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ON ALEXANDROFF BASE COMPACTIFICATIONS

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In [**13**] we characterized the completely regular Hausdorff spaces as the class of spaces whose topology is generated by an Alexandroff base. A space may have more than one Alexandroff base and each such base $\mathscr{A}$ determines a Hausdorff compactification $\alpha_{\mathscr{A}}X$. It was shown in [**13**] that each Wallman compactification $\omega_{\mathscr{Z}}X$ where $\mathscr{Z}$ is a normal base for X (see [**3**] for appropriate definitions) is obtainable as an Alexandroff base compactification. Recent works ([**1**; **4**; **7**; **8**, and **9**]) have shown that many compactifications are Wallman and consequently of the type $\alpha_{\mathscr{A}}X$. It is possible that all compact Hausdorff extensions can be obtained this way but we have not been able to settle this question.

The purpose of the present paper is to characterize those compactifications which are Alexandroff base. We shall do this by relating the base $\mathscr{A}$ to the proximity and uniformity aspects of the compactification it determines. In addition, we give a characterization directly in terms of the embedding of X in one of its compactifications; this yields necessary and sufficient conditions for the space $\alpha_{\mathscr{A}}X$ to be Wallman and also simplifies some previous results. We begin by recalling the necessary results from [**13**].

Definition. Let $\mathscr{A}$ be a family of subsets of a set X. For $A, B \subset X$ we define $A < B$ (rel $\mathscr{A}$) if and only if there exist $G, H \in \mathscr{A}$ with $A \subset G$, $X - B \subset H$, and $G \cap H = \emptyset$.

The relation $A < B$ (rel $\mathscr{A}$) is read "A is well-inside B relative to $\mathscr{A}$". When no confusion can result, reference to the family $\mathscr{A}$ will be dropped.

Definition. An *Alexandroff base* for a space X is a base, $\mathscr{A}$, for the open subsets of X satisfying:

(1) $\mathscr{A}$ is closed under finite unions and intersections. (i.e. $\mathscr{A}$ is a ring of sets)

(2) If $p \in G \in \mathscr{A}$, then $p \in H < G$ for some $H \in \mathscr{A}$.

(3) If $G, H \in \mathscr{A}$ with $G < H$, then $G < U < H$ for some $U \in \mathscr{A}$.

By a *regular* $\mathscr{A}$-filter we mean a non-void set $\sigma \subset \mathscr{A}$ satisfying: (1) no member of σ is empty. (2) σ is closed under finite intersections. (3) if $G \in \sigma$ and $H \in \mathscr{A}$ with $G \subset H$, then $H \in \sigma$. (4) If $G \in \sigma$, then $H < G$ for some $H \in \sigma$. A maximal regular $\mathscr{A}$-filter is called an $\mathscr{A}$-*cluster* and the set of all $\mathscr{A}$-clusters is designated by $\alpha_{\mathscr{A}}X$ or merely αX when the base reference is unnecessary.

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THEOREM 1. *A space X is a completely regular Hausdorff space if and only if its topology is generated by an Alexandroff base. Moreover, if $\mathscr{A}$ is such a base for Hausdorff X, then for each $G \in \mathscr{A}$, define $G^* = \{\sigma \in \alpha X : G \in \sigma\}$; $\{G^* : G \in \mathscr{A}\}$ is then a base for a compact Hausdorff topology on αX which contains a dense subspace homeomorphic with X.*

Proof. The proof can be found in [**13**]. We point out that if $\mathscr{A}^*$ denotes the ring in αX generated by $\{G^* : G \in \mathscr{A}\}$, then $\mathscr{A}^*$ is an Alexandroff base for αX and for $U, V \in \mathscr{A}^*$ we have $U < V$ (rel $\mathscr{A}^*$) if and only if $U \cap X < V \cap X$ (rel $\mathscr{A}$); moreover the trace in X of $\mathscr{A}^*$ is precisely $\mathscr{A}$.

1. Proximities. It is well-known (Smirnov [**6**]) that there is a one-to-one correspondence between the class of Hausdorff compactifications of a completely regular space and the class of proximities compatible with the topology of the space. Our immediate aim is to establish the exact relationship between the Alexandroff base $\mathscr{A}$ and the proximity on X associated with $\alpha_{\mathscr{A}}X$.

By a (*separated*) *proximity* on X is meant a relation δ between the subsets of X satisfying:

(P1) $A\delta B$ implies $B\delta A$.

(P2) $A\delta(B \cup C)$ if and only if $A\delta B$ or $A\delta C$.

(P3) $\{a\}\ \delta\ \{b\}$ if and only if $a = b$.

(P4) $\emptyset \not\delta X$.

(P5) $A \not\delta B$ implies there exist $C, D \subset X$ so that $C \cup D = X$, $A \not\delta C$ and $B \not\delta D$.

Following Smirnov one defines $A \ll B$ if and only if $A \not\delta (X - B)$; the relation $\ll$ is called the *subordination* associated with δ. A proximity is *compatible* with a topology if the topology consists precisely of those sets $G \subset X$ for which $p \in G$ implies $\{p\} \ll G$. The subordination relation satisfies:

(S1) $A \ll B$ implies $A \subset B$.

(S2) $A \ll (B \cap C)$ if and only if $A \ll B$ and $A \ll C$.

(S3) $a \neq b$ implies $\{a\} \ll X - \{b\}$.

(S4) $\emptyset \ll A$ for every $A \subset X$.

(S5) $A \ll B$ implies $A \ll C \ll B$ for some $C \subset X$.

(S1) thru (S5) are sufficient to characterize the subordination of a proximity and can be used as an alternative approach to proximity structures; we shall find this approach more convenient for our purpose.

A cover, β, in a proximity space is called a *p-cover* (see Engelking [**2**]) if and only if there is a cover α so that $A \in \alpha$ implies $A \ll B$ for some $B \in \beta$. If all the elements of a cover belong to a certain family $\mathscr{A}$, then we call the cover an $\mathscr{A}$-cover. In particular, we introduce the concept of a *regular* $\mathscr{A}$-cover.

Definition. Let $\mathscr{A}$ be a family of subsets of X. An $\mathscr{A}$-cover β is called *regular* with respect to $\mathscr{A}$ if and only if there is an $\mathscr{A}$-cover α so that $G \in \alpha$ implies $G < H$ (rel $\mathscr{A}$) for some $H \in \beta$. α is then said to *well-refine* β.

Definition. Let δ be a proximity on X and $\ll$ its associated subordination. By a *base* for δ we mean a ring, $\mathscr{A}$, of open subsets of X satisfying:

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(1) If $G, H \in \mathscr{A}$, then $G < H$ if and only if $G \ll H$.

(2) If $A, B \subset X$, then $A \ll B$ if and only if there are $G, H \in \mathscr{A}$ with $A \subset G < H \subset B$.

For any covering γ of X and any set $A \subset X$, we use the notation $\gamma^*(A)$ to represent $\bigcup \{G \in \gamma : G \cap A \neq \emptyset\}$. The next theorem provides great utility in dealing with proximity bases.

THEOREM 2. *Let $\mathscr{A}$ be a base for the proximity δ on X. Then for subsets A and B of X, one has the following mutually equivalent statements:*

(1) *$A \ll B$ if and only if $A \subset G \ll H \subset B$ for some $G, H \in \mathscr{A}$.*

(2) *$A \ll B$ if and only if $A \ll G \ll B$ for some $G \in \mathscr{A}$.*

(3) *$A \ll B$ if and only if $A \ll G \ll H \ll B$ for some $G, H \in \mathscr{A}$.*

(4) *$A \ll B$ if and only if $\gamma^*(A) \ll B$ for some finite regular $\mathscr{A}$-cover γ.*

(5) *$A\delta B$ if and only if every finite regular $\mathscr{A}$-cover contains a member which meets both A and B.*

(6) *$A\delta B$ if and only if each $G \in \mathscr{A}$ either $A\delta G$ or $B\delta X - G$.*

(7) *$A\delta B$ if and only if A and B meet every member of σ for some $\sigma \in \alpha_{\mathscr{A}}X$.*

(8) *$A\not\delta B$ if and only if $A \subset G$, $B \subset H$, $G\not\delta H$ for some $G, H \in \mathscr{A}$.*

(9) *$A\not\delta B$ if and only if $A \ll G$, $B \ll H$, $G\not\delta H$ for some $G, H \in \mathscr{A}$.*

(10) *$A\not\delta B$ if and only if there is a finite regular $\mathscr{A}$-cover no member of which meets both A and B.*

(11) *$A\not\delta B$ if and only if there is a regular $\mathscr{A}$-cover $\{G, H\}$ with $A\not\delta G$ and $B\not\delta H$.*

(12) *$A\not\delta B$ if and only if there is a finite regular $\mathscr{A}$-cover γ so that $\gamma^*(A)\not\delta\gamma^*(B)$.*

(13) *$A\not\delta B$ if and only if there are $G_i \in \mathscr{A}$ $(i = 1, 2, 3, 4)$ with $A \subset G_1 < G_2$, $B \subset G_3 < G_4$ and $G_2 \cap G_4 = \emptyset$.*

Proof. The proof of this theorem, although tedious, follows directly from the definitions given and the properties (P1) thru (P5) and (S1) thru (S5) so we shall not present it here. We remark, however, that if γ is any finite $\mathscr{A}$-cover satisfying (4) then there is a finite regular $\mathscr{A}$-cover also satisfying (4). In addition, condition (13) leads to a neat rephrasing of the definition of a proximity base by noting that two members of the base are far apart if and only if they are well-inside disjoint members of the base; hence, two arbitrary subsets are far apart if and only if they are separated by members of the base which are far apart.

LEMMA 1. *Let Y be a compact Hausdorff space and $\mathscr{A}$ an Alexandroff base for Y. Then for any $G, H \in \mathscr{A}$ the following are equivalent:* (i) $G < H$. (ii) $\bar{G} \subset H$. (iii) $G \ll H$.

Proof. If $G < H$, let $U, V \in \mathscr{A}$ so that $G \subset U$, $Y - H \subset V$, and $U \cap V = \emptyset$. Then $G \subset U \subset Y - V \subset H$ and since $Y - V$ is closed $\bar{G} \subset H$.

If $\bar{G} \subset H$, then $\bar{G}$ and $Y - H$ are disjoint closed sets in a compact space; such a space has a unique proximity structure defined by $A\not\delta B$ if and only if $\bar{A} \cap \bar{B} = \emptyset$ (see Smirnov [**6**, page 11],). Thus $G\not\delta Y - H$ and it follows that $G \ll H$.

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Finally, if $G \ll H$ then $\bar{G}$ and $Y - H$ are disjoint compact subsets in a Hausdorff space and can therefore be separated by disjoint members of any basic ring of open sets; thus, $G < H$.

THEOREM 3. *Let $\mathscr{A}$ be an Alexandroff base for X and δ the proximity on X associated with the compactification $\alpha_{\mathscr{A}}X$. Then $\mathscr{A}$ is a proximity base for δ.*

Proof. Let $G, H \in \mathscr{A}$ with $G < H$; for some $U \in \mathscr{A}$ we have $G < U < H$. By lemma 4, page 367 of [**13**], $\bar{G} \subset U^*$ (closure in αX); moreover, for some $V \in \mathscr{A}$, $X - H \subset V$ and $U \cap V = \emptyset$. Thus $U^* \cap V^* = \emptyset$ and we now see that $V^* \subset \alpha X - U^*$, hence $\overline{(X - H)} \subset \alpha X - U^*$, therefore $\bar{G} \cap \overline{(X - H)} = \emptyset$ so $G \ll H$. Conversely, if $G \ll H$, then $\bar{G} \cap \overline{(X - H)} = \emptyset$ (closures in αX), thus there are $U, V \in \mathscr{A}^*$ with $G \subset \bar{G} \subset U \subset \bar{U} \subset V \subset \alpha X - \overline{(X - H)}$. By Lemma 1, $U < V$ (rel $\mathscr{A}^*$) and from our observation in the proof of Theorem 1, $G \subset U \cap X < V \cap X \subset H$ (rel $\mathscr{A}$), thus $G < H$.

Now if $A, B \subset X$ with $A \ll B$ then $\bar{A} \cap \overline{(X - B)} = \emptyset$ (closures in αX) so there are sets $U, V \in \mathscr{A}^*$ with $\bar{A} \subset U \subset \bar{U} \subset V \subset \alpha X - \overline{(X - B)}$; again by Lemma 1 and our observation in Theorem 1, this yields $A \subset U \cap X < V \cap X \subset B$. Since $\mathscr{A}$ is a ring of open sets the proof is complete.

THEOREM 4. *If $\mathscr{A}$ is a base for the proximity δ on X then $\mathscr{A}$ is an Alexandroff base on X and δ is exactly the proximity on X associated with $\alpha_{\mathscr{A}}X$.*

Proof. $\mathscr{A}$ is a ring of open sets by definition. If U is any open subset of X and $p \in U$ then $\{p\} \ll U$ so there are sets $G, H \in \mathscr{A}$ with $p \in G < H \subset U$; this shows that $\mathscr{A}$ is a topological base and also satisfies condition (2) of the definition of Alexandroff base. That $\mathscr{A}$ is densely ordered by the well-inside relation follows at once from condition (2) of Theorem 2 and the fact that $<$ and $\ll$ agree on $\mathscr{A}$; thus $\mathscr{A}$ is an Alexandroff base.

To see that δ is the proximity on X associated with $\alpha_{\mathscr{A}}X$ we employ (7) of Theorem 2 which shows that $A\delta B$ if and only if $\bar{A} \cap \bar{B} \neq \emptyset$ (closure in $\alpha_{\mathscr{A}}X$).

By combining the results of Theorems 3 and 4 we arrive at the following characterization of Alexandroff base compactifications:

(I) *A compactification, Y, of X is an Alexandroff base compactification of X if and only if the proximity on X induced by Y has a base.*

2. Uniformities. We now focus our attention on the relationship between Alexandroff bases and precompact separated uniformities (see e.g. Engelking [**2**] for basic definitions). A (*separated*) *uniformity* on a set X is a filter, $\mathscr{U}$, on $X \times X$ satisfying: (i) $E^{-1} \in \mathscr{U}$ whenever $E \in \mathscr{U}$. (ii) $\bigcap \mathscr{U} = \Delta$ (Δ denotes the diagonal in $X \times X$). (iii) Whenever $F \in \mathscr{U}$, then $E \circ E \subset F$ for some $E \in \mathscr{U}$. We shall drop the adjective "separated" throughout.

For $A \subset X$ and $E \in \mathscr{U}$ let $E(A) = \{x \in X : (a, x) \in E$ for some $a \in A\}$. A *uniform cover* (relative to $\mathscr{U}$) of X is any cover which is refined by a cover of the form $\{E(x) : x \in X\}$ for some $E \in \mathscr{U}$; the family of uniform covers will be denoted by $\mathscr{C}$ and satisfies:

(UC1) If β is refined by some $\alpha \in \mathscr{C}$, then $\beta \in \mathscr{C}$.

(UC2) If $\alpha, \beta \in \mathscr{C}$, then some $\gamma \in \mathscr{C}$ refines both α and β.

(UC3) Every cover in $\mathscr{C}$ is star-refined by a cover in $\mathscr{C}$.

(UC4) Whenever $x \neq y$, there is a cover in $\mathscr{C}$ no member of which contains both x and y.

Any class of covers satisfying (UC1) thru (UC4) is the class of uniform covers of a uniformity on X. A uniformity is *precompact* if and only if every uniform cover has a finite subcover and it is known (e.g. [**5**]) there is a one-to-one correspondence between compactifications of a space and the precompact uniformities compatible with the topology of the space. Thus there is a bijective correspondence between precompact uniformities and proximities. A direct connection is given by any of the following where $\mathscr{U}$ is a precompact uniformity and δ its associated proximity:

(UP1) $A\delta B$ if and only if $E(A) \cap E(B) \neq \emptyset$ for every $E \in \mathscr{U}$.

(UP2) $A\delta B$ if and only if $E(A) \cap B \neq \emptyset$ for every $E \in \mathscr{U}$.

(UP3) $A \not\delta B$ if and only if $E \cap (A \times B) = \emptyset$ for some $E \in \mathscr{U}$.

(UP4) $A \not\delta B$ if and only if $\gamma^*(A) \cap \gamma^*(B) = \emptyset$ for some uniform cover γ.

(UP5) $A \ll B$ if and only if $E(A) \subset B$ for some $E \in \mathscr{U}$.

(UP6) $A \ll B$ if and only if $\gamma^*(A) \subset B$ for some uniform cover γ.

Definition. Let $\mathscr{C}$ be a precompact uniformity (in the covering sense) on X. A ring of open sets, $\mathscr{A}$, is a *base* for $\mathscr{C}$ if and only if

(1) if $G, H \in \mathscr{A}$ with $G < H$, then for some finite regular $\mathscr{A}$-cover β, $\beta^*(G) \subset H$; and

(2) $\gamma \in \mathscr{C}$ if and only if γ is refined by a finite regular $\mathscr{A}$-cover.

THEOREM 5. *If $\mathscr{A}$ is a base for the uniformity $\mathscr{C}$ on X and δ is the proximity on X associated with $\mathscr{C}$, then $\mathscr{A}$ is a proximity base for δ.*

Proof. By (2) in the above definition, every finite regular $\mathscr{A}$-cover is uniform, thus if $G, H \in \mathscr{A}$ with $G < H$, then condition (UP6) yields $G \ll H$. Now if A, $B \subset X$ with $A \ll B$, let $D \subset X$ so that $A \ll D \ll B$. By (UP6), let $\gamma_1, \gamma_2 \in \mathscr{C}$ with $\gamma_1^*(A) \subset D$ and $\gamma_2^*(D) \subset B$. Now (2) yields the existence of finite regular $\mathscr{A}$-covers β_1 and β_2 refining γ_1 and γ_2 respectively; let α_1 and α_2 be finite $\mathscr{A}$-covers which well-refine β_1 and β_2 respectively. We then have $A \subset \alpha_1^*(A) < \beta_1^*(A) \subset \gamma_1^*(A) \subset D \subset \alpha_2^*(D) < \beta_2^*(D) \subset \gamma_2^*(D) \subset B$. Since $\mathscr{A}$ is a ring and α_1 and α_2 are finite, this yields $A \subset \alpha_1^*(A) < \alpha_2^*(D) \subset B$. Hence $\mathscr{A}$ is a proximity base.

THEOREM 6. *Let $\mathscr{A}$ be a base for the proximity δ on X and let $\mathscr{C}$ denote the precompact covering uniformity on X associated with δ. Then $\mathscr{A}$ is a uniform base for $\mathscr{C}$.*

Proof. $\mathscr{C}$ is precisely the class of covers of X which are refined by the finite

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p-covers of δ (see Engelking [**2**]); hence, if $G, H \in \mathscr{A}$ with $G < H$ then $G \ll H$. Thus for some $\gamma \in \mathscr{C}$, $\gamma^*(G) \subset H$. Now let $c_1 = \{A_i : 1 \leqq i \leqq n\}$ and $c_2 = \{B_i : 1 \leqq i \leqq n\}$ so that $A_i \ll B_i$ for each i and c_2 refines γ. Since $\mathscr{A}$ is a proximity base there are sets $G_i, H_i \in \mathscr{A}$ with $A_i \subset G_i < H_i \subset B_i$ for each i. The family $\beta = \{H_i : 1 \leqq i \leqq n\}$ is thus seen to be a finite regular $\mathscr{A}$-cover with $\beta^*(G) \subset H$. This argument also shows that each cover in $\mathscr{C}$ is refined by a finite regular $\mathscr{A}$-cover, hence $\mathscr{A}$ is a uniform base for $\mathscr{C}$.

Theorems 5 and 6 show the equivalence of proximity bases and uniform bases and we therefore have a characterization of Alexandroff base compactifications in terms of the associated precompact uniformity:

(II) *A compactification Y of X is an Alexandroff base compactification if and only if the precompact uniformity on X induced by Y has a base.*

3. Traces and envelopes. While the characterizations derived in §1 and §2 are useful, it is often convenient to have a direct connection between a given compactification, Y, of X and Alexandroff bases in X. We provide such a connection in this section.

Definition. Let X be a dense subset of Y and G an open set in X; the set $Y - \overline{(X - G)}$ (closure in Y) will be denoted by $Y(G)$ and, following Tamano [**10**], called the *envelope* of G in Y.

It is easily shown that $Y(G \cap H) = Y(G) \cap Y(H)$ and that $Y(G)$ is the largest open subset of Y whose trace in X is G.

LEMMA 2. *Let Y be a compactification of X and $\mathscr{B}$ a ring of open sets which is a base for Y. Then the ring generated by $\{Y(G \cap X) : G \in \mathscr{B}\}$ is also a base for Y.*

This result can be found in [**12**] and we have recently discovered that it had previously been shown by F. J. Wagner in [**11**].

Proof. Let U be an open set in Y and take $y \in Y$. Then there are sets G and H in $\mathscr{B}$ satisfying $y \in H \subset \bar{H} \subset G \subset U$; we claim that $Y(H \cap X) \subset G$. By the definition of $Y(H \cap X)$ this containment is clearly equivalent with $Y - G \subset \overline{X - H}$. Now if $z \in Y - G$, then $z \in Y - \bar{H}$ and for any open set O with $z \in O$, $O \cap Y - \bar{H}$ is a nonvoid open set in Y and consequently $O \cap (Y - \bar{H}) \cap X \neq \emptyset$. Since $X - \bar{H} \subset X - H$, it follows that $O \cap (X - H) \neq \emptyset$ thus $z \in \overline{X - H}$.

The lemma shows that a basic ring in a compact Hausdorff space may always be taken as the ring generated by the envelopes of its traces in a dense subset. We shall say that such a ring is *envelope generated.* In addition, Lemma 1 shows that any basic ring of open sets in a compact space is an Alexandroff base.

THEOREM 6. *Let Y be a compactification of X and $\mathscr{A}^*$ be an Alexandroff base in Y. Let $\mathscr{A}$ denote the trace of $\mathscr{A}^*$ in X and assume that $\mathscr{A}^*$ is the ring generated* *by* $\{Y(G) : G \in \mathscr{A}\}$ *and suppose that for all* $G, H \in \mathscr{A}$ $G < H$ *if and only if* $\overline{Y(G)} \subset Y(H)$. *Then* $\mathscr{A}$ *is an Alexandroff base in* X *and* Y *and* $\alpha_{\mathscr{A}}X$ *are equal compactifications of* X.

Proof. The trace of a basic ring is a basic ring. Now if $p \in G \in \mathscr{A}$, then $p \in G \subset Y(G) \in \mathscr{A}^*$ so for some $H \in \mathscr{A}^*, p \in H < Y(G)$ (rel $\mathscr{A}^*$). It follows that $p \in H \cap X < G$ (rel $\mathscr{A}$). Now suppose $G, H \in \mathscr{A}$ with $G < H$ (rel $\mathscr{A}$) Then $\overline{Y(G)} \subset Y(H)$ and by Lemma 1, $Y(G) < Y(H)$ (rel $\mathscr{A}^*$), hence there is $U \in \mathscr{A}^*$ with $Y(G) < U < Y(H)$ (rel $\mathscr{A}^*$) and we have $G < U \cap X < H$ (rel $\mathscr{A}$) with $U \cap X \in \mathscr{A}$. This shows that $\mathscr{A}$ is an Alexandroff base.

Now for any A and B in X we have $\bar{A}^Y \cap \bar{B}^Y = \emptyset$ if and only if there are sets $G, H, U, V \in \mathscr{A}^*$ with $\bar{A}^Y \subset G < H$ (rel $\mathscr{A}^*$), $\bar{B}^Y \subset U < V$ (rel $\mathscr{A}^*$), and $H \cap V = \emptyset$ if and only if $A \subset G \cap X < H \cap X$ (rel $\mathscr{A}$), $B \subset U \cap X < V \cap X$ (rel $\mathscr{A}$) and $(H \cap X) \cap (V \cap X) = \emptyset$ if and only if $\bar{A} \cap \bar{B} = \emptyset$ (closure in αX). This shows that Y and $\alpha_{\mathscr{A}}X$ are the same compactification since they induce the same proximity on X.

To yield a characterization, we observe that if $\mathscr{A}$ is an Alexandroff base on X, then for each $G \in \mathscr{A}$, $G^* = \alpha X(G)$; clearly $G^* \subset \alpha X(G)$. For the reverse containment, let $\sigma \in \alpha_{\mathscr{A}}X - \overline{(X - G)}$, then $\sigma \notin \overline{X - G}$ so $H \cap (X - G) = \emptyset$ for some $H \in \sigma$ but this means $H \subset G$ so $G \in \sigma$ hence $\sigma \in G^*$.

Combining this observation with the above theorem yields:

THEOREM 7. *Y is an Alexandroff base compactification of X if and only if Y has an envelope generated basic ring, $\mathscr{A}^*$, satisfying $G < H$ (rel $\mathscr{A}$) if and only if $\overline{Y(G)} \subset Y(H)$.*

It is clear from the arguments presented that we also have

COROLLARY 1. *If Y is a compactification of X and $\mathscr{A}$ is an Alexandroff base in X then Y and $\alpha_{\mathscr{A}}X$ are equal compactifications if and only if, $\mathscr{A}^*$, the ring in Y generated by $\{Y(G) : G \in \mathscr{A}\}$ is a base for Y such that $G < H$ if and only if $\overline{Y(G)} \subset Y(H)$.*

The theorem also provides the following useful results.

COROLLARY 2. *Let Y be a compactification of X which has an envelope generated basic ring $\mathscr{A}^*$. If $G, H \in \mathscr{A}^*$ with $X = (G \cap X) \cup (H \cap X)$ implies $Y = Y(G \cap X) \cup Y(H \cap X)$, then Y is an Alexandroff base compactification of X.*

Proof. Let $\mathscr{A}$ be the trace of $\mathscr{A}^*$ in X. If $G < H$ (rel $\mathscr{A}$), let $G \subset U$, $X - H \subset V$, and $U \cap V = \emptyset$. Then $\{H, V\}$ covers X so $\{Y(H), Y(V)\}$ covers Y and it follows that $Y(G) \subset Y(U)$, $Y - \underline{Y(H)} \subset Y(V)$, and $Y(U) \cap Y(V) = \emptyset$ so $Y(G) < Y(H)$ (rel $\mathscr{A}^*$). Therefore $\overline{Y(G)} \subset Y(H)$.

COROLLARY 3. *If a compact Hausdorff space Y has a basic ring of regular open sets, then Y is an Alexandroff base compactification of each of its dense subspaces.*

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Proof. Let $\mathscr{A}$ be such a ring, then for any dense subset X and any $U \in \mathscr{A}$, $U = Y(U \cap X)$. Moreover, if $X = (X \cap G) \cup (X \cap H)$ for any $G, H \in \mathscr{A}$, then $Y = \overline{(X \cap G)} \cup \overline{(X \cap H)} = \bar{G} \cup \bar{H}$. But $G \cup H \in \mathscr{A}$, therefore $G \cup H$ is regular; since Y is also a regular open set $Y = \overline{G \cup H}$ if and only if $Y = G \cup H$. Now just apply Corollary 2.

4. Conclusion. Corollary 3 is not surprising, for spaces possessing basic rings of regular open sets are regular Wallman spaces (see Steiner [**7**]) hence are also Alexandroff base compactifications of each dense subspace. The proof presented here is somewhat easier than Steiner's.

Corollary 2 is more interesting since it in fact characterizes those Alexandroff base compactifications which are Wallman; this can be seen by noting that the complements of the members of a base satisfying the condition in Corollary 2 form a normal base with the trace property given by Steiner in [**7**] and conversely. The condition in Theorem 7, however, seems to be weaker than the requirement that base members cover X if and only if their envelopes cover Y. One sees the difference more clearly by comparing our notion of a uniform base with the uniform base defined in [**9**]. Our definition requires only the regular finite covers by base members to be uniform whereas in [**9**] it is necessary that all finite base covers be uniform. It may be therefore that there are Alexandroff base compactifications which are not Wallman.

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