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The roles of nonlinear diffusion, haptotaxis and ECM remodelling in determining the global solvability of a cancer invasion model

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In this paper, we consider the following PDE-ODE system modelling cancer invasion with slow diffusion and ECM remodelling,

$$\begin{cases} u_t = \Delta u^m - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla \omega) + \mu u(1 - u - \omega), \\ v_t = \Delta v + u - v, \\ \omega_t = -v\omega + \eta \omega (1 - u - \omega). \end{cases}$$

For the special case $\eta = 0$, fruitful results have been achieved since Tao and Winkler's work in 2011. However, there is no any progress for the general case $\eta > 0$ in the past ten years. In this paper, we analysed some commonly used research methods when $\eta = 0$, and found that these methods are completely unsuitable for situations where $\eta > 0$. By introducing some new forms of functionals, we reconstruct the relationship between the haptotactic term and the nonlinear diffusion term, and ultimately prove the global existence of weak solutions. This result improves and perfects a series of works previously presented in the literature.

Keywords: haptotaxis; nonlinear diffusion; ECM remodelling; global solution

2020 *Mathematics Subject Classification:* 35M13; 35K65; 92C17

1. Introduction

As a major disease threatening human life safety, cancer has always been the focus of social attention. The combination of mathematical models and various medical data will help people turn cancer research into a quantitative and predictable science. The mathematical modelling of tumour growth and its theoretical research have always been the concern of biologists and mathematicians. Generally speaking, the growth of a tumour usually goes through two stages: vascular phase and vascularization. That is, if there are no blood vessels to provide enough nutrients, the tumour will stop growing when it grows to a certain size. In order to continue to grow, some tumours will secrete a chemical substance, which is called urokinase Plasminogen Activator (uPA), to recruit vascular factors, and to build a vascular network around themselves to meet the growing nutritional needs. At the same

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time, blood vessels also provide a way for tumours to travel to other parts of the body, which is called metastasis. In recent years, some reaction-diffusion-taxis models are proposed to characterize the process of tumour growth and invasion [**1**, **3**, **4**, **17**]. In particular, the following model proposed by Chaplain and Lolas [**4**] has attracted extensive attention of mathematicians in recent years.

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla\omega) + \mu u(1-u-\omega), \\ v_t = \Delta v + u - v, \\ \omega_t = -v\omega + \eta\omega(1-u-\omega), \end{cases} \tag{1.1}$$

which describes the invasion and diffusion process of solid tumours during the vascular growth stage. In this system, u, v, ω represent the cancer cell density, urokinase Plasminogen Activator (uPA) concentration, and the extracellular matrix (ECM) density respectively. $D(u)$ denotes the mobility of cells, the positive constants χ, ξ denote the uPA-mediated chemotaxis, ECM-mediated haptotaxis coefficients respectively, $\mu u(1-u-\omega)$ denotes the proliferation or death of cancer cells. In the second equation, $+u$ represents the production of uPA by cancer cells, $-v$ represents decay of uPA. As for the ECM density ω, it is known that it does not diffuse, therefore, one can omit the random motion. $-v\omega$ denotes the degradation of ECM, $\eta\omega(1-u-\omega)$ represents the remodelling of ECM components.

In the past ten years, this model has been widely studied. When $D(u)\equiv 1$, $\eta = 0$, we refer to [**5**, **13**, **18**, **23**] for the study of global existence of bounded solutions, and refer to [**6**, **20**, **22**] for the study of large time behaviour. While, if the remodelling of ECM is considered, the calculation of the haptotaxis term will bring some essential difficulties. Therefore, the following transformation is introduced to avoid the estimation of the haptotaxis term

$$\rho = ue^{-\xi\omega},$$

and the first equation of (1.1) is transformed into

$$e^{\xi\omega}\rho_t - \mathrm{div}(e^{\xi\omega}\nabla\rho) = -\chi\mathrm{div}(\rho e^{\xi\omega}\nabla v) + \xi\rho v\omega e^{\xi\omega} + \rho e^{\xi\omega}(1-\omega-\rho e^{\xi\omega})(\mu - \xi\eta\omega). \tag{1.2}$$

Noticing that ω is bounded, then $\|u\|_{L^p} \sim \|\rho\|_{L^p}$. Using this method, the global existence and boundedness of classical solution for any initial datum in two dimensional space is proved [**10**, **15**]. While in three dimensional space, only a small global classical solution is established [**11**, **16**]. In fact, even for the haptotaxis-only system, the global bounded solution for any μ, $\eta > 0$ in dimension 3 is still open.

Considering the mechanism of avoiding crowding between individual cells, the mobility of cells should be related to density, so a model with nonlinear diffusion is also very practical, and the most representative one is the porous medium diffusion model. For example, in [**1**, **2**], the authors use porous medium diffusion to describe the process of trophoblast cells invading the uterine tissue, that is,

$$\begin{cases} u_t = D_u\nabla\cdot(u^{m-1}\nabla u) - \xi\nabla\cdot(u\nabla v) + k_1 u(1-u-\omega), \\ n_t = \Delta n + k_2 nu(1-u) - k_3 nv, \\ v_t = D_v\Delta v + k_4 n\omega - k_3 nv, \\ \omega_t = -\delta n\omega + \eta\omega(1-u-\omega). \end{cases} \tag{1.3}$$

In 2011, Tao and Winkler [19] first studied the global solvability of (1.1) by ignoring the remodelling of ECM, that is, for

$$\begin{cases} u_t = \Delta u^m - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla\omega) + \mu u(1-u-\omega), \\ v_t = \Delta v + u - v, \\ \omega_t = -v\omega, \end{cases} \tag{1.4}$$

they proved the global existence of weak solutions when

$$m > m^* = \begin{cases} \dfrac{2N^2+4N-4}{N(N+4)}, & \text{if } N \leqslant 8, \\ \dfrac{2N^2+3N+2-\sqrt{8N(N+1)}}{N(N+1)}, & \text{if } N \geqslant 9. \end{cases}$$

After that, the global existence of weak solutions to this problem began to be widely studied, see for example [7, 12, 21, 24] etc. For this case, from the third equation of (1.4), one observed that

$$\omega = \omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s}. \tag{1.5}$$

At this time, there are two commonly used methods to assist in dealing with the haptotactic term.

Method I: A direct calculation from (1.5) leads to

$$|\nabla\omega(x,t)|^2 \leqslant 2|\nabla\omega_0(x)|^2 e^{-2\int_0^t v(x,s)\mathrm{d}s} + 2|w_0(x)|^2 e^{-2\int_0^t v(x,s)\mathrm{d}s}\Big|\int_0^t \nabla v(x,s)\mathrm{d}s\Big|^2.$$

Therefore,

$$\begin{aligned} \int_\Omega |\nabla\omega(x,t)|^2\mathrm{d}x &\leqslant C + C\int_\Omega e^{-2\int_0^t v(x,s)\mathrm{d}s}\Big|\int_0^t \nabla v(x,s)\mathrm{d}s\Big|^2\mathrm{d}x \\ &= C - \frac{C}{2}\int_\Omega \nabla e^{-2\int_0^t v(x,s)\mathrm{d}s}\cdot\Big(\int_0^t \nabla v(x,s)\mathrm{d}s\Big)\mathrm{d}x \\ &= C + \frac{C}{2}\int_\Omega e^{-2\int_0^t v(x,s)\mathrm{d}s}\cdot\Big(\int_0^t \Delta v(x,s)\mathrm{d}s\Big)\mathrm{d}x \\ &= C + \frac{C}{2}\int_\Omega e^{-2\int_0^t v(x,s)\mathrm{d}s}\cdot\Big(\int_0^t (v_t+v-u)\mathrm{d}s\Big)\mathrm{d}x \\ &\leqslant C + \frac{C}{2}\int_\Omega e^{-2\int_0^t v(x,s)\mathrm{d}s}\cdot\Big(v(x,t)+\int_0^t v(x,s)\mathrm{d}s\Big)\mathrm{d}x \\ &\leqslant \tilde{C} + \frac{C}{2}\int_\Omega v(x,t)\mathrm{d}x. \end{aligned}$$

Similarly, one can use iteration to sequentially obtain $\|\nabla\omega(\cdot,t)\|_{L^4}$, $\|\nabla\omega(\cdot,t)\|_{L^6}$, $\|\nabla\omega(\cdot,t)\|_{L^{2n}}\cdots$.

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Method II: A direct calculation from (1.5) leads to

$$\begin{aligned}
-\Delta\omega(x,t) &= -\Delta\omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s} + 2e^{-\int_0^t v(x,s)\mathrm{d}s}\nabla\omega_0 \cdot \int_0^t \nabla v(x,s)\mathrm{d}s \\
&\quad - \omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s}\left|\int_0^t \nabla v(x,s)\mathrm{d}s\right|^2 + \omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s}\int_0^t \Delta v(x,s)\mathrm{d}s \\
&= -\Delta\omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s} - \omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s}\left(\frac{\nabla\omega_0}{\omega_0} - \int_0^t \nabla v(x,s)\mathrm{d}s\right)^2 \\
&\quad + \frac{|\nabla\omega_0|^2}{\omega_0} e^{-\int_0^t v(x,s)\mathrm{d}s} + \omega_0 e^{-\int_0^t v(x,s)\mathrm{d}s}\int_0^t (v_t + v - u)\mathrm{d}s \\
&\leqslant \|\Delta\omega_0\|_{L^\infty} + 4\|\nabla\sqrt{\omega_0}\|_{L^\infty}^2 + \|\omega_0\|_{L^\infty}(1 + v(x,t)) \\
&\leqslant \|\omega_0\|_{L^\infty} v(x,t) + K
\end{aligned}$$

where K only depends on ω_0 [**19**]. Therefore, the above two methods can make the haptotactic term easy to handle, however, the two methods are obviously not suitable for situations where $\eta > 0$. In fact, if the remodelling of ECM is considered, that is, the third equation of (1.4) is replaced with

$$\omega_t = -v\omega + \eta\omega(1 - u - \omega),$$

the problem becomes much more complex since the regularity of ω completely depends on the regularity of u. Actually, due to that the ODE has no regularization effect, the regularity of ω is greatly reduced by $-\eta u\omega$ in remodelling term. Therefore, although a lot of results for the case $\eta = 0$ have been achieved in the past ten years, there is no any progress in the case of $\eta > 0$.

In the present paper, we consider the initial and boundary value problem for the system (1.1) with nonlinear diffusion, that is

$$\begin{cases}
u_t = \Delta u^m - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla\omega) + \mu u(1 - u - \omega), & \text{in } Q, \\
v_t = \Delta v + u - v, & \text{in } Q, \\
\omega_t = -v\omega + \eta\omega(1 - u - \omega), & \text{in } Q, \\
\left.\dfrac{\partial u^m}{\partial \mathbf{n}} - \chi u\dfrac{\partial v}{\partial \mathbf{n}} - \xi u\dfrac{\partial \omega}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0,\ \frac{\partial v}{\partial \mathbf{n}}\big|_{\partial\Omega} = 0, & \\
u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad \omega(x,0) = \omega_0(x), & x \in \Omega,
\end{cases} \tag{1.6}$$

where $Q = \Omega \times \mathbb{R}^+$, $m > 1$, $\Omega \subset \mathbb{R}^N$ ($N \geqslant 2$) is a smooth bounded domain. χ, ξ, μ, η are positive constants.

According to previous research experience for some chemotaxis models, the larger m is, the easier it is to prove the uniform boundedness of the solution, but this experience seems to be invalid for the current model. On the one hand, due to the existence of remodelling term $\eta\omega(1 - u - \omega)$, the methods (Method I and Method II) used in the study of equations (1.4) is completely invalid. On the other hand, due to the lack of a good coupling structure similar to the linear diffusion case

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between the diffusion term and the haptotactic term, the method used for the linear diffusion case (see (1.2)) is also not suitable for the current nonlinear diffusion model. Therefore, we have to try new methods. Fortunately, we found a complex relationship between the haptotactic term and the nonlinear diffusion term. In addition to constructing classical functional such as

$$F(u,\omega) = \int_\Omega \left(u \ln u + \frac{|\nabla \omega|^2}{\omega} \right) \mathrm{d}x,$$

we also construct the following new form of functionals

$$\begin{aligned} F_1(u,\omega) &= \int_\Omega \left(\frac{1}{m-1}(u+\varepsilon)^m - \xi u \omega \right) \mathrm{d}x, \\ G_k(u,\omega) &= \left(\frac{m}{m-1} \right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \\ &\quad \times \int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1} \omega^i \mathrm{d}x, \end{aligned}$$

which allows us to reconstruct the relationship between nonlinear diffusion term and haptotactic term. In particular, by taking the derivative of the above functional, we can use the diffusion term to neutralize the 'bad' effect brought by the haptotactic term. That is (see lemma 3.6)

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} G_k(u,\omega) + (2k+1) \int_\Omega (u+\varepsilon) \left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega \right)^{2k} \\ &\quad \times \left| \nabla \left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega \right) \right|^2 \mathrm{d}x \cdots . \end{aligned}$$

However, this also reduces the original good effect of the diffusion term. Fortunately, we have the logistic term, with the help of this term, we can finally get the L^p-norm ($\forall\ p > 1$)uniform estimation of u and the $W^{1,\infty}$-norm estimation of v. However, it is hard to get the L^∞-norm estimation of u since the Moser's iteration technique is no longer applicable due to the diffusion term is not working.

Although the haptotactic term caused the main difficulties in the proof, we prove that the chemotaxis still plays a leading role in determining whether the solution can exist globally. More precisely, we prove that the weak solution will exists globally for any $m > \frac{2N}{N+2}$ if $\chi > 0$, while for the haptotaxis-only model, that is the case $\chi = 0$. We prove that for any $m > 1$, the solution always exists globally. This work obviously improves the results in references[**12**, **19**, **21**, **24**], in which, only the special case $\eta = 0$ is studied.

In what follows, we give the assumptions of this paper.

$$(H) \qquad \begin{cases} u_0, \omega_0 \in L^\infty(\Omega), \nabla\sqrt{\omega_0} \in L^2(\Omega), \\ \quad v_0 \in W^{1,\infty}(\Omega) \cap W^{2,p}(\Omega), & \text{for any } p > 1, \\ u_0, v_0, \omega_0 \geqslant 0 & . \end{cases}$$

We state the main results as follows.

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THEOREM 1.1. *Assume that (H) holds, $\mu > 0$ and $m > \frac{2N}{N+2}$. Then the problem (1.6) admits a global nonnegative weak solution $(u,\, v,\, \omega) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$. In particular, $u,\, v,\, \omega$ are bounded uniformly in the following sense*

$$\sup_{0<t<\infty} (\|v(\cdot,t)\|_{W^{1,\infty}} + \|\omega(\cdot,t)\|_{L^\infty} + \|u(\cdot,t)\|_{L^r}) \leqslant C_r, \quad \textit{for any } r > 1\ , \tag{1.7}$$

where C_r only depends on $r,\, \chi,\, \xi,\, m,\, \mu,\, \eta,\, u_0,\, v_0,\, \omega_0,\, \Omega$. Here

$$\begin{aligned}
\mathcal{X}_1 = \Big\{ & u \in L^\infty(\mathbb{R}^+; L^r(\Omega)) \quad \textit{for any } r > 1; \nabla u^m \in L^p_{loc}(\mathbb{R}^+; L^p(\Omega)), \\
& \quad u_t \in L^2_{loc}(\mathbb{R}^+; W^{-1,p}(\Omega)), \\
& \textit{for any } p \in (1,2), \nabla u^{m-\frac{1}{2}} \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega)) \Big\}; \\
\mathcal{X}_2 = \{ & v \in L^\infty(\mathbb{R}^+; W^{1,\infty}(\Omega)); v_t, D^2 v \in L^p_{loc}(\mathbb{R}^+; L^p(\Omega)) \textit{ for any } p > 1\}; \\
\mathcal{X}_3 = \{ & \omega \in L^\infty(Q); \nabla\sqrt{\omega} \in L^\infty_{loc}(\mathbb{R}^+; L^2(\Omega)), \sqrt{u}\nabla\sqrt{\omega} \in L^2_{loc}(\mathbb{R}^+; L^2(\Omega)), \\
& \omega_t \in L^r_{loc}(\mathbb{R}^+; L^r(\Omega)) \quad \textit{for any } r > 1\}.
\end{aligned}$$

Although the haptotactic term caused the main difficulties in the proof, the requirement of this index $m > \frac{N+2}{2N}$ was still caused by the chemotactic term. In fact, if $\chi = 0$, we have the global existence result for any slow diffusion case $m > 1$.

THEOREM 1.2. *Assume $\chi = 0$, $\mu > 0$, $m > 1$ and (H) holds. Then the problem (1.6) admits a global nonnegative weak solution $(u,\, v,\, \omega) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$. In particular, $u,\, v,\, \omega$ are bounded uniformly in the following sense*

$$\sup_{0<t<\infty} (\|v(\cdot,t)\|_{W^{1,\infty}} + \|\omega(\cdot,t)\|_{L^\infty} + \|u(\cdot,t)\|_{L^r}) \leqslant C_r, \quad \textit{for any } r > 1, \tag{1.8}$$

where C_r only depends on $r,\, \chi,\, \xi,\, m,\, \mu,\, \eta,\, u_0,\, v_0,\, \omega_0,\, \Omega$.

REMARK 1.3. As for the stability of the equilibrium points, the local stability of equilibrium point (1, 1, 0) is easy to obtain by linear stability analysis, see for example [**11**]. However, the research on global asymptotic stability of the equilibrium point is a very challenging problem. In fact, this problem is still unsolved even for linear diffusion case $m = 1$.

2. Preliminaries

We first give the definition of weak solutions.

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Definition 2.1. *(u, v, ω) is called a nonnegative weak solution of (1.6), if $(u, v, \omega) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_2$, such that for any $T > 0$,*

$$\begin{aligned}
&-\iint_{Q_T} u\varphi_{1t}\mathrm{d}x\mathrm{d}t - \int_\Omega u(x,0)\varphi_1(x,0)\mathrm{d}x + \iint_{Q_T}(\nabla u^m - \chi u\nabla v - \xi u\nabla\omega)\nabla\varphi_1\mathrm{d}x\mathrm{d}t\\
&\quad = \mu\iint_{Q_T} u(1-u-\omega)\varphi_1\mathrm{d}x\mathrm{d}t,\\
&-\iint_{Q_T} v\varphi_{2t}\mathrm{d}x\mathrm{d}t - \int_\Omega v(x,0)\varphi_2(x,0)\mathrm{d}x + \iint_{Q_T}\nabla v\nabla\varphi_2\mathrm{d}x\mathrm{d}t\\
&\quad = \iint_{Q_T}(u-v)\varphi_2\mathrm{d}x\mathrm{d}t,\\
&-\iint_{Q_T} \omega\varphi_{3t}\mathrm{d}x\mathrm{d}t - \int_\Omega \omega(x,0)\varphi_3(x,0)\mathrm{d}x + \iint_{Q_T} v\omega\varphi_3\mathrm{d}x\mathrm{d}t\\
&\quad = \eta\iint_{Q_T}\omega(1-u-\omega)\varphi_3\mathrm{d}x\mathrm{d}t,
\end{aligned}$$

for any $\varphi_1, \varphi_2, \varphi_3 \in C^\infty(\overline{Q}_T)$ with $\varphi_i(x, T) = 0$.

As a preparation, we introduce the following inequalities [**8**, **9**, **14**], which is useful in the calculation of energy estimation.

Lemma 2.2. *Assume that Ω is bounded with smooth boundary, and let $\omega \in C^2(\overline{\Omega})$ satisfies $\frac{\partial\omega}{\partial\nu}\Big|_{\partial\Omega} = 0$. Then we have*

$$\frac{\partial|\nabla\omega|^2}{\partial\nu} \leqslant 2\kappa|\nabla\omega|^2 \quad \text{on } \partial\Omega, \tag{2.1}$$

where $\kappa > 0$ is an upper bound for the curvatures of Ω. In particular, we have the following inequalities

$$\int_\Omega \frac{|\nabla\omega|^4}{\omega^3}\mathrm{d}x \leqslant (2+\sqrt{N})^2\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x, \tag{2.2}$$

$$\int_\Omega |\nabla\omega|^{r+2}\mathrm{d}x \leqslant (\sqrt{N}+r)^2\|\omega\|_{L^\infty}^2\int_\Omega|\nabla\omega|^{r-2}|D^2\omega|^2\mathrm{d}x, \quad \text{for any } r \geqslant 2, \tag{2.3}$$

and

$$\int_{\partial\Omega}\frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2\mathrm{d}S \leqslant \delta\int_\Omega\omega|D^2\ln\omega|^2\mathrm{d}x + C_\delta, \tag{2.4}$$

where $\delta > 0$ is an arbitrary small constant, and C_δ is a constant depending on δ.

Using Neumann heat semigroup theory, we prove the following lemma.

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LEMMA 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. Assume that $u \in L^q(\Omega \times (0, T))$, $v_0 \in W^{2,q}(\Omega)$ with $q \geqslant 2$. Then the following problem admits*

$$\begin{cases} v_t - \Delta v + v = u, & (x,t) \in \Omega \times (0,T) \\ \left.\dfrac{\partial v}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0 \\ v(x,0) = v_0(x), & x \in \Omega \end{cases}$$

admits a unique strong solution $v \in W_q^{2,1}(\Omega \times (0, T))$, such that for any $r < \frac{Nq}{(N+2-q)_+}$,

$$\sup_{t\in(0,T)} \|\nabla v(\cdot,t)\|_{L^r} \leqslant C_1 + C_2 \left(\sup_{t\in(\tau,T)} \int_{t-\tau}^{t} \|u(s)\|_{L^q}^q \mathrm{d}s \right)^{\frac{1}{q}}, \tag{2.5}$$

and

$$\sup_{t\in(\tau,T)} \int_{t-\tau}^{t} \|\nabla v\|_{L^{q+\frac{rq}{N}}}^{q+\frac{rq}{N}} \mathrm{d}s \leqslant C_3 \left(\sup_{t\in(\tau,T)} \int_{t-\tau}^{t} \|u(s)\|_{L^q}^q \mathrm{d}s \right)^{\frac{r}{N}+1} + C_4, \tag{2.6}$$

where $\tau = \min\left\{1, \frac{T}{2}\right\}$, $C_i (i = 1, 2, 3, 4)$ are constants depending only on v_0, Ω.

Proof. By Duhamel's principle, v can be expressed as follows

$$v = e^{-t} e^{t\Delta} v_0 + \int_0^t e^{-(t-s)} e^{(t-s)\Delta} u(s) \mathrm{d}s,$$

where $\{e^{t\Delta}\}_{t \geqslant 0}$ is the Neumann heat semigroup in Ω. For any $r \in (1, +\infty)$, we have

$$\begin{aligned} \|\nabla v(\cdot,t)\|_{L^r} &\leqslant e^{-t} \|\nabla v_0\|_{L^r} + \int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\frac{N}{2}(\frac{1}{q}-\frac{1}{r})-\frac{1}{2}}\right) \|u(s)\|_{L^q} \mathrm{d}s \\ &\leqslant e^{-t} \|\nabla v_0\|_{L^r} + \left(\int_0^t e^{-(t-s)} \left(1 + (t-s)^{-\left(\frac{N}{2}(\frac{1}{q}-\frac{1}{r})+\frac{1}{2}\right)\frac{q}{q-1}}\right) \mathrm{d}s \right)^{\frac{q-1}{q}} \\ &\quad \times \left(\int_0^t e^{-(t-s)} \|u(s)\|_{L^q}^q \mathrm{d}s \right)^{\frac{1}{q}} \\ &\leqslant e^{-t} \|\nabla v_0\|_{L^r} + \left(\int_0^t e^{-s} \left(1 + s^{-\left(\frac{N}{2}(\frac{1}{q}-\frac{1}{r})+\frac{1}{2}\right)\frac{q}{q-1}}\right) \mathrm{d}s \right)^{\frac{q-1}{q}} \\ &\quad \times \left(\int_0^t e^{-(t-s)} \|u(s)\|_{L^q}^q \mathrm{d}s \right)^{\frac{1}{q}}. \end{aligned}$$

A direct calculation gives

$$\left(\frac{N}{2}(\frac{1}{q} - \frac{1}{r}) + \frac{1}{2} \right) \frac{q}{q-1} < 1 \Leftrightarrow \frac{1}{r} > \frac{N+2-q}{Nq}.$$

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Then when $r < \frac{Nq}{(N+2-q)_+}$,

$$\|\nabla v(\cdot,t)\|_{L^r} \leqslant e^{-t}\|\nabla v_0\|_{L^r} + C\left(\int_0^t e^{-(t-s)}\|u(s)\|_{L^q}^q \mathrm{d}s\right)^{\frac{1}{q}}.$$

We also note that for any $t > 0$, there exists an integer N such that $t = N\tau + \sigma$ with $0 < \sigma < \tau$. Therefore, we have

$$\begin{aligned}\int_0^t e^{-(t-s)}\|u(s)\|_{L^q}^q \mathrm{d}s &= e^{-t}\left(\sum_{k=1}^{N}\int_{(k-1)\tau}^{k\tau} e^s\|u(s)\|_{L^q}^q \mathrm{d}s + \int_{N\tau}^{N\tau+\sigma} e^s\|u(s)\|_{L^q}^q \mathrm{d}s\right)\\ &\leqslant e^{-t}\left(\sum_{k=1}^{N} e^{k\tau}\int_{(k-1)\tau}^{k\tau}\|u(s)\|_{L^q}^q \mathrm{d}s + e^t\int_{N\tau}^{t}\|u(s)\|_{L^q}^q \mathrm{d}s\right)\\ &\leqslant \frac{e^{2\tau}}{e^\tau - 1}\sup_{\tau<a<t}\int_{a-\tau}^{a}\|u(s)\|_{L^q}^q \mathrm{d}s,\end{aligned}$$

which implies that (2.5). By L^p theory of linear parabolic equations, we obtain

$$\sup_{\tau<t<T_{\max}}\int_{t-\tau}^{t}\|D^2 v(\cdot,t)\|_{L^q}^q \leqslant C_3 + C_4\sup_{\tau<t<T_{\max}}\int_{t-\tau}^{t}\|u(s)\|_{L^q}^q \mathrm{d}s,\ \text{for any } q > 1. \tag{2.7}$$

From Gagliardo-Nirenberg interpolation inequality, we infer that

$$\|\nabla v\|_{L^{q+\frac{rq}{N}}}^{q+\frac{rq}{N}} \leqslant C_5\|\nabla v\|_{L^r}^{\frac{rq}{N}}\|D^2 v\|_{L^q}^q + C_6\|\nabla v\|_{L^r}^{q+\frac{rq}{N}}. \tag{2.8}$$

Using (2.5), (2.7) and (2.8), we derive (2.6). □

3. Uniform energy estimations and global solvability of chemotaxis-haptotaxis system

In order to prove the existence of the weak solution of problem (1.6), some prior estimates are necessary. Since problem (1.6) is degenerate at $u = 0$, in order to facilitate us to obtain various prior estimates later, we first use the standard method to build a framework, that is, consider its regularization problem

$$\begin{cases} u_t = m\nabla\cdot((u+\varepsilon)^{m-1}\nabla u) - \chi\nabla\cdot(u\nabla v) - \xi\nabla\cdot(u\nabla\omega) + \mu u(1-u-\omega), \quad \text{in } Q,\\ v_t = \Delta v + u - v, \quad \text{in } Q,\\ \omega_t = \varepsilon\Delta\omega - v\omega + \eta\omega(1-u-\omega),\ \text{in } Q,\\ \left.\dfrac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0,\ \left.\dfrac{\partial v}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0,\ \left.\dfrac{\partial \omega}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0,\\ u(x,0) = u_{\varepsilon 0}(x), \quad v(x,0) = v_{\varepsilon 0}(x), \quad \omega(x,0) = \omega_{\varepsilon 0}(x), \quad x\in\Omega, \end{cases} \tag{3.1}$$

for any $\varepsilon \in (0,\,1)$, and $(u_{\varepsilon 0},\, v_{\varepsilon 0},\, \omega_{\varepsilon 0})$ is smooth approximation of $(u_0,\, v_0,\, \omega_0)$ with $(u_{\varepsilon 0},\, v_{\varepsilon 0},\, \omega_{\varepsilon 0})$ sufficiently smooth. By standard fixed point method, it is easy to

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obtain the local existence of classical solutions to the above problem, see for example [19]. That is

LEMMA 3.1. *Assume B, $m > 1$, $\varepsilon \in (0, 1)$. Then there exists $T_{\max} \in (0, +\infty]$ such that the problem (3.1) admits a classical solution $(u_\varepsilon, v_\varepsilon, \omega_\varepsilon) \in C^{2,1}(Q_{T_{\max}}) \cap C^0(\overline{\Omega} \times [0, T_{\max}))$ with*

$$u_\varepsilon > 0, \quad v_\varepsilon > 0, \quad \omega_\varepsilon > 0 \quad \textit{for all } (x,t) \in \Omega \times (0, T_{\max}).$$

Moreover, we have the following dichotomy: Either $T_{\max} = \infty$, or

$$\limsup_{t \nearrow T_{\max}} (\|u_\varepsilon(\cdot, t)\|_{L^\infty} + \|v_\varepsilon\|_{W^{1,\infty}} + \|\omega_\varepsilon\|_{W^{1,\infty}}) = \infty.$$

Throughout this paper, we denote

$$\tau := \min\left\{1, \frac{T_{\max}}{2}\right\} \leqslant 1.$$

In what follows, we focus on the uniform energy estimates of $(u_\varepsilon, v_\varepsilon, \omega_\varepsilon)$. For simplicity, we omit the subscript ε of the approximate solutions $(u_\varepsilon, v_\varepsilon, \omega_\varepsilon)$ in the subsequent energy estimate calculations.

Firstly, it is easy to obtain the following estimates.

LEMMA 3.2. *Let (u, v, ω) be the classical solution of (3.1) in $[0, T_{\max})$. Then*

$$\sup_{t\in(0,T_{\max})} \|u(\cdot,t)\|_{L^1} + \mu \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^{t} \int_\Omega (u^2 + u\omega) \mathrm{d}x\mathrm{d}s \leqslant C_1, \tag{3.2}$$

$$\sup_{t\in(0,T_{\max})} \left(\|\omega\|_{L^\infty} + \|v(\cdot,t)\|_{H^1}^2\right) + \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^{t} \|v(\cdot,s)\|_{H^2}^2 \mathrm{d}s \leqslant C_2, \tag{3.3}$$

where the constants C_1, C_2 only depend on μ, u_0, v_0, ω_0 and Ω.

Proof. By comparison lemma, it is easy to see that

$$\omega \leqslant \max\{1, \|\omega_{\varepsilon 0}\|_{L^\infty}\} \leqslant \max\{1, 2\|\omega_0\|_{L^\infty}\}.$$

By direct integration over Ω for the first equation of (3.1), (3.2) is easy to be obtained. By L^2 theory of linear parabolic equations, (3.3) is arrived. □

LEMMA 3.3. *Let (u, v, ω) be the classical solution of (3.1) in $[0, T_{\max})$. Then for any $T < T_{\max}$,*

$$\begin{aligned}
&\sup_{0<t<T} \int_\Omega \left(u \ln u + \frac{|\nabla\omega|^2}{\omega}\right) \mathrm{d}x \\
&\quad + \int_0^T \int_\Omega \left(\frac{(u+\varepsilon)^{m-1}}{u} |\nabla u|^2 + \frac{|\nabla\omega|^2}{\omega}(u+v) + u^2 |\ln u|\right) \mathrm{d}x\mathrm{d}t \\
&\quad + \varepsilon \int_0^T \int_\Omega \omega |D^2 \ln\omega|^2 \mathrm{d}x\mathrm{d}t \leqslant C_T,
\end{aligned} \tag{3.4}$$

where C_T is independent of ε, it depends only on χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω, and T.

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In particular, we also have

$$\begin{aligned}
&\sup_{0<t<T_{\max}} \int_\Omega \left(u\ln u + \frac{|\nabla\omega|^2}{\omega} \right) \mathrm{d}x \\
&+ \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \int_\Omega \left(\frac{(u+\varepsilon)^{m-1}}{u}|\nabla u|^2 + \frac{|\nabla\omega|^2}{\omega}(u+v) + u^2|\ln u| \right) \mathrm{d}x\mathrm{d}t \\
&+ \varepsilon \sup_{t\in(\tau,T_{\max})} \int_{t-\tau}^t \int_\Omega \omega |D^2 \ln\omega|^2 \mathrm{d}x\mathrm{d}t \leqslant \frac{C}{\varepsilon},
\end{aligned} \tag{3.5}$$

where C is independent of ε and $T_{\max}$, it depends only on χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω.

It should be noted that the estimation (3.4) is time-dependent, but (3.5) is not time-dependent. Although (3.5) depends on ε, it is useful for us to obtain uniform estimations independent of time and ε.

Proof. By direct calculation,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla\omega|^2}{\omega}\mathrm{d}x &= \int_\Omega \frac{\nabla\omega}{\omega}\nabla\omega_t \mathrm{d}x - \frac{1}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega^2}\omega_t \mathrm{d}x \\
&= -\int_\Omega \omega_t \left(\frac{\Delta\omega}{\omega} - \frac{|\nabla\omega|^2}{\omega^2} \right) \mathrm{d}x - \frac{1}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega^2}\omega_t \mathrm{d}x \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega^2}\omega_t \mathrm{d}x - \int_\Omega \frac{\Delta\omega}{\omega}\omega_t \mathrm{d}x \\
&= \frac{1}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega^2}(\varepsilon\Delta\omega - v\omega + \eta\omega(1-u-\omega))\mathrm{d}x \\
&\quad - \int_\Omega \frac{\Delta\omega}{\omega}(\varepsilon\Delta\omega - v\omega + \eta\omega(1-u-\omega))\mathrm{d}x \\
&= \frac{\varepsilon}{2}\int_\Omega \left(\frac{|\nabla\omega|^2}{\omega^2}\Delta\omega - 2\frac{|\Delta\omega|^2}{\omega} \right) \mathrm{d}x - \int_\Omega \nabla\omega\nabla v \mathrm{d}x - \eta\int_\Omega \nabla\omega\nabla u \mathrm{d}x \\
&\quad + \frac{1}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}(\eta - 3\eta\omega - \eta u - v)\mathrm{d}x.
\end{aligned}$$

Similar to the proof of lemma 2.6 in [8], we have

$$\frac{\varepsilon}{2}\int_\Omega \left(\frac{|\nabla\omega|^2}{\omega^2}\Delta\omega - 2\frac{|\Delta\omega|^2}{\omega} \right) \mathrm{d}x = -\varepsilon\int_\Omega \omega|D^2\ln\omega|^2 \mathrm{d}x + \frac{\varepsilon}{2}\int_{\partial\Omega} \frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2 \mathrm{d}S.$$

Putting the above two equalities together, we get the following conclusion

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{|\nabla\omega|^2}{\omega}\mathrm{d}x + \varepsilon\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x + \frac{1}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}(3\eta\omega + \eta u + v)\mathrm{d}x \\
&\quad = -\int_\Omega \nabla\omega\nabla v\mathrm{d}x - \eta\int_\Omega \nabla\omega\nabla u\mathrm{d}x + \frac{\eta}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}\mathrm{d}x + \frac{\varepsilon}{2}\int_{\partial\Omega} \frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2\mathrm{d}S.
\end{aligned} \tag{3.6}$$

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Multiplying both sides of the first equation of (3.1) by $1+\ln u$, and integrating the resultant equation over Ω yields

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega u\ln u\mathrm{d}x+m\int_\Omega\frac{(u+\varepsilon)^{m-1}}{u}|\nabla u|^2\mathrm{d}x\\&\quad=\chi\int_\Omega\nabla v\nabla u\mathrm{d}x+\xi\int_\Omega\nabla\omega\nabla u\mathrm{d}x+\mu\int_\Omega u(1+\ln u)(1-u-\omega)\mathrm{d}x.\end{aligned}\tag{3.7}$$

The combination of (3.6) and (3.7) leads to

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\eta u\ln u+\frac{\xi}{2}\frac{|\nabla\omega|^2}{\omega}\right)\mathrm{d}x+\eta m\int_\Omega\frac{(u+\varepsilon)^{m-1}}{u}|\nabla u|^2\mathrm{d}x+\xi\varepsilon\int_\Omega\omega|D^2\ln\omega|^2\mathrm{d}x\\&+\frac{\xi}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}(3\eta\omega+\eta u+v)\mathrm{d}x+\mu\eta\int_\Omega u(1+\ln u)(u+\omega)\mathrm{d}x\\&\quad=\frac{\varepsilon\xi}{2}\int_{\partial\Omega}\frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2\mathrm{d}S+\frac{\xi\eta}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}\mathrm{d}x+\chi\eta\int_\Omega\nabla v\nabla u\mathrm{d}x\\&\qquad-\xi\int_\Omega\nabla\omega\nabla v\mathrm{d}x+\mu\eta\int_\Omega u(1+\ln u)\mathrm{d}x\\&\quad=\frac{\varepsilon\xi}{2}\int_{\partial\Omega}\frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2\mathrm{d}S+\frac{\xi\eta}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}\mathrm{d}x-\chi\eta\int_\Omega u\Delta v\mathrm{d}\\&\qquad+\xi\int_\Omega\omega\Delta v\mathrm{d}x+\mu\eta\int_\Omega u(1+\ln u)\\&\quad\leqslant\frac{\varepsilon\xi}{2}\int_{\partial\Omega}\frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2\mathrm{d}S+\frac{\xi\eta}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}\mathrm{d}x+\int_\Omega(\chi^2\eta^2u^2+\xi^2\omega^2\\&\qquad+|\Delta v|^2)\mathrm{d}x+\frac{\mu\eta}{4}\int_\Omega u^2\ln u\mathrm{d}x+C.\end{aligned}\tag{3.8}$$

Recalling (2.4), we have

$$\frac{\varepsilon\xi}{2}\int_{\partial\Omega}\frac{1}{\omega}\frac{\partial}{\partial\mathbf{n}}|\nabla\omega|^2\mathrm{d}S\leqslant\frac{\varepsilon\xi}{2}\int_\Omega\omega|D^2\ln\omega|^2\mathrm{d}x+C\varepsilon.\tag{3.9}$$

We substitute (3.9) into (3.8) to obtain

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\eta u\ln u+\frac{\xi}{2}\frac{|\nabla\omega|^2}{\omega}\right)\mathrm{d}x+\eta m\int_\Omega\frac{(u+\varepsilon)^{m-1}}{u}|\nabla u|^2\mathrm{d}x+\frac{\xi\varepsilon}{2}\int_\Omega\omega|D^2\ln\omega|^2\mathrm{d}x\\&+\frac{\xi}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}(3\eta\omega+\eta u+v)\mathrm{d}x+\frac{\mu\eta}{2}\int_\Omega u^2(1+|\ln u|)\mathrm{d}x\\&\quad\leqslant\frac{\xi\eta}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}\mathrm{d}x+\int_\Omega(\chi^2\eta^2u^2+\xi^2\omega^2+|\Delta v|^2)\mathrm{d}x+\hat{C}.\end{aligned}\tag{3.10}$$

Using (3.2) and (3.3), by direct calculation, we complete the proof of (3.4).

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On the other hand, we note that

$$\frac{\xi(\eta+1)}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}\mathrm{d}x = -\frac{\xi(\eta+1)}{2}\int_\Omega \omega\Delta\ln\omega\mathrm{d}x \leqslant \frac{\varepsilon\xi}{4}\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x + \frac{C_1}{\varepsilon}, \tag{3.11}$$

which together with (3.10) yields

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\eta u\ln u + \frac{\xi}{2}\frac{|\nabla\omega|^2}{\omega}\right)\mathrm{d}x + \eta m\int_\Omega \frac{(u+\varepsilon)^{m-1}}{u}|\nabla u|^2\mathrm{d}x + \frac{\xi\varepsilon}{2}\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x\\
&\quad + \frac{\xi}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}(3\eta\omega+\eta u+v+1)\mathrm{d}x + \frac{\mu\eta}{2}\int_\Omega u^2(1+|\ln u|)\mathrm{d}x\\
&\leqslant \frac{\xi(\eta+1)}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}\mathrm{d}x + \int_\Omega(\chi^2\eta^2u^2+\xi^2\omega^2+|\Delta v|^2)\mathrm{d}x + \hat{C}\\
&\leqslant \frac{\xi\varepsilon}{4}\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x + \int_\Omega(\chi^2\eta^2u^2+\xi^2\omega^2+|\Delta v|^2)\mathrm{d}x + \frac{\tilde{C}}{\varepsilon},
\end{aligned} \tag{3.12}$$

where $\tilde{C}$ is independent of ε. Denote

$$f(t) = \int_\Omega \left(\eta u\ln u + \frac{\xi}{2}\frac{|\nabla\omega|^2}{\omega}\right)\mathrm{d}x,$$

clearly,

$$\begin{aligned}
&\eta m\int_\Omega \frac{(u+\varepsilon)^{m-1}}{u}|\nabla u|^2\mathrm{d}x + \frac{\xi\varepsilon}{4}\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x\\
&\quad + \frac{\xi}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}(3\eta\omega+\eta u+v+1)\mathrm{d}x\\
&\quad + \frac{\mu\eta}{2}\int_\Omega u^2(1+|\ln u|)\mathrm{d}x \geqslant \frac{\min\{\mu,\xi\}}{4}f(t) - C.
\end{aligned}$$

Then (3.12) implies that

$$f'(t) + \frac{\min\{\mu,\xi\}}{4}f(t) \leqslant \int_\Omega(\chi^2\eta^2u^2+\xi^2\omega^2+|\Delta v|^2)\mathrm{d}x + \frac{\tilde{C}}{\varepsilon} + C.$$

Then the uniform boundedness of f is derived from (3.2) and (3.3). Furthermore, (3.5) is obtained by integrating (3.12) from $t-\tau$ to t. □

Next, our purpose is to continuously improve the regularity of the solution. In this process, the key is to deal with the haptotaxis term. We try to offset the influence of the haptotaxis term with the diffusion term. First, we can get the following lemma.

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LEMMA 3.4. *Let* (u, v, ω) *be the classical solution of* (3.1) *in* $[0, T_{\max})$, *and assume* $m > \frac{2N}{N+2}$. *Then*

$$\sup_{0<t<T_{\max}} \int_\Omega u^m \mathrm{d}x + \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega \Bigg((u+\varepsilon) \left| \nabla \left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega \right) \right|^2 + u^{m+1} \Bigg) \mathrm{d}x\mathrm{d}s \leqslant C, \tag{3.13}$$

where C *is independent of* ε *and* $T_{\max}$, *and it depends only on* χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω.

In addition, we also have

$$\int_0^T \int_\Omega (u+\varepsilon)^{2m-3} |\nabla u|^2 \mathrm{d}x\mathrm{d}t \leqslant C_T, \tag{3.14}$$

and for any $1 < p < 2$,

$$\int_0^T \int_\Omega |\nabla (u+\varepsilon)^m|^p \mathrm{d}x\mathrm{d}t \leqslant C_{Tp}, \tag{3.15}$$

where C_T, C_{Tp} *are independent of* ε, *and* C_T *depends on* T, C_{Tp} *depends on* T *and* p.

Proof. Noticing that

$$\begin{aligned} \omega |D^2 \ln \omega|^2 &= \frac{|D^2\omega|^2}{\omega} + \frac{|\nabla\omega|^4}{\omega^3} - 2\frac{\nabla\omega D^2\omega\nabla\omega}{\omega^2} \\ &\geqslant \frac{|D^2\omega|^2}{\omega} + \frac{|\nabla\omega|^4}{\omega^3} - \frac{1}{2}\frac{|D^2\omega|^2}{\omega} - 2\frac{|\nabla\omega|^4}{\omega^3}, \end{aligned}$$

that is

$$\frac{|D^2\omega|^2}{\omega} \leqslant 2\omega |D^2 \ln\omega|^2 + 2\frac{|\nabla\omega|^4}{\omega^3}.$$

Recalling (2.2), we conclude

$$\begin{aligned} \int_\Omega \frac{|D^2\omega|^2}{\omega}\mathrm{d}x &\leqslant 2\int_\Omega \omega|D^2\ln\omega|^2\mathrm{d}x + 2\int_\Omega \frac{|\nabla\omega|^4}{\omega^3}\mathrm{d}x \\ &\leqslant \left(2 + 2(2+\sqrt{N})^2\right) \int_\Omega \omega |D^2 \ln\omega|^2 \mathrm{d}x. \end{aligned} \tag{3.16}$$

Noticing that

$$\begin{aligned} &m\nabla\cdot((u+\varepsilon)^{m-1}\nabla u) - \xi\nabla\cdot(u\nabla\omega) \\ &= \nabla\cdot\left(\frac{m}{m-1}(u+\varepsilon)\nabla(u+\varepsilon)^{m-1} - \xi(u+\varepsilon)\nabla\omega\right) + \xi\varepsilon\Delta\omega \\ &= \nabla\cdot\left((u+\varepsilon)\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right) + \xi\varepsilon\Delta\omega, \end{aligned} \tag{3.17}$$

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multiplying both sides of the first equation of (3.1) by $\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega$, and using (3.17), (3.5) and lemma 3.2, we arrive at

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{m-1}(u+\varepsilon)^m-\xi u\omega\right)\mathrm{d}x+\int_\Omega(u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\
&\quad=-\varepsilon\xi\int_\Omega\nabla\omega\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\mathrm{d}x\\
&\qquad+\chi\int_\Omega u\nabla v\nabla\left(\frac{m}{m-1}u^{m-1}-\xi\omega\right)\mathrm{d}x\\
&\qquad+\int_\Omega\mu u(1-u-\omega)\left(\frac{m}{m-1}u^{m-1}-\xi\omega\right)\mathrm{d}x\\
&\qquad+\xi\int_\Omega u(-\varepsilon\Delta\omega+v\omega-\eta\omega(1-u-\omega))\mathrm{d}x\\
&\quad\leqslant\frac{\varepsilon\xi^2\|\omega\|_{L^\infty}}{2}\int_\Omega\frac{|\nabla\omega|^2}{\omega}\mathrm{d}x+\frac{\varepsilon}{2}\int_\Omega\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\
&\qquad+\frac{\chi^2}{2}\int_\Omega u|\nabla v|^2\mathrm{d}x+\frac{1}{2}\int_\Omega u\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\
&\qquad-\frac{3m\mu}{4(m-1)}\int_\Omega u^{m+1}\mathrm{d}x+\varepsilon^2\int_\Omega|\Delta\omega|^2\mathrm{d}x+C\\
&\quad\leqslant\frac{1}{2}\int_\Omega(u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\
&\qquad-\frac{m\mu}{2(m-1)}\int_\Omega u^{m+1}\mathrm{d}x+C\int_\Omega|\nabla v|^{\frac{2(m+1)}{m}}\mathrm{d}x+N\varepsilon^2\int_\Omega|D^2\omega|^2\mathrm{d}x+\tilde{C},
\end{aligned}$$

namely,

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{m-1}(u+\varepsilon)^m-\xi u\omega\right)\mathrm{d}x\\
&\quad+\frac{1}{2}\int_\Omega(u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\
&\quad+\frac{m\mu}{2(m-1)}\int_\Omega u^{m+1}\mathrm{d}x\leqslant C\int_\Omega|\nabla v|^{\frac{2(m+1)}{m}}\mathrm{d}x+N\varepsilon^2\int_\Omega|D^2\omega|^2\mathrm{d}x+\tilde{C}.
\end{aligned}\tag{3.18}$$

Taking advantage of Gagliardo-Nirenberg interpolation inequality and (3.3), we have

$$\begin{aligned}
\|\nabla v\|_{L^{\frac{(N+2)(m+1)}{N}}}^{\frac{(N+2)(m+1)}{N}}&\leqslant C_1\|\nabla v\|_{L^2}^{\frac{2(m+1)}{N}}\|D^2v\|_{L^{m+1}}^{m+1}+C_2\|\nabla v\|_{L^2}^{\frac{(N+2)(m+1)}{N}}\\
&\leqslant C_3\|D^2v\|_{L^{m+1}}^{m+1}+C_4.
\end{aligned}\tag{3.19}$$

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Noticing that $\frac{(N+2)(m+1)}{N} > \frac{2(m+1)}{m}$ when $m > \frac{2N}{N+2}$, we infer from (3.18), (3.19),

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{m-1}(u+\varepsilon)^m-\xi u\omega\right)\mathrm{d}x+\frac{1}{2}\int_\Omega(u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x$$
$$+\frac{m\mu}{2(m-1)}\int_\Omega u^{m+1}\mathrm{d}x\leqslant\sigma\int_\Omega|D^2v|^{m+1}\mathrm{d}x+N\varepsilon^2\int_\Omega|D^2\omega|^2\mathrm{d}x+C_\sigma \tag{3.20}$$

for any small constant $\sigma>0$, with C_σ depending on σ. Noticing that

$$\int_\Omega u^{m+1}\mathrm{d}x\geqslant\int_\Omega\left(\frac{1}{m-1}(u+\varepsilon)^m-\xi u\omega\right)\mathrm{d}x-C,$$

and using L^p theory of linear parabolic equations, we conclude that

$$\begin{aligned}
&\sup_{0<t<T_{\max}}\int_\Omega(u+\varepsilon)^m\mathrm{d}x+\frac{1}{2}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega(u+\varepsilon)\\
&\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\mathrm{d}s\\
&+\frac{m\mu}{2(m-1)}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{m+1}\mathrm{d}x\mathrm{d}s\\
&\leqslant C_5\sigma\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega|D^2v|^{m+1}\mathrm{d}x\mathrm{d}s\\
&\quad+C_6\varepsilon^2\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega|D^2\omega|^2\mathrm{d}x\mathrm{d}s+\tilde{C}_\sigma\\
&\leqslant C_7\sigma\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega|u|^{m+1}\mathrm{d}x\mathrm{d}s\\
&\quad+C_6\varepsilon^2\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega|D^2\omega|^2\mathrm{d}x\mathrm{d}s+\hat{C}_\sigma.
\end{aligned}$$

Here, all these constants C_i, $\hat{C}$, $\tilde{C}$ are independent of ε. By the arbitrariness of σ, using (3.5), (3.16), and (3.13) is proved by taking σ appropriately small in the above inequality.

In addition, using (3.3), (3.4) and (3.13), and noticing that

$$\begin{aligned}
(u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}\right)\right|^2&\leqslant2(u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\\
&\quad+2\xi^2(u+\varepsilon)\left|\nabla\omega\right|^2,
\end{aligned}$$

we arrive at (3.14).

For any $1 < p < 2$, we also notice that

$$
\begin{aligned}
\|\nabla(u+\varepsilon)^m\|_{L^p}^p &= \|m(u+\varepsilon)^{\frac{1}{2}}(u+\varepsilon)^{m-\frac{3}{2}}\nabla u\|_{L^p}^p \\
&\leqslant \|m(u+\varepsilon)^{m-\frac{3}{2}}\nabla u\|_{L^2}^p \|u+\varepsilon\|_{L^{\frac{p}{2-p}}}^{\frac{p}{2}} \\
&\leqslant \|m(u+\varepsilon)^{m-\frac{3}{2}}\nabla u\|_{L^2}^2 + C_p.
\end{aligned}
$$

Then (3.15) is derived from (3.14) and the above inequality. □

To improve the regularity of u, we need the following estimate on ω. Although this estimation does not depend on $T_{\max}$, it depends on ε. In order to obtain some subsequent uniform estimates that does not depend on ε, we need to the exact order of its dependence on ε.

Lemma 3.5. *Let (u, v, ω) be the classical solution of (3.1) in $[0, T_{\max})$. Then for any $r > 2$,*

$$
\begin{aligned}
&\sup_{0<t<T_{\max}} \int_\Omega |\nabla\omega|^r \mathrm{d}x + \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega \int_\Omega \varepsilon|\nabla\omega|^{r+2} \mathrm{d}x\mathrm{d}s \\
&\quad \leqslant \frac{\tilde{C}}{\varepsilon^{\frac{r}{2}}} \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega (u^r + v^r + 1)\mathrm{d}x\mathrm{d}s,
\end{aligned} \tag{3.21}
$$

where $\tilde{C}$ is independent of ε and $T_{\max}$, it depends only on χ, ξ, m, μ, η, u_0, v_0, ω_0, r, Ω.

Proof. Applying ∇ to the third equation of (3.1), multiplying both sides of the resultant equation by $|\nabla\omega|^{r-2}\nabla\omega$ for any $r > 2$, and using lemma 2.2, we obtain

$$
\begin{aligned}
&\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\omega|^r \mathrm{d}x + \varepsilon\int_\Omega |\nabla\omega|^{r-2}|\nabla^2\omega|^2 \mathrm{d}x \\
&\quad + (r-2)\varepsilon\int_\Omega |\nabla\omega|^{r-2}(\nabla|\nabla\omega|)^2 \mathrm{d}x + \int_\Omega |\nabla\omega|^r \mathrm{d}x \\
&\quad = \int_\Omega (v\omega - \eta\omega(1-u-\omega))\nabla\cdot(|\nabla\omega|^{r-2}\nabla\omega)\mathrm{d}x \\
&\qquad + \frac{\varepsilon}{2}\int_{\partial\Omega} \frac{\partial(|\nabla\omega|^2)}{\partial\mathbf{n}}|\nabla\omega|^{r-2}\mathrm{d}S + \int_\Omega |\nabla\omega|^r \mathrm{d}x \\
&\quad \leqslant \frac{\varepsilon}{4}\int_\Omega |\nabla\omega|^{r-2}|\nabla^2\omega|^2 \mathrm{d}x + \frac{(r-2)\varepsilon}{4}\int_\Omega |\nabla\omega|^{r-2}(\nabla|\nabla\omega|)^2 \mathrm{d}x \\
&\qquad + \frac{C_1}{\varepsilon}\int_\Omega |\nabla\omega|^{r-2}(u^2+v^2)\mathrm{d}x + \int_\Omega |\nabla\omega|^r \mathrm{d}x + \kappa\varepsilon\int_{\partial\Omega} |\nabla\omega|^r \mathrm{d}S.
\end{aligned} \tag{3.22}
$$

Recalling (2.3), and noticing that ω is bounded, then there exists a constant ρ such that

$$
\frac{\varepsilon}{4}\int_\Omega |\nabla\omega|^{r-2}|\nabla^2\omega|^2 \mathrm{d}x \geqslant \rho\varepsilon\int_\Omega |\nabla\omega|^{r+2}\mathrm{d}x. \tag{3.23}
$$

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By the boundary trace embedding inequalities, we conclude that for any small $\delta > 0$,

$$\kappa\varepsilon \int_{\partial\Omega} |\nabla\omega|^r \mathrm{d}S \leqslant \frac{(r-2)\varepsilon}{4} \int_\Omega |\nabla\omega|^{r-2}(\nabla|\nabla\omega|)^2 \mathrm{d}x + C_2 \int_\Omega |\nabla\omega|^r \mathrm{d}x. \tag{3.24}$$

The combination of (3.22)–(3.24) leads to

$$\begin{aligned}
&\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\omega|^r \mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega |\nabla\omega|^{r-2}|\nabla^2\omega|^2 \mathrm{d}x + \frac{(r-2)\varepsilon}{2}\int_\Omega |\nabla\omega|^{r-2}(\nabla|\nabla\omega|)^2 \mathrm{d}x \\
&\quad + \int_\Omega |\nabla\omega|^r \mathrm{d}x + \rho\varepsilon \int_\Omega |\nabla\omega|^{r+2} \mathrm{d}x \\
&\quad \leqslant \frac{C_1}{\varepsilon}\int_\Omega |\nabla\omega|^{r-2}(u^2+v^2)\mathrm{d}x + (C_2+1)\int_\Omega |\nabla\omega|^r \mathrm{d}x \\
&\quad \leqslant C_3 \int_\Omega |\nabla\omega|^r \mathrm{d}x + \frac{C_4}{\varepsilon^{\frac{r}{2}}}\int_\Omega (u^r+v^r)\mathrm{d}x \\
&\quad \leqslant \frac{\rho}{2}\varepsilon \int_\Omega |\nabla\omega|^{r+2}\mathrm{d}x + \frac{C_4}{\varepsilon^{\frac{r}{2}}}\int_\Omega (u^r+v^r)\mathrm{d}x + \frac{C_5}{\varepsilon^{\frac{r}{2}}}.
\end{aligned} \tag{3.25}$$

It implies that

$$\sup_{0<t<T_{\max}} \int_\Omega |\nabla\omega|^r \mathrm{d}x \leqslant \frac{C_6}{\varepsilon^{\frac{r}{2}}} \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega (u^r+v^r+1)\mathrm{d}x\mathrm{d}s. \tag{3.26}$$

Using (3.26), and integrating (3.25) directly, we arrive at

$$\sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega \varepsilon|\nabla\omega|^{r+2}\mathrm{d}x\mathrm{d}s \leqslant \frac{C_7}{\varepsilon^{\frac{r}{2}}} \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega (u^r+v^r+1)\mathrm{d}x\mathrm{d}s. \tag{3.27}$$

Here these constants C_i are independent of ε. Then (3.21) is derived from (3.26) and (3.27). □

Based on the above lemmas, we can improve the regularity of the solution, and obtain the following result.

LEMMA 3.6. *Let $(u,\, v,\, \omega)$ be the classical solution of (3.1) in $[0, T_{\max})$. Assume that $m > \frac{2N}{N+2}$. Then for any positive integer k, we have*

$$\begin{aligned}
&\sup_{0<t<T_{\max}} \int_\Omega (u+\varepsilon)^{(m-1)(2k+1)+1}\mathrm{d}x + \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s \\
&\quad + \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)^{2k} \\
&\quad \times \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2 \mathrm{d}x\mathrm{d}s \leqslant C,
\end{aligned} \tag{3.28}$$

where C is independent of ε and $T_{\max}$, it depends only on k, χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω.

Proof. For any given positive integer k, multiplying both sides of the first equation of (3.1) by $(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega)^{2k+1}$, recalling (3.17), and noticing that ω is bounded, then we arrive at

$$\begin{aligned}
&\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k+1} u_t \mathrm{d}x \\
&\quad + (2k+1)\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} \\
&\quad \times \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2 \mathrm{d}x \\
&\quad = -\varepsilon\xi(2k+1)\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} \\
&\qquad \times \nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\nabla\omega \mathrm{d}x \\
&\qquad + \chi(2k+1)\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} \\
&\qquad \times u\nabla v\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\mathrm{d}x \\
&\qquad + \int_\Omega \mu u(1-u-\omega)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k+1}\mathrm{d}x \\
&\quad \leqslant \frac{2k+1}{2}\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} \\
&\qquad \times \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2 \mathrm{d}x \\
&\qquad + \varepsilon\xi^2(2k+1)\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} |\nabla\omega|^2 \mathrm{d}x \\
&\qquad + \chi^2(2k+1)\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} u|\nabla v|^2 \mathrm{d}x \\
&\qquad - \frac{3\mu}{4}\left(\frac{m}{m-1}\right)^{2k+1}\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x + C_1.
\end{aligned}$$

That is

$$\begin{aligned}
&\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k+1} u_t \mathrm{d}x + \frac{3\mu}{4}\left(\frac{m}{m-1}\right)^{2k+1}\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x \\
&\quad + \frac{2k+1}{2}\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}
\end{aligned}$$

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$$
\begin{aligned}
&\times \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2 \mathrm{d}x \\
&\leqslant \varepsilon\xi^2(2k+1)\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} |\nabla\omega|^2 \mathrm{d}x \\
&\quad + \chi^2(2k+1)\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k} u|\nabla v|^2 \mathrm{d}x + C_1 \\
&= I + II + C_1. \qquad (3.29)
\end{aligned}
$$

We first calculate the first term on the left of the above inequality.

$$
\begin{aligned}
&\int_\Omega \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k+1} u_t \mathrm{d}x \\
&= \int_\Omega \sum_{i=0}^{2k+1} (-\xi)^i C_{2k+1}^i \left(\frac{m}{m-1}(u+\varepsilon)^{m-1}\right)^{2k+1-i} \omega^i u_t \mathrm{d}x \\
&= \left(\frac{m}{m-1}\right)^{2k+1} \int_\Omega \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \left((u+\varepsilon)^{(m-1)(2k+1-i)+1}\right)_t \omega^i \mathrm{d}x \\
&= \left(\frac{m}{m-1}\right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i \mathrm{d}x\right. \\
&\quad \left. -i\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^{i-1}\omega_t \mathrm{d}x\right) \\
&= \left(\frac{m}{m-1}\right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i \mathrm{d}x \\
&\quad - \left(\frac{m}{m-1}\right)^{2k+1} \sum_{i=1}^{2k+1} \frac{i(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^{i-1}\omega_t \mathrm{d}x \\
&\geqslant \left(\frac{m}{m-1}\right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i \mathrm{d}x \\
&\quad - C\int_\Omega \left(u^{2k(m-1)+2} + |\omega_t|^{2k(m-1)+2} + 1\right) \mathrm{d}x \\
&\geqslant \left(\frac{m}{m-1}\right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i \mathrm{d}x \\
&\quad - \frac{\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1} \int_\Omega u^{(2k+1)(m-1)+2} \mathrm{d}x - C_2\int_\Omega \left(|\omega_t|^{2k(m-1)+2} + 1\right) \mathrm{d}x. \qquad (3.30)
\end{aligned}
$$

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In what follows, we calculate I, II respectively. Using Young's inequality and recalling (3.5), it is not difficult to see that

$$\begin{aligned} I &\leqslant C_3 \int_\Omega u^{2k(m-1)+2} + \varepsilon^{k(m-1)+1} \int_\Omega |\nabla \omega|^{2k(m-1)+2} \mathrm{d}x + C_4 \varepsilon \int_\Omega |\nabla \omega|^2 \mathrm{d}x \\ &\leqslant \frac{\mu}{8} \left(\frac{m}{m-1} \right)^{2k+1} \int_\Omega u^{(2k+1)(m-1)+2} \mathrm{d}x + \varepsilon^{k(m-1)+1} \int_\Omega |\nabla \omega|^{2k(m-1)+2} \mathrm{d}x + C_5. \end{aligned} \tag{3.31}$$

Noticing that ω is bounded, and using (3.3) gives

$$II \leqslant \frac{\mu}{8} \left(\frac{m}{m-1} \right)^{2k+1} \int_\Omega u^{(2k+1)(m-1)+2} \mathrm{d}x + C_6 \int_\Omega |\nabla v|^{\frac{4k(m-1)+2(m+1)}{m}} \mathrm{d}x + C_7. \tag{3.32}$$

Next, we estimate the second term $C_6 \int_\Omega |\nabla v|^{\frac{4k(m-1)+2(m+1)}{m}} \mathrm{d}x$ in (3.32). From Gagliardo-Nirenberg inequality, we infer that

$$\|\nabla v\|_{L^p}^p \leqslant C \|D^2 v\|_{L^{q_1}}^{p\alpha} \|\nabla v\|_{L^{q_2}}^{p(1-\alpha)} + C \|\nabla v\|_{L^2}^2, \text{ with } \frac{1}{p} = \left(\frac{1}{q_1} - \frac{1}{N} \right) \alpha + \frac{1-\alpha}{q_2}. \tag{3.33}$$

By direct calculation, it is easy to obtain

$$p\alpha < q_1 \Leftrightarrow \frac{p}{q_1} < 1 + \frac{q_2}{N}.$$

When $N = 2$, take

$$p = \frac{4k(m-1)+2(m+1)}{m}, \quad q_1 = 2k(m-1)+m+1, \quad q_2 = 2$$

in (3.33). Then $p\alpha < q_1$. Using (3.3), we infer from (3.33) that for any small constant $\rho > 0$,

$$\begin{aligned} C_6 \int_\Omega |\nabla v|^{\frac{4k(m-1)+2(m+1)}{m}} \mathrm{d}x &\leqslant C_8 \|D^2 v\|_{L^{2k(m-1)+m+1}}^{p\alpha} + C_9 \\ &\leqslant \rho \int_\Omega |D^2 v|^{2k(m-1)+m+1} \mathrm{d}x + C_\rho, \quad \text{when } N = 2, \end{aligned} \tag{3.34}$$

since $p\alpha < q_1$.

When $N \geqslant 3$. By (2.5) and (3.13), we see that

$$\sup_{0<t<T_{\max}} \|\nabla v\|_{L^r}^r \leqslant C, \quad \forall r < \frac{N(m+1)}{(N+1-m)_+}. \tag{3.35}$$

We take

$$\begin{aligned} p &= \frac{4k(m-1)+2(m+1)}{m}, \quad q_1 = 2k(m-1)+m+1, \\ q_2 &= \frac{N(m+\frac{5}{2N})}{(N+1-m)_+} \quad (N \geqslant 3) \end{aligned}$$

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in (3.33). Then $\|\nabla v\|_{L^{q_2}}$ is uniformly bounded by (3.35), and

$$p\alpha < q_1 \Leftrightarrow \frac{p}{q_1} < 1 + \frac{q_2}{N} \Leftrightarrow \frac{2}{m} < 1 + \frac{m + \frac{5}{2N}}{(N+1-m)_+} \quad (N \geqslant 3).$$

By direct verification, the above inequality $\frac{2}{m} < 1 + \frac{m+\frac{5}{2N}}{(N+1-m)_+}$ holds when $m > \frac{2N}{N+2}$. Then

$$C_6 \int_\Omega |\nabla v|^{\frac{4k(m-1)+2(m+1)}{m}} \mathrm{d}x \leqslant \rho \int_\Omega |D^2 v|^{2k(m-1)+m+1} \mathrm{d}x + \tilde{C}_\rho, \quad \text{when } N \geqslant 3 \tag{3.36}$$

for any small constant $\rho > 0$.

Substituting (3.34) and (3.36) into (3.32) yields

$$II \leqslant \frac{\mu}{8} \left(\frac{m}{m-1}\right)^{2k+1} \int_\Omega u^{(2k+1)(m-1)+2} \mathrm{d}x + \rho \int_\Omega |D^2 v|^{2k(m-1)+m+1} \mathrm{d}x + \hat{C}_\rho \tag{3.37}$$

for any small constant $\rho > 0$ with $\hat{C}_\rho$ depending on ρ.

By substituting the inequalities (3.30), (3.31) and (3.37) into (3.29) yields

$$\begin{aligned}
&\left(\frac{m}{m-1}\right)^{2k+1} \sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1} \omega^i \mathrm{d}x \\
&\quad + \frac{2k+1}{2} \int_\Omega (u+\varepsilon) \left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)^{2k} \\
&\quad \times \left|\nabla \left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2 \mathrm{d}x \\
&\quad + \frac{\mu}{8} \left(\frac{m}{m-1}\right)^{2k+1} \int_\Omega u^{(2k+1)(m-1)+2} \mathrm{d}x \\
&\qquad \leqslant C_2 \int_\Omega |\omega_t|^{2k(m-1)+2} \mathrm{d}x + \varepsilon^{k(m-1)+1} \int_\Omega |\nabla\omega|^{2k(m-1)+2} \mathrm{d}x \\
&\qquad\quad + \rho \int_\Omega |D^2 v|^{2k(m-1)+m+1} \mathrm{d}x + \widehat{C}_\rho.
\end{aligned} \tag{3.38}$$

Noticing that

$$\begin{aligned}
&\sum_{i=0}^{2k+1} \frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1} \int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1} \omega^i \mathrm{d}x \\
&\quad \leqslant \frac{\mu}{8} \int_\Omega u^{(2k+1)(m-1)+2} \mathrm{d}x + C
\end{aligned}$$

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and for any small constant $\rho > 0$, there exists a constant C_ρ such that

$$\begin{aligned}\int_\Omega |v|^{2k(m-1)+2}\mathrm{d}x &\leqslant \rho\int_\Omega |D^2 v|^{2k(m-1)+2}\mathrm{d}x + C_\rho\int_\Omega |v|^2\mathrm{d}x\\ &\leqslant \tilde{\rho}\int_\Omega |D^2 v|^{2k(m-1)+m+1}\mathrm{d}x + \tilde{C}_\rho,\end{aligned}$$

combining (3.21), and using L^p theory of linear parabolic equations, from (3.38) we derive

$$\begin{aligned}&\left(\frac{m}{m-1}\right)^{2k+1}\sum_{i=0}^{2k+1}\frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1}\sup_{0<t<T_{\max}}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i\mathrm{d}x\\
&\quad+\frac{2k+1}{2}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}\\
&\quad\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\mathrm{d}s\\
&\quad+\frac{3\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s\\
&\quad\leqslant C_{10}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |\omega_t|^{2k(m-1)+2}\mathrm{d}x\mathrm{d}s + C_{11}\varepsilon^{k(m-1)+1}\\
&\qquad\times\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |\nabla\omega|^{2k(m-1)+2}\mathrm{d}x\mathrm{d}s\\
&\qquad+C_{12}\rho\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |D^2 v|^{2k(m-1)+m+1}\mathrm{d}x\mathrm{d}s + M_\rho\\
&\quad\leqslant C_{13}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega \left(|u|^{2k(m-1)+2}+|v|^{2k(m-1)+2}+1\right)\mathrm{d}x\mathrm{d}s\\
&\qquad+C_{14}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega \left(|u|^{2k(m-1)}+|v|^{2k(m-1)}+1\right)\mathrm{d}x\mathrm{d}s\\
&\qquad+C_{12}\rho\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |D^2 v|^{2k(m-1)+m+1}\mathrm{d}x\mathrm{d}s + M_\rho\\
&\quad\leqslant \frac{\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s\\
&\qquad+C_{15}\rho\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |D^2 v|^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s + \widetilde{M}_\rho\\
&\quad\leqslant \frac{\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s\\
&\qquad+C_{16}\rho\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |u|^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s + \widetilde{M}_\rho.\end{aligned}$$

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Take ρ appropriately small in the above inequality, such that $C_{16}\rho < \frac{\mu}{8}(\frac{m}{m-1})^{2k+1}$. Then we finally arrive that

$$\begin{aligned}
&\left(\frac{m}{m-1}\right)^{2k+1}\sum_{i=0}^{2k+1}\frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1}\sup_{0<t<T_{\max}}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i\mathrm{d}x\\
&+\frac{2k+1}{2}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}\\
&\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\mathrm{d}s\\
&+\frac{\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s\leqslant C,
\end{aligned}$$

which implies (3.28). □

By (3.28), there exists k sufficiently large such that $(m-1)(2k+1)+1>N$, then by Neumann heat semigroup theory, it is easy to obtain the L^∞ estimation of ∇v. By L^p theory of linear parabolic equations, we also have $W_p^{2,1}$ estimation of v and L^p estimation of ω_t.

Lemma 3.7. *Let $(u,\,v,\,\omega)$ be the classical solution of (3.1) in $[0,\,T_{\max})$. Assume that $m>\frac{2N}{N+2}$. Then*

$$\sup_{0<t<T_{\max}}\|v(\cdot,t)\|_{W^{1,\infty}}<C,$$

and for any $p>1$,

$$\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\left(\|v_t\|_{L^p}^p+\|\omega_t\|_{L^p}^p+\|D^2v\|_{L^p}^p\right)\mathrm{d}x\mathrm{d}s\leqslant\tilde{C},$$

where C, $\tilde{C}$ are independent of ε and $T_{\max}$, it depends only on k, χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω.

In addition, by (3.28), and combining with Neumann heat semigroup theory, it is also easy to obtain the L^∞ estimation of $\nabla\omega$ (depending on ε). Furthermore, by (3.28) and Lemma 3.7, using standard Morser iterative technique, the L^∞ estimation of u is also easy to obtain.

Lemma 3.8. *Let $(u,\,v,\,\omega)$ be the classical solution of (3.1) in $[0,\,T_{\max})$. Assume that $m>\frac{2N}{N+2}$. Then*

$$\sup_{0<t<T_{\max}}\left(\|u(\cdot,t)\|_{L^\infty}+\|\omega(\cdot,t)\|_{W^{1,\infty}}\right)<C_\varepsilon,$$

where C_ε depends on ε, and is independent of $T_{\max}$.

Using lemmas 3.1, 3.7, and 3.8, the solution of the regularized problem (3.1) exists globally for any $\varepsilon>0$, and these estimations in lemma 3.2-Lemma 3.7 are independent of ε. To obtain the global solution to the original problem (1.6), we also need to make some a priori estimates for u_t.

LEMMA 3.9. *Let* (u, v, ω) *be the classical solution of* (3.1) *in* $[0, T_{\max})$. *Assume that* $m > \frac{2N}{N+2}$. *Then for any* $1 < r < 2$, *and any* $T > 0$,

$$\|u_t\|_{L^2((0,T);W^{-1,r}(\Omega))} \leqslant C_{T,r}, \tag{3.39}$$

where $C_{T,r}$ *depends on* T, r, *and it is independent of* ε.

Proof. For any $\varphi \in C^\infty(Q_T)$, we see that

$$\int_0^T\int_\Omega u_t\varphi \mathrm{d}x\mathrm{d}t = -\int_0^T\int_\Omega (u+\varepsilon)\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\nabla\varphi \mathrm{d}x$$
$$+\chi\int_0^T\int_\Omega u\nabla v\nabla\varphi\mathrm{d}x\mathrm{d}t + \mu\int_0^T\int_\Omega u(1-u-\omega)\varphi\mathrm{d}x\mathrm{d}t.$$

Then for any $q > 2$, using (3.13), (3.28) and lemma 3.7, we derive that

$$\left|\int_0^T\int_\Omega u_t\varphi\mathrm{d}x\mathrm{d}t\right| \leqslant \int_0^T \left\|\sqrt{u+\varepsilon}\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right\|_{L^2}$$
$$\times \|u+\varepsilon\|_{L^{\frac{q}{q-2}}}^{\frac{1}{2}}\|\nabla\varphi\|_{L^q}\mathrm{d}t$$
$$+\chi\int_0^T \|u\|_{L^2}\|\nabla v\|_{L^\infty}\|\nabla\varphi\|_{L^2}\mathrm{d}t + \mu\int_0^T \|u(1-u-\omega)\|_{L^2(Q_T)}\|\varphi\|_{L^2}\mathrm{d}t$$
$$\leqslant C\left(\int_0^T\left\|\sqrt{u+\varepsilon}\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right\|_{L^2}^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\|\nabla\varphi\|_{L^q}^2\mathrm{d}t\right)^{\frac{1}{2}}$$
$$+\chi\left(\int_0^T\|u\|_{L^2}^2\|\nabla v\|_{L^\infty}^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\|\nabla\varphi\|_{L^2}^2\mathrm{d}t\right)^{\frac{1}{2}}$$
$$+\mu\left(\int_0^T\|u(1-u-\omega)\|_{L^2}^2\mathrm{d}t\right)^{\frac{1}{2}}\left(\int_0^T\|\varphi\|_{L^2}^2\mathrm{d}t\right)^{\frac{1}{2}}$$
$$\leqslant C\left(\int_0^T\|\varphi\|_{W^{1,q}}^2\mathrm{d}t\right)^{\frac{1}{2}},$$

namely for any $q > 2$,

$$\|u_t\|_{L^2\left((0,T);W^{-1,\frac{q}{q-1}}(\Omega)\right)} \leqslant C_q,$$

which implies (3.39). □

Combining lemmas 3.1 -3.8, and noticing (3.16), we have the following proposition.

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Proposition 3.10. *Assume that* $m > \frac{2N}{N+2}$. *For any* $\varepsilon \in (0, 1)$, *the problem* (3.1) *in* $[0, T_{\max})$ *admits a unique positive global classical solution* $(u_\varepsilon, v_\varepsilon, \omega_\varepsilon) \in C^{2,1}(Q) \cap C^0(\overline{\Omega} \times [0, \infty))$ *such that*

$$\sup_{0<t<\infty} (\|v_\varepsilon(\cdot,t)\|_{W^{1,\infty}} + \|\omega_\varepsilon(\cdot,t)\|_{L^\infty} + \|u_\varepsilon(\cdot,t)\|_{L^r}) \leqslant C_r, \quad \text{for any } r > 1, \tag{3.40}$$

and for any $T > 0$,

$$\sup_{0<t<T} \int_\Omega \left(\frac{|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon}\right) \mathrm{d}x + \int_0^T \int_\Omega \left((u_\varepsilon+\varepsilon)^{2m-3}|\nabla u_\varepsilon|^2 + \frac{u_\varepsilon|\nabla\omega_\varepsilon|^2}{\omega_\varepsilon} + \varepsilon|D^2\omega_\varepsilon|^2\right) \mathrm{d}x\mathrm{d}t \leqslant C_T, \tag{3.41}$$

$$\int_0^T \int_\Omega \left(|D^2 v_\varepsilon|^r + |\partial_t v_\varepsilon|^r + |\partial_t \omega_\varepsilon|^r\right) \mathrm{d}x\mathrm{d}t \leqslant C_{Tr}, \quad \text{for any } r > 1, \tag{3.42}$$

$$\|\nabla(u_\varepsilon+\varepsilon)^m\|_{L^p(Q_T)} + \|\partial_t u_\varepsilon\|_{L^2((0,T);W^{-1,p}(\Omega))} \leqslant C_{Tp}, \quad \text{for any } 1 < p < 2, \tag{3.43}$$

where C_r, C_T, C_{Tp}, *and* C_{Tr} *are independent of* ε, *they depend on* χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω, C_r *depends on* r, C_T *depends on* T, C_{Tp} *depends on* T, p, *and* C_{Tr} *depends on* T, r.

Proof of theorem 1.1. Since $(u_\varepsilon, v_\varepsilon, \omega_\varepsilon)$ are classical solutions of (3.1), they obviously satisfy the equations (3.1) in the sense of distribution. By proposition 3.10, using Aubin-Lions lemma, and Sobolev compact embedding theorem, there exists a subsequence of $\{(u_\varepsilon, v_\varepsilon, \omega_\varepsilon)\}$, which we still remember as itself for convenience, such that as $\varepsilon \to 0$,

$$\begin{aligned}
&u_\varepsilon, \omega_\varepsilon \to u, \omega \text{ in } L^r(Q_T) \text{ for any } r > 1,\\
&\nabla(u_\varepsilon+\varepsilon)^m \rightharpoonup \nabla u^m, \text{ in } L^p(Q_T) \text{ for any } 1 < p < 2,\\
&\varepsilon\Delta\omega \to 0, \text{ in } L^2(Q_T),\\
&v_\varepsilon \rightharpoonup v, \text{ in } W_r^{2,1}(Q_T) \text{ for any } r > 1,\\
&\nabla v_\varepsilon \to \nabla v, \text{ in } L^r(Q_T) \text{ for any } r > 1,\\
&v_\varepsilon \to v, \text{ uniformly.}
\end{aligned}$$

Letting $\varepsilon \to 0$ and utilizing the above convergence results, we can ultimately verify that (u, v, ω) satisfies definition 2.1, that is, $(u, v, \omega) \in \mathcal{X}_1 \times \mathcal{X}_2 \times \mathcal{X}_3$ is a weak solution. Theorem 1.1 is proved. □

4. Global solution to the haptotaxis-only system

In this section, we consider the haptotaxis-only system. Similar to § 2, to prove the existence of weak solutions, we consider the following regularized problem

$$\begin{cases} u_t = m\nabla\cdot(u+\varepsilon)^{m-1}\nabla u - \xi\nabla\cdot(u\nabla\omega) + \mu u(1-u-\omega), \quad \text{in } Q, \\ v_t = \Delta v + u - v, \quad \text{in } Q, \\ \omega_t = \varepsilon\Delta\omega - v\omega + \eta\omega(1-u-\omega), \ \text{in } Q, \\ \left.\dfrac{\partial u}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \left.\dfrac{\partial v}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \left.\dfrac{\partial \omega}{\partial \mathbf{n}}\right|_{\partial\Omega} = 0, \\ u(x,0) = u_{\varepsilon 0}(x), \quad v(x,0) = v_{\varepsilon 0}(x), \quad \omega(x,0) = \omega_{\varepsilon 0}(x), \quad x\in\Omega. \end{cases} \tag{4.1}$$

Similar to § 3, in the subsequent energy estimate calculations, we omit the subscript ε of the approximate solutions $(u_\varepsilon,\, v_\varepsilon,\, \omega_\varepsilon)$ in the subsequent energy estimate calculations.

It is clear that lemmas 3.1, 3.2, and 3.3 also hold for $\chi = 0$. Similar to lemma 3.4, we also have

Lemma 4.1. *Let $(u,\, v,\, \omega)$ be the classical solution of (4.1) in $[0,\, T_{\max})$. Then*

$$\sup_{0<t<T_{\max}} \int_\Omega u^m \mathrm{d}x + \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega \Bigg((u+\varepsilon) \times \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2 + u^{m+1}\Bigg)\mathrm{d}x\mathrm{d}s \leqslant C, \tag{4.2}$$

where C is independent of ε and $T_{\max}$, it depends only on χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω. In addition, we also have

$$\int_0^T \int_\Omega (u+\varepsilon)^{2m-3}|\nabla u|^2 \mathrm{d}x\mathrm{d}t \leqslant C_T, \tag{4.3}$$

and for any $1<p<2$,

$$\int_0^T \int_\Omega |\nabla(u+\varepsilon)^m|^p \mathrm{d}x\mathrm{d}t \leqslant C_{Tp}, \tag{4.4}$$

where C_T, C_{Tp} are independent of ε, and C_T depends on T, C_{Tp} depends on T and p.

Proof. Similar to the proof of (3.18), we see that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{m-1}(u+\varepsilon)^m - \xi u\omega\right)\mathrm{d}x + \int_\Omega (u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2 \mathrm{d}x$$

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$$
\begin{aligned}
&= -\varepsilon\xi \int_\Omega \nabla\omega\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\mathrm{d}x \\
&\quad + \int_\Omega \mu u(1-u-\omega)\left(\frac{m}{m-1}u^{m-1} - \xi\omega\right)\mathrm{d}x \\
&\quad + \xi\int_\Omega u(-\varepsilon\Delta\omega + v\omega - \eta\omega(1-u-\omega))\mathrm{d}x \\
&\leqslant \frac{\varepsilon\xi^2\|\omega\|_{L^\infty}}{2}\int_\Omega \frac{|\nabla\omega|^2}{\omega}\mathrm{d}x + \frac{\varepsilon}{2}\int_\Omega \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2\mathrm{d}x \\
&\quad - \frac{3m\mu}{4(m-1)}\int_\Omega u^{m+1}\mathrm{d}x + \varepsilon^2\int_\Omega |\Delta\omega|^2\mathrm{d}x + C \\
&\leqslant \frac{\varepsilon}{2}\int_\Omega \left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2\mathrm{d}x - \frac{3m\mu}{4(m-1)}\int_\Omega u^{m+1}\mathrm{d}x \\
&\quad + N\varepsilon^2\int_\Omega |D^2\omega|^2\mathrm{d}x + \tilde{C},
\end{aligned}
$$

that is

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{m-1}(u+\varepsilon)^m - \xi u\omega\right)\mathrm{d}x + \frac{1}{2}\int_\Omega (u+\varepsilon)\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1} - \xi\omega\right)\right|^2\mathrm{d}x \\
&\quad + \frac{3m\mu}{4(m-1)}\int_\Omega u^{m+1}\mathrm{d}x \leqslant N\varepsilon^2\int_\Omega |D^2\omega|^2\mathrm{d}x + \tilde{C}. \qquad (4.5)
\end{aligned}
$$

Then completely similar to the proof of lemma 3.4, we complete the proof. □

Completely similar to lemma 3.5, we also have that

Lemma 4.2. *Let (u, v, ω) be the classical solution of (4.1) in $[0, T_{\max})$. Then for any $r > 2$,*

$$
\begin{aligned}
&\sup_{0<t<T_{\max}} \int_\Omega |\nabla\omega|^r\mathrm{d}x + \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega \varepsilon|\nabla\omega|^{r+2}\mathrm{d}x\mathrm{d}s \\
&\quad \leqslant \frac{\tilde{C}}{\varepsilon^{\frac{r}{2}}} \sup_{\tau<t<T_{\max}} \int_{t-\tau}^t \int_\Omega (u^r + v^r + 1)\mathrm{d}x\mathrm{d}s, \qquad (4.6)
\end{aligned}
$$

where $\tilde{C}$ is independent of ε and $T_{\max}$, it depends only on χ, ξ, m, μ, η, u_0, v_0, ω_0, r, Ω.

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Then similar to lemma 3.6, we also have that

LEMMA 4.3. *Let* (u, v, ω) *be the classical solution of* (4.1) *in* $[0, T_{\max})$. *Then for any positive integer* k, *we have that*

$$\begin{aligned}&\sup_{0<t<T_{\max}}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1)+1}\mathrm{d}x+\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s\\&\quad+\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}\\&\quad\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\mathrm{d}s\leqslant C, \qquad (4.7)\end{aligned}$$

where C *is independent of* ε *and* $T_{\max}$, *it depends only on* k, χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω.

Proof. For any given positive integer k, multiplying both sides of the first equation of (4.1) by $(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega)^{2k+1}$, and noticing that ω is bounded, then we arrive at

$$\begin{aligned}&\int_\Omega\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k+1}u_t\mathrm{d}x\\&\quad+(2k+1)\int_\Omega(u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}\\&\quad\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\&\quad=-\varepsilon\xi(2k+1)\int_\Omega\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}\\&\qquad\times\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\nabla\omega\mathrm{d}x\\&\qquad+\int_\Omega\mu u(1-u-\omega)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k+1}\mathrm{d}x\\&\quad\leqslant\frac{2k+1}{2}\int_\Omega\varepsilon\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\\&\qquad+\frac{\varepsilon\xi^2(2k+1)}{2}\int_\Omega\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}|\nabla\omega|^2\mathrm{d}x\\&\qquad-\frac{3\mu}{4}\left(\frac{m}{m-1}\right)^{2k+1}\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x+C_1.\end{aligned}$$

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Combining with (3.30), and using (3.31) yields

$$\left(\frac{m}{m-1}\right)^{2k+1}\sum_{i=0}^{2k+1}\frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i\mathrm{d}x$$
$$+\frac{2k+1}{2}\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}$$
$$\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x+\frac{5\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x$$
$$\leqslant \frac{\varepsilon\xi^2(2k+1)}{2}\int_\Omega\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}|\nabla\omega|^2\mathrm{d}x$$
$$+C_2\int_\Omega |\omega_t|^{2k(m-1)+2}\mathrm{d}x+C_3$$
$$\leqslant \frac{\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x+\varepsilon^{k(m-1)+1}\int_\Omega |\nabla\omega|^{2k(m-1)+2}\mathrm{d}x$$
$$+C_2\int_\Omega |\omega_t|^{2k(m-1)+2}\mathrm{d}x+C_4. \tag{4.8}$$

By (4.6), (2.5), and using L^p theory of linear parabolic equations, we arrive at

$$\left(\frac{m}{m-1}\right)^{2k+1}\sum_{i=0}^{2k+1}\frac{(-\frac{m-1}{m}\xi)^i C_{2k+1}^i}{(m-1)(2k+1-i)+1}\sup_{0<t<T_{\max}}\int_\Omega (u+\varepsilon)^{(m-1)(2k+1-i)+1}\omega^i\mathrm{d}x$$
$$+\frac{2k+1}{2}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega (u+\varepsilon)\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)^{2k}$$
$$\times\left|\nabla\left(\frac{m}{m-1}(u+\varepsilon)^{m-1}-\xi\omega\right)\right|^2\mathrm{d}x\mathrm{d}s$$
$$+\frac{\mu}{2}\left(\frac{m}{m-1}\right)^{2k+1}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s$$
$$\leqslant C_5\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |\omega_t|^{2k(m-1)+2}\mathrm{d}x\mathrm{d}s+C_6\varepsilon^{k(m-1)+1}$$
$$\times\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega |\nabla\omega|^{2k(m-1)+2}\mathrm{d}x\mathrm{d}s+C_7$$
$$\leqslant C_8\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega\left(|u|^{2k(m-1)+2}+|v|^{2k(m-1)+2}+1\right)\mathrm{d}x\mathrm{d}s$$
$$+C_9\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega\left(|u|^{2k(m-1)}+|v|^{2k(m-1)}+1\right)\mathrm{d}x\mathrm{d}s+C_{10}$$
$$\leqslant\frac{\mu}{8}\left(\frac{m}{m-1}\right)^{2k+1}\sup_{\tau<t<T_{\max}}\int_{t-\tau}^t\int_\Omega u^{(2k+1)(m-1)+2}\mathrm{d}x\mathrm{d}s+C_{11}.$$

Then (4.7) is proved. □

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Completely similar to lemmas 3.7– 3.9. We have the following results.

PROPOSITION 4.4. *For any* $\varepsilon \in (0, 1)$, *the problem* (4.1) *admits a unique positive global classical solution* $(u_\varepsilon, v_\varepsilon, \omega_\varepsilon) \in C^{2,1}(Q) \cap C^0(\overline{\Omega} \times [0, \infty))$ *such that*

$$\sup_{0<t<\infty} (\|v_\varepsilon(\cdot, t)\|_{W^{1,\infty}} + \|\omega_\varepsilon(\cdot, t)\|_{L^\infty} + \|u_\varepsilon(\cdot, t)\|_{L^r}) \leqslant C_r, \quad \textit{for any } r > 1, \tag{4.9}$$

and for any $T > 0$,

$$\sup_{0<t<T} \int_\Omega \left(\frac{|\nabla \omega_\varepsilon|^2}{\omega_\varepsilon} \right) \mathrm{d}x + \int_0^T \int_\Omega \Bigg((u_\varepsilon + \varepsilon)^{2m-3} |\nabla u_\varepsilon|^2 + \frac{u_\varepsilon |\nabla \omega_\varepsilon|^2}{\omega_\varepsilon} + \varepsilon |D^2 \omega_\varepsilon|^2 \Bigg) \mathrm{d}x\mathrm{d}t \leqslant C_T, \tag{4.10}$$

$$\int_0^T \int_\Omega \left(|D^2 v_\varepsilon|^r + |\partial_t v_\varepsilon|^r + |\partial_t \omega_\varepsilon|^r \right) \mathrm{d}x\mathrm{d}t \leqslant C_{Tr}, \quad \textit{for any } r > 1, \tag{4.11}$$

$$\|\nabla (u_\varepsilon + \varepsilon)^m\|_{L^p(Q_T)} + \|\partial_t u_\varepsilon\|_{L^2((0,T);W^{-1,p}(\Omega))} \leqslant C_{Tp}, \quad \textit{for any } 1 < p < 2, \tag{4.12}$$

where C_r, C_T, C_{Tp}, *and* C_{Tr} *are independent of* ε, *they depend on* χ, ξ, m, μ, η, u_0, v_0, ω_0, Ω, C_r *depends on* r, C_T *depends on* T, C_{Tp} *depends on* T, p, *and* C_{Tr} *depends on* T, r.

Letting $\varepsilon \to 0$, we complete the proof of Theorem 1.2.

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Statements and declarations
The author declares that there is no competing interest, and confirms that the data supporting the findings of this study are available within the paper.

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