) operator-valued function ϕ such that

$$\Box\, \phi = m^2\phi$$

and the commutator $[\phi(x), \phi(x')] = -iD(x - x')$, where D is a certain singular function, and x is written in place of the 4-tuple (x, y, z, t). No such operator-valued function is known to exist, and actually there is practically conclusive evidence that it cannot exist in any literal sense; and in any event it could not be unique within unitary equivalence even if its range of values formed an irreducible set of operators.

In order to deal in a mathematically clear and physically conservative way with such a matter, it is appropriate to make first a purely mathematical construction and development; next to make a statement of what *mathematical* objects in this construction represent the observable or theoretically relevant *physical* objects associated with the physical system motivating the construction; and finally to establish the essential agreement between the resulting physical theory and the conventional one and/or experimental indications. The first part of this procedure should be mathematically rigorous; the second part should be precise, but necessarily only in the sense of legal rather than mathematical definitions; while the third part may well involve quite heuristic elements, and in fact this is necessarily the case when dealing with a theory whose conventional form is heuristic, as is quantum field theory. Accordingly we proceed as follows.

Mathematical construction. There is no compelling reason not to use terminology here that is indicative of the physical object being considered, and a great deal of circumlocution may be avoided in this way. In particular, the term "field" will be so used, but several different mathematical objects related to various heuristic types of fields must be distinguished. It will suffice here to deal with *concrete*, *general*, *clothed*, and *zero-interaction* linear boson fields.

Definition 1. Let $\mathfrak{H}$ be a real linear vector space, and let B be a given skew-symmetric bilinear form over $\mathfrak{H}$.

(a) A *concrete* LBF (LBF = linear boson field) over $(\mathfrak{H}, B)$ is a map $z \to R(z)$ from to $\mathfrak{H}$ the self-adjoint operators in a complex Hilbert space $\mathfrak{K}$ such that

(x) $$e^{iR(z)}e^{iR(z')} = e^{iR(z+z')}e^{iB(z,z')} \qquad (z, z' \text{ arbitrary in } \mathfrak{H}).$$

Two such fields, $R(.)$ on $\mathfrak{K}$ and $R'(.)$ on $\mathfrak{K}'$, over the same $(\mathfrak{H}, B)$, are *unitarily equivalent* in case there exists a unitary operator V from $\mathfrak{K}$ onto $\mathfrak{K}'$ such that

$$VR(z)V^{-1} = R'(z), \text{ for all } z \text{ in } \tilde{H}.$$

(b) A *bounded field observable* of a given concrete LBF is defined as a bounded operator on $\mathfrak{K}$ that is a limit of a uniformly convergent sequence of operators, each of which is in the weakly closed ring of operators generated by the $e^{iR(z)}$, for z ranging over some finite-dimensional (but otherwise arbitrary) linear subspace of $\mathfrak{H}$. The set of all bounded field observables is then a uniformly closed self-adjoint algebra of operators on $\mathfrak{K}$ (cf. **(12)**, referred to henceforth as "I").

(c) Two concrete LBF's $R(.)$ and $R(.)'$ on $\mathfrak{K}$ and $\mathfrak{K}'$, over the same $(\mathfrak{H}, B)$, are said to be *physically equivalent* in case there is a one-to-one correspondence between their respective bounded hermitian field observables preserving the operations of addition and squaring. A general LBF over $(\mathfrak{H}, B)$ is defined as a physical-equivalence class of concrete LBFs. We note that when $\mathfrak{H}$ is a complex Hilbert space and B is the canonically associated skew form (cf. below), there is only one general LBF.

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(d) A *clothed* LBF over $(\mathfrak{H}, B)$ is a couple consisting of a general LBF over $(\mathfrak{H}, B)$, together with a given state E of the (abstract) algebra of field observables. E is said to be *regular* in case its restriction to the weakly closed ring of operators generated by the $e^{iR(z)}$, as z ranges over a finite-dimensional subspace of $\mathfrak{H}$ on which B is non-degenerate, is weakly continuous relative to the unit sphere of this ring, for every such finite-dimensional subspace, and some concrete representative for the general LBF. An LBF is *properly clothed* if the associated state E is regular.

(f) When a real linear vector space $\mathfrak{H}$ has in addition a designated structure as a complex Hilbert space, compatible with its real-linear structure, the notation $(\mathfrak{H}, B)$ will be understood to refer to $\mathfrak{H}$ as a real linear vector space, with $B(z, z') = \frac{1}{2}Im[(z, z')]$; and the notation $\mathfrak{H}$ alone may refer to the couple $(\mathfrak{H}, B)$, when it is clear from the context that it is this couple that is relevant.

(g) As an example of a clothed LBF the *zero-interaction* LBF is defined as that clothed LBF over the complex Hilbert space $\mathfrak{H}$ for which the given state E is invariant under the induced action of all unitary operators on $\mathfrak{H}$ (cf. I, and especially Cor. 3.1 showing the uniqueness of the free LBF).

Physical interpretation. If ϕ is for example the conventional real Klein-Gordon field and f is any smooth function on space-time that vanishes at infinity, the field average $\int \phi(x)f(x)d_4x$ is just such an $R(z)$. The appropriate $\mathfrak{H}$ is just that defined above, consisting of all normalizable solutions of the Klein-Gordon equation; the appropriate z is that function on the mass hyperboloid (that is, the manifold $k^2 = m^2$) that coincides there with the complex conjugate of the Fourier transform of f; and the appropriate skew-symmetric form $B(z, z')$ is just that defined equivalently as $B(z, z') = \frac{1}{2}\int\int D(x - x') f(x)f'(x')d_4xd_4x'$, where z' is related to f' in the same fashion as z to f, and D is the conventional singular function such that $[\phi(x), \phi(x')] = -iD(x - x')$, or, in a form in which the finiteness of $B(z, z')$ is more apparent,

$$B(z, z') = -\frac{i}{8\pi^3}\int_M \operatorname{sgn} k_0\, z'(k)z(-k)d\lambda(k)$$

where M denotes the mass hyperboloid. The equation (x) is the bounded (Weyl) form of the infinitesimal relation

$$[R(z), R(z')] = -2iB(z, z'),$$

to which it is formally equivalent.

Conventionally it was assumed, then, that the Klein-Gordon field is concrete and irreducible, and that this sufficed to define the field uniquely. The discovery that in actuality this was very far from being the case led to the introduction of general linear fields in I, in which the more sophisticated form of quantum phenomenology developed originally in **(13)** is employed. The notion of physical equivalence above seems at first glance insufficiently restrictive, but it is shown in **(13)** that it implies that the two systems have corresponding pure states, corresponding observables have identical spectral values and probability distributions in given states, etc., so that the two systems are in fact in all observable respects the same.

The distinguished state of a clothed LBF is physically the vacuum state, and a more conventional formulation may be obtained from the correspondence between states and representations of operator algebras, which leads to the result (cf. I) that for any properly clothed LBF there is a concrete LBF with a distinguished vector v, whose transforms under the $e^{iR(z)}$ span the representation space $\mathfrak{K}$, and plays the role of the conventional physical vacuum state vector, in that the vacuum expectation value of the observable represented by the operator A is (Av, v). Conversely, such a concrete LBF with distinguished vector v gives rise to a properly clothed LBF associated with it in the foregoing fashion; and the concrete LBF-with-vector is uniquely determined, within unitary equivalence, by the clothed LBF.

The definition of regular state is rather technical, but is admissible from a purely physical point of view, since only those values of the state on the field observables are involved; and is justified by the existence of various equivalent formulations. It is surely reasonable from an empirical-physical viewpoint to require that for a physical state E, $E[e^{itR(z)}]$ be a continuous function of t, for any fixed vector z in $\mathfrak{H}$, and the regular states are precisely those that have this property, and in addition are determined in a natural way by the expectation values $E[e^{iR(z)}]$ for all z. Alternatively, a regular state is one whose restriction to any subsystem of a finite number of degrees of freedom (that is, the ring of operators generated by the $e^{iR(z)}$ as z ranges over a finite-dimensional subspace of $\mathfrak{H}$) is a normalizable (possibly mixed) state in essentially the conventional sense, that is, it has the form $E(A) = \text{tr}\,(AD)$ for some operator D of absolutely convergent trace. It should be noted that the present notion of regularity is more stringent than that employed in I, which permitted a theoretical generality that is physically not entirely appropriate. In fact Corollary 3.1 of I is correct (at least in proof) only with the present notion of regular state, by virtue of the possible existence of a pathological state other than the zero-interaction vacuum, which would agree with the zero-interaction vacuum on all sufficiently smooth observables, in particular on all observables that are uniform limits of products of those of the form $f(R(z))$, for some z and continuous function f that vanishes at infinity, but not on the weak limits of such. A state that is not determined by its values on such observables could be fully determined only through the use of infinite fields; it could never be obtained as a limit of states in cut-off theories. The restriction to regular states thus amounts to a type of universally covariant cut-off; in place of it one could substantially limit the observables to those "smooth" ones obtainable in the fashion indicated, which it can reasonably be argued are the only ones that can actually be observed even conceptually.

An interacting relativistic field on a particular space-like surface, in the interaction representation, gives a formal example of a linear boson field clothed by the physical vacuum; in conventional theory, this clothing degenerates as the space-like surface recedes or advances into the infinite past or future, and the *zero-interaction* LBF is obtained.

Formal equivalence of the present and the conventional formalisms. It must be shown how to define the conventional quantized field $\phi(x)$; that this satisfies the relevant partial differential equation and also the canonical commutation relations. To this end let δ_x' denote the projection of the delta-function at the point x on the manifold of solutions of the relevant partial differential equation, for example, δ_x' is the reciprocal Fourier transform of the function in momentum space that agrees with $e^{ik.x}$ on the mass hyperboloid and vanishes outside the hyperboloid, in the case of the Klein-Gordon equation. Then set $\phi(x) = R(\delta_x')$; δ_x' is an improper element of $\mathfrak{H}$, but this is inevitable since $\phi(x)$ is an improper operator. That ϕ as thus defined satisfies the given partial differential equations is a simple deduction from the Parseval formula for Fourier transforms. That the canonical commutation relations hold follows by substitution of δ_x' and $\delta_{x'}'$ for z and z' in the relation $[R(z), R(z')] = -2iB(z, z')$.

It must also be shown that conversely, from such a conventional quantized field ϕ, the present operators $R(z)$ can be constructed. If f is any smooth function on space-time that vanishes at infinity, the operator $\int\phi(x)f(x)d_4x$ is defined as $R(z)$, with z equal to the projection of f on the space of solutions of the relevant partial differential equation. For a non-scalar field this definition extends with the use of the Lorentz-invariant inner product in the finite-dimensional spin space for the field in question. Although the projection in question is singular as an operator in a Hilbert space, the z's may be analytically well defined for appropriate f, and thereby the $R(z)$ also.

3. Characterization and uniqueness of the generating functional. For any state E of a general linear boson field over $(\mathfrak{H}, B)$, the functional $\mu(z) = E[e^{iR(z)}]$ is well defined, and may be called the *generating functional* for the state. From it, the expectation values of arbitrary products of the field (at distinct points) may be obtained by differentiation, at least heuristically, when such expectation values are finite. For foundational purposes what is essential is

THEOREM 1. *A (complex-valued) function μ on $\mathfrak{H}$ is the generating functional of a regular state E of the general linear boson field over $(\mathfrak{H}, B)$ with non-degenerate B if and only if the restrictions of μ to arbitrary finite-dimensional subspaces of $\mathfrak{H}$ are continuous, $\mu(0) = 1$, and*

$$\sum_{j,k\in F} \mu(z_j - z_k)e^{iB(z_j,z_k)}\bar{\alpha}_k\alpha_j \geqslant 0$$

for arbitrary z_j in $\mathfrak{H}$ and complex numbers α_j, F being any finite index set. The functional μ uniquely determines E.

The "only if" part follows from the obvious fact that

$$\Big(\sum_j \alpha_j e^{iR(z_j)}\Big)^*\Big(\sum_j \alpha_j e^{iR(z_j)}\Big) \geqslant 0,$$

together with the relation (x). To prove the "if" part, let μ be given satisfying the stated conditions. Put K_0 for the set of all complex-valued functions on $\mathfrak{H}$ that vanish except at a finite set of points, with the inner product

$$(f, g) = \sum_{z, z'} f(z)\bar{g}(z')\mu(z - z')e^{iB(z,z')}$$

(f and g in $\mathfrak{K}_0$). This inner product is a positive semi-definite hermitian form, so the set of all f in $\mathfrak{K}_0$ with $(f,f) = 0$ forms a linear subspace $\mathfrak{K}_0'$ of $\mathfrak{K}_0$, and the quotient $\mathfrak{K}_0/\mathfrak{K}_0' = \mathfrak{K}_1$ (say) has canonically defined on it a strictly positive definite hermitian form:

$$(f', g') = (f, g);\ f' = f + \mathfrak{K}_0' \text{ and } g' = g + \mathfrak{K}_0'.$$

Now let $U_0(z')$, for z' in $\mathfrak{H}$, denote the transformation on $\mathfrak{K}_0$:

$$f(z) \to e^{iB(z',z)}f(z - z').$$

Then $U_0(z')$ is a linear operator with the inverse $U_0(-z')$. Also,

$$(U_0(z')f,\ U_0(z')g) = (f, g)$$

for arbitrary f, g, and z'. It follows that the map

$$U_1(z'): f' \to U_0(z')f + \mathfrak{K}_0$$

is a well-defined linear transformation on $\mathfrak{K}_1$, that it has the inverse $U_1(-z')$, and that

$$(U_1(z')f',\ U_1(z')g') = (f', g').$$

Hence $U_1(z)$ extends uniquely to a unitary transformation $U(z)$ on the completion $\mathfrak{K}$ of $\mathfrak{K}_1$. Now $U_0(z)U_0(z') = e^{iB(z,z')}U_0(z + z')$, from which it follows readily that

$$U(z)U(z') = e^{iB(z,z')}U(z + z').$$

In particular, $[U(tz): -\infty < t < \infty]$ is a one-parameter group of unitary operators in $\mathfrak{K}$. This one-parameter group is continuous; to show this it suffices, by the density of $\mathfrak{K}_1$ in $\mathfrak{K}$, to show that $(U(tz)f', g')$ is continuous for arbitrary f' and g' in $\mathfrak{K}_1$. But

$$(U(tz)f', g') = \sum_{u,u'} e^{iB(tz,u)}f(u - tz)\bar{g}(u')\mu(u - u')e^{iB(u,u')}.$$

Now if f has the values $\sigma_1, \ldots, \sigma_n$ at $v_1, \ldots, v_n$, respectively, and g has the values $\tau_1, \ldots, \tau_n$ at these points, and both functions vanish elsewhere, this sum is

$$\sum_{j,k} \mu(v_j + tz - v_k)\, e^{iB(tz, v_j + tz)}\, e^{iB(v_j + tz, v_k)}\sigma_j\bar{\tau}_k,$$

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which represents a continuous function of t by the assumption on μ. Thus the one-parameter group has a self-adjoint generator $R(z)$, and the $R(z)$ satisfy the relations (x).

If f_0 denotes the function defined by the equations $f_0(0) = 1, f_0(z) = 0$ for $z \neq 0$, then $(f_0', f_0') = 1$, and

$$(U(z)f_0', f_0') = \mu(z).$$

Thus setting

$$E(A) = (Af_0', f_0'),$$

E is regular and has characteristic functional μ.

To show that μ determines E uniquely, let E' be an arbitrary regular state with characteristic functional μ. Then E' is weakly continuous relative to the unit sphere of the ring $\mathfrak{A}_{\mathfrak{M}}$ generated by the $e^{iR(z)}$ for z in $\mathfrak{M}$, for all $\mathfrak{M}$ on which B is non-degenerate, and some concrete LBF. If E is similarly weakly continuous, etc., relative to the same concrete LBF, then the unicity follows from the circumstance that the finite linear combinations of the $e^{iR(z)}$, z in $\mathfrak{M}$, form an algebra $\mathfrak{A}_{0,\mathfrak{M}}$ whose weak closure is $\mathfrak{A}_{\mathfrak{M}}$; this implies that the unit sphere in $\mathfrak{A}_{0,\mathfrak{M}}$ is weakly dense in that of $\mathfrak{A}_{\mathfrak{M}}$, according to a variant of an argument due to von Neumann **(14)** (for full details cf. **(15)**). The assumed weak continuity of E and E' on $\mathfrak{A}_{\mathfrak{M}}$ relative to the unit sphere, together with their agreement on the unit sphere of $\mathfrak{A}_{0,\mathfrak{M}}$, then implies their equality.

To conclude the proof it therefore suffices to show the

LEMMA. *If a state E is weakly continuous on $\mathfrak{A}_{\mathfrak{M}}$ relative to the unit sphere for one concrete* LBF, *then the same is true for all concrete* LBFs.

To prove this, observe that as B is non-degenerate on $\mathfrak{M}$, co-ordinates may be chosen so that in $\mathfrak{M}$, $z = (x_1, \ldots, x_n) \oplus (y_1, \ldots, y_n)$ ($\mathfrak{M}$ being of dimension $2n$), and B has the form $B(z, z') = \sum_k (x_k y_k' - x_k' y_k)$. The relations (x) then imply those on which von Neumann's proof of the uniqueness of the Schrødinger operators is based, so that by his result, $\mathfrak{A}_{\mathfrak{M}}$ is, within multiplicity, and unitary equivalence, the conventional system of bounded observables in quantum mechanics for a particle with a $2n$-dimensional phase space. That is to say, $\mathfrak{A}_{\mathfrak{M}}$ is unitarily equivalent to an n-fold copy, for some finite or infinite cardinal number n, of the ring of operators generated by the $\exp(isq_j)$ and the $\exp(itp_k)$ $(-\infty < s, t < \infty; j, k = 1, 2, \ldots, n)$ in their action on the space $L_2(E_n)$ of all complex-valued square-integrable functions on Euclidean n-space. All that is relevant here is that the weak topology on the unit sphere of an operator ring is easily seen to be independent of the multiplicity n of its representation. Now for any two concrete LBFs, the resulting $\mathfrak{A}_{\mathfrak{M}}$ are

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within unitary equivalence multiples of the same ring of operators, and the lemma follows.

This concludes the proof of the theorem, but it also follows and seems worth pointing out explicitly that we have the

COROLLARY. *If $\mathfrak{M}$ is a linear subspace of $\mathfrak{H}$ on which B is non-degenerate, then the ring $\mathfrak{A}_{\mathfrak{M}}$ of all field observables based on M is a factor of type I_∞, and the restriction of E to $\mathfrak{A}_{\mathfrak{M}}$ has the form*

$$E(X) = \mathrm{tr}(XD_{\mathfrak{M}})$$

for some operator $D_{\mathfrak{M}}$ of absolutely convergent trace relative to $\mathfrak{A}_{\mathfrak{M}}$.

This is an immediate consequence of von Neumann's result used as above, together with the known form of the states of the ring of all bounded operators on a Hilbert space that are weakly continuous relative to the unit sphere. For this last result cf. for example (**16**, Theorem 14).

4. Some examples. The *generating functional of the zero-interaction linear boson field* over a (complex) Hilbert space may be computed explicitly as follows. If $C(z)$ denotes the creation operator for a particle with wave function z, as defined for example in (**5**), and $R(z)$ denotes the closure of $(C(z) + C(z)^*)/\sqrt{2}$, while E denotes the *zero-interaction* vacuum state, then the evaluation of $E[e^{iR(z)}]$ reduces to the case when $\mathfrak{H}$ is one-dimensional (see Cor. 3.6 of (**17**)). The representation of $e^{iR(z)}$ in terms of the one-parameter unitary groups generated by the canonical p's and q's in one dimension leads to a familiar type of integral involving the normal distribution, and ultimately to the result

$$\mu(z) = e^{-\frac{1}{4}|z|^2}$$

for the *zero-interaction* vacuum generating functional.

This may be used to obtain zero-interaction vacuum expectation values of arbitrary products of field values by noting that formally one has for any state E,

$$E[R(z_1)\dots R(z_n)] = i^{-n}\{\partial^n/\partial t_1 \dots \partial t_n\}E[e^{it_1R(z_1)}\dots e^{it_nR(z_n)}]|_{t_1=\dots=t_n=0},$$

while

$$e^{it_1R(z_1)}\dots e^{it_nR(z_n)} = e^{iR(t_1z_1+\dots+t_nz_n)+i\sum_{j<k} t_jt_kB(z_j,z_k)}$$

which has zero-interaction vacuum expectation value

$$\exp[-\tfrac{1}{2}\sum_{j,k} t_jt_k(z_j, z_k)].$$

Of course, in general the indicated derivative as well as the expectation value of the product of field operators will fail to exist. However, it is easy to justify in the bare field case the foregoing formal equality, and thereby obtain explicit expressions for the bare vacuum expectation values of products of fields. In the simplest non-vanishing case, there results the formula

$$E[R(z_1)R(z_2)] = \tfrac{1}{2}(z_1, z_2)$$

(also easily obtainable directly) having the conventional interpretation that the *zero-interaction* vacuum expectation value of the product of two field values is (1/2) the singular function providing the kernel for the symmetric form giving the Lorentz-invariant inner product, as is well known for special fields.

Turning to general states, if all the regular states could readily be expressed in terms of zero-interaction field quantites, the use of the general LBF might be avoidable. Theoretically this would be rather extraordinary, in view of the general situation in quantum fields, and in fact quite explicit examples can be constructed to show that this is not the case. To show simply *the existence of regular pure states that are not normalizable in the Fock-Cook representation*, that is, not of the form $E(A) = (Av, v)$ for some normalizable vector v, one may proceed as follows. Take the representation of the bare field in terms of the space $L_2(\mathfrak{H}_r, n)$ of square-integrable functionals over a real subspace $\mathfrak{H}_r$ of $\mathfrak{H}$ such that $\mathfrak{H} = \mathfrak{H}_r + i\mathfrak{H}_r$, as given in **(17)**. Let θ denote the automorphism of the algebra of field observables over $\mathfrak{H}$ taking each canonical p into $2p$ and each canonical q into $q/2$ (this exists by Theorem 2 of I), and define E_θ as the transform of the bare vacuum state under the induced action of θ, that is, $E_\theta(A) = E(A^\theta)$, where E denotes the bare vacuum state and $A \to A^\theta$ the action of θ. Then E_θ is evidently regular and pure, but may be seen to be non-normalizable as follows.

As a basis for an indirect argument, assume that $E_\theta(A) = (Au, u)$, for some u in $L_2(\mathfrak{H}_r)$. Then for arbitrary x in $\mathfrak{H}_r$, using the notation and Theorem 3 of **(17)**, with $c = \frac{1}{2}$,

$$E_\theta(e^{iQ(x)}) = \int_{\mathfrak{H}_r} e^{ix\cdot y}|u(y)|^2 dn(y),$$

and since $E_\theta(e^{iQ(x)}) = E(e^{iQ(x)/2})$, which is readily evaluated as

$$e^{-1/8|x|^2},$$

it suffices to show

$$\int_{\mathfrak{H}_r} e^{ix\cdot y} F(y) dn(y) \neq e^{-1/8|x|^2}$$

if F is an arbitrary element of $L_1(\mathfrak{H}_r, n)$. Such an F is an L_1-limit of polynomial functions, so it suffices to show that if $G(.)$ is any polynomial and

$$g(x) = \int e^{ix\cdot y} G(y) dn(y),$$

then

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$$\sigma = \sup_x |g(x) - e^{-1/8|x|^2}| \geqslant \delta,$$

where δ is > 0 and independent of G. But $g(x)$ has the form

$$g(x) = p(x)e^{-\frac{1}{4}|x|^2},$$

where p is a polynomial, based, say, on the finite-dimensional subspace $\mathfrak{M}$ of $\mathfrak{H}$. Now if x is in the orthocomplement of $\mathfrak{M}$, $p(x) = p(0)$, whence

$$\sigma \geqslant \inf_\alpha \sup_{0<r<\infty} |\alpha e^{-\frac{1}{4}|x|^2} - e^{-1/8|x|^2}|.$$

Simple calculus leads from this to the bound $\sigma \geqslant \frac{1}{8}$.

5. Special dynamics in terms of generating functionals. For the special but significant case of a Hamiltonian that is quadratic in the canonical variables, there is a remarkably simple formulation of the corresponding quantum dynamics. It gives the explicit time development of the generating functional, and hence of the system, in terms of the corresponding classical dynamics.

If a classical *motion with Hamiltonian quadratic in the canonical variables takes*

$$z \to V_t z (-\infty < t < \infty; t = \text{time})$$

then the corresponding quantum-mechanical *motion transforms generating functionals as follows:*

$$\mu(z) \to \mu(V_t z).$$

To prove this, note that if the Hamiltonian is quadratic, then the motion in phase space $\mathfrak{H}$ is linear. That is to say, for any fixed t, V_t is a non-singular transformation preserving the fundamental skew form $B(z, z') = \sum_k (p_k' q_k - p_k q_k')$, where $z = (p_1, \ldots, p_n) \oplus (q_1, \ldots, q_n)$, where n may be finite, or there may be infinitely many degrees of freedom, in which case the appropriate modifications in the notation are obvious (cf. **(18)** for the infinitesimal situation in a finite number of dimensions and **(19)** for the global situation in any number of dimensions). Now for any such transformation, the corresponding quantum-mechanical motion may be uniquely given by the condition that it transform $R(z)$ into $R(Tz)$ (cf. I), so that it has precisely the stated effect on the generating functional.

6. Restriction to regular observables instead of regular states. Instead of dealing with a restricted class of states of an extensive system of observables, one may contemplate dealing with all states of a restricted subsystem of observables. A general bounded self-adjoint operator is an observable only in a quite theoretical way; a class of operators slightly closer to actual measurements than those merely generated by the canonical variables would be those expressible explicitly in terms of them. It turns out that for systems of a finite number of degrees of freedom, which may be used to approximate infinite systems, there is a natural way to make this idea effective.

Definition 2. A *regular observable* over a finite-dimensional linear space $\mathfrak{H}$ with distinguished skew-symmetric form B, relative to a concrete LBF over $(\mathfrak{H}, B)$, is one of the form

$$\int_{\mathfrak{H}} e^{iR(z)} f(z) dz$$

or uniformly approximable by such. Here f is integrable over $\mathfrak{H}$, while dz is the element of measure determined by B.

The foregoing integral may be taken in the strong or weak operator topologies (in the sense of (**20**)). The resulting class of operators is unaffected by a change in the measure employed, within absolute continuity. It is not difficult to verify that the notion of regular observable is invariant under physical equivalence, and therefore may be applied to elements of the general LBF over a finite-dimensional space.

THEOREM 2. *The regular observables form a uniformly closed self-adjoint algebra, every state of which extends uniquely to a regular state of the general* LBF *over* $(\mathfrak{H}, B)$, $\mathfrak{H}$ *being finite-dimensional; and every regular state arises in this way.*

That the regular observables form an algebra follows without difficulty from the relations (x), the general properties of the integrals involved, and the Fubini theorem. Now for any state E,

$$\left| E\left[\int_{\mathfrak{H}} e^{iR(z)} f(z) dz\right]\right| \leqslant \int_{\mathfrak{H}} |f(z)| dz,$$

so that by the known form of the general continuous linear functional in L_1, these exists a bounded measurable function μ on $\mathfrak{H}$ such that

$$E\left[\int_{\mathfrak{H}} e^{iR(z)} f(z) dz\right] = \int \mu(z) f(z) dz.$$

By the positivity of E, μ satisfies the inequality

$$\iint \mu(z - z') e^{iB(z,z')} f(z) \bar{f}(z') dz\, dz' \geqslant 0.$$

If μ were continuous, this would imply that μ is a generating functional. Now it is well-known that a measurable integrally-positive-definite function on a locally compact group differs from a continuous positive definite function on the group on a null set (in the sense of Haar measure on the group). An argument similar to that involved in the proof of this result yields the corresponding result here, or this result may be derived from the theory of positive definite functions on groups; here we take the latter course, as this is illuminating in certain additional respects.

Let G be the group (cf. (**18**)) of all pairs (z, s) with z in $\mathfrak{H}$ and s real, and the multiplication

$$(z, s) \,.\, (z', s') = (z + z', s + s' + B(z, z')).$$

Then G is a Lie group with the obvious manifold structure. Set $\lambda(z, s) = \mu(z)e^{-is}$; then λ is a measurable and integrally positive definite function on G, as is readily verified. By the result cited, it differs on a null set from a continuous positive definite function. Now $\mathfrak{H}$ may be identified with the subset of G consisting of all elements of the form $(z, 0)$, the map $z \rightarrow (z, 0)$ being a homeomorphism into. It follows that μ differs on a null set in $\mathfrak{H}$ from a continuous function μ'.

Now μ' will satisfy the positivity condition given in Theorem 1, as follows from the integral positivity condition by a simple approximation argument. Since $(\mu'(z_j - z_k)e^{iB(z_j,z_k)}; j, k = 1, \ldots, n)$ is a positive semi-definite hermitian matrix for arbitrary $z_1, \ldots, z_n$, and μ' cannot vanish identically since $E \neq 0$, $\mu(0)$ must be positive. Setting $\mu'' = \mu'/\mu'(0)$, μ'' satisfies all the conditions of Theorem 1, so there exists a regular state E' of the general LBF, say $\mathfrak{A}$, over $(\mathfrak{H}, B)$, whose generating functional is μ''. Now the unit sphere of the algebra $\mathfrak{R}$ of regular observables is weakly dense in that of $\mathfrak{A}$, by the result cited above, together with the easily established fact that the weak closure of $\mathfrak{R}$ is $\mathfrak{A}$. In particular,

$$\sup_{X \in \mathfrak{A},\, |X| \leqslant 1} E'(X^*X) = \sup_{X \in \mathfrak{R},\, |X| \leqslant 1} E\,(X^*X)$$

which shows, since E' is a state of $\mathfrak{A}$, that the right side of the foregoing equality has the value unity. On the other hand,

$$E'\left[\int_{\mathfrak{H}} e^{iR(z)} f(z)dz\right] = (\mu'(0))^{-1}\int_{\mathfrak{H}} \mu'(z)f(z)dz$$

for arbitrary continuous f vanishing outside a compact set, as the integrals may then be taken in the Riemann sense; and by a simple approximation argument, it results that $E'(X) = (\mu'(0))^{-1}E(X)$ for arbitrary X in $\mathfrak{R}$. Now as E is a state of $\mathfrak{R}$,

$$\sup_{X \in \mathfrak{R},\, |X|=1} E(X^*X) = 1,$$

and it follows that $\mu'(0) = 1$.

Thus there exists a regular state E' of the full general LBF extending the given state E of the regular observables. That E' is unique follows from the density of the unit sphere of $\mathfrak{R}$ in that of $\mathfrak{A}$. Conversely, if E' is a given regular state of $\mathfrak{A}$, its restriction to $\mathfrak{R}$ is a positive linear functional which by the argument just given must have unit norm, and so be a state E.

It should be remarked that the connection with the theory of positive definite functions can be misleading, if not utilized with care. For example, when $B = 0$, the condition of Theorem 1 becomes ordinary positive definiteness (apart from the normalization), but the theorem is then irremediably false, as it asserts essentially that an arbitrary positive definite function is the Fourier-Stieltjes transform of an *absolutely continuous* measure. The introduction of a non-degenerate B thus has roughly the qualitative effect of eliminating the possible discontinuous and continuous but singular parts of the associated state.

7. Possible further developments. A number of problems emerge from the foregoing work. Some typical ones of interest are as follows.

1. The *zero-interaction* vacuum is the only regular state of the general LBF over a complex Hilbert space $\mathfrak{H}$ that is invariant under all unitary operators on $\mathfrak{H}$. Presumably it is, more cogently, the only such state invariant under a physically relevant representation of the Lorentz group, say, that associated with a relativistic particle of integral spin. For a particle of positive mass (= minimum proper value of the infinitesimal generator of translations it time), it is clear that there exist no normalizable states in the Fock-Cook representation other than the zero-interaction vacuum that are invariant, but it remains to be proved that this is the case for all regular states. In the vanishing mass case the situation is less clear, and correspondingly more interesting.

2. It seems probable that any symplectic transformation in a complex Hilbert space $\mathfrak{H}$ (that is, a real-linear transformation leaving invariant the imaginary part of the inner product) will effect a transformation of the zero-interaction vacuum of the general LBF over $\mathfrak{H}$ into a state that is not normalizable in the Fock-Cook representation, except when the transformation is unitary; this would generalize the example given above of such a state.*

3. When $\mathfrak{H}$ is finite-dimensional, $\mu(z)$ is essentially the Fourier transform of Wigner's quasi-probability distribution (**(21)**; cf. also **(22)** and the literature cited there, especially the paper by Moyal). Now a classical motion takes a z into a z', while a quantum-mechanical one takes a generating function μ into another μ'. The determination of the precise relation between $\mu'(z)$ and $\mu(z')$, shown above to be identical in the case of a quadratic Hamiltonian, is connected with the problems of interpretation considered by Wigner and later authors. Although it seems fairly clear that no exact result is to be hoped for, even a simple approximate relation between the two functions might well be quite useful.

4. The difficulty forming the basis of the preceding problem also suggests the more extensive question of the extent to which a theory similar to the present one, but covariant under the entire group of classical contact transformations rather than merely the symplectic group, can be set up. In such a theory the analogue to the smoothed field operators $R(z)$ would perhaps be a function $R(Z)$ defined for infinitesimal contact transformations Z, and satisfying in place of the commutation relations involved above, the relations

$$[R(Z), R(Z')] = R([Z, Z']) + \Omega\,(Z, Z'),$$

where Ω denotes the fundamental second-order differential form on $\mathfrak{H}$ (that is, the well-known form $\sum_k dp_k dq_k$ in the case of a finite number of degrees of freedom, while in the infinite case it is an analogous form determined by the field commutators). The main difficulty here is not the presence of the term

**Remark added in proof.* This result has been established in the meantime by David Shale.

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$R([Z, Z'])$, which was absent above because Z and Z' were essentially infinitesimal translations in $\mathfrak{H}$ and so had vanishing commutator, but rather the circumstance that the $\Omega\ (Z, Z')$ are not constant numbers, but scalar functions on $\mathfrak{H}$, the multiplications by which naturally do not commute with the Z's.

A possible way around this difficulty is the employment of a suitable analogue to the group G employed above, such as the group whose Lie algebra (that is, associated infinitesimal group) consists of all pairs (Z, f), where Z is an infinitesimal classical contact transformation and f is a function on phase space, with the commutation relations

$$[(Z,f),\ (Z',f')] = ([Z, Z'],\ Zf' - Z'f + \Omega\ (Z, Z')).$$

That these define a Lie algebra (that is, notably that the Jacobi conditions hold) follows from the fact that Ω is a closed form, using the expression for the derivative of a form in terms of the form itself, together with brackets of vector fields and the operations of the vector fields on values of the form (cf. **(23)**, § 1). A pair such as (Z, f) may be interpreted as the generator of a contact transformation in the tangent bundle of the phase space $\mathfrak{H}$, a construction that has been suggested in another form and connection in **(24)** for the finite-dimensional case, and which leads to difficulties of interpretation as pointed out there. On the other hand, the linearity of $\mathfrak{H}$ has ceased to play a role; the same construction can be made for any manifold $\mathfrak{H}$ (endowed with a suitable form Ω, determined in physics from the equations defining $\mathfrak{H}$). The approach therefore opens up a possible way of quantizing non-linear systems covariantly with respect to the group of all classical contact transformations.

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