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TORSION IN THE ADDITIVE GROUP OF RELATIVELY FREE LIE RINGS

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Let $L = L(X)$ be the free Lie ring of countable rank and let p be prime. Then $L(V_p) = L/[(L')^p, L]$ is the relatively free ring for the variety of Lie rings $V_p = [N_{p-1}A, E]$ and V_p is defined by the identity

$$[[[x_1,x_2],\ldots,[x_{2p-1},x_{2p}]],x_{2p+1}] = 0.$$

The purpose of this note is to establish that there exist elements of order p in the additive group of $L(V_p)$. Previously, the existence of p-torsion was proved by Kuz'min for $p = 2$ only. Similar results were obtained for varieties of groups by Gupta when $p = 2$ and by Stöhr when $p = 3$.

Introduction

Let $H(x_1,\ldots,x_c)$ be a "multilinear" commutator of length c, that is, the commutator brackets are placed in the monomial $x_1\ldots x_c$ in an arbitrary, but fixed way. Let V be the variety of groups determined by the identity $H(x_1,\ldots,x_c) = 1$. Denote by $F(V)$ the relatively free group of countable rank in V and let $S_n(F(V))$ be the n-th element of the lower central series of $F(V)$. In the classical case, when $V = N_{c_1}\ldots N_{c_k}$ is a polynilpotent variety, the factors $S_n(F(V))/S_{n+1}(F(V))$ are torsion-free abelian groups. It was a surprising result due to Gupta [2], that there are elements of order 2 in the centre of $F(V_2)$, the relatively free group of the centre-by-metabelian variety $V_2 = [A^2, E]$, defined by

$$[[[x_1,x_2],[x_3,x_4]],x_5] = 1.$$

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Kuz'min [3], applying methods of homological algebra, found 2-torsion in the relatively free ring $L/[L'',L]$ of the centre-by-metabelian variety of Lie rings. Here $L = L(X)$ is the free Lie ring of countable rank. Stöhr [5], using ideas of Kuz'min, proved the existence of 3-torsion in the centre of $F(V_3)$ where $V_3 = [N_2A,E]$. We refer to [6] for a survey of results in this field.

In this note we investigate the variety of Lie rings $V_p = [N_{p-1}A,E]$, which is defined by the polynomial identity

$$[[[x_1,x_2],\ldots,[x_{2p-1},x_{2p}]],x_{2p+1}] = 0 \tag{1}$$

and p is prime. The purpose is to generalize Kuz'min's result and to prove that there exist elements of order p in the additive group of the relatively free ring

$$L(V_p) = L/[(L')^p,L].$$

The following result is obtained:

THEOREM. *Let p be prime. Then there exists a non-zero multilinear element $u(x_1,\ldots,x_{2p+1}) \in L(V_p)$ such that $pu(x_1,\ldots,x_{2p+1}) = 0$.*

1. Preliminaries

We denote by Z the ring of integers, by Q the field of rationals and by Z_p the field with p elements, where p is a fixed prime integer. We consider the free Lie ring $L = L(X)$, $X = \{x_1,x_2,\ldots\}$, canonically embedded in the free associative ring $Z\langle X\rangle$. All tensor products are over Z. For a given field K, the algebras $L_K = K \otimes L(X)$ and $K \otimes Z\langle X\rangle$ are isomorphic to the free Lie K-algebra and the free associative K-algebra, respectively. For convenience, we identity $a \otimes u \in K \otimes Z\langle X\rangle$ and $au \in K\langle X\rangle$, where $a \in K$, $u \in Z\langle X\rangle$. Let

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$f(x_1,\ldots,x_c)$ be a multilinear polynomial from L and let W be the verbal ideal generated by $f(x_1,\ldots,x_c)$. It is well-known that $K \otimes L/W$ is the relatively free algebra of the variety of Lie K-algebras, defined by the identity $f(x_1,\ldots,x_c) = 0$. All details concerning varieties of Lie algebras can be found in Bahturin [1]. In addition, all commutators are left-normed:

$$[x_1,x_2] = x_1x_2-x_2x_1,\ [x_1,\ldots,x_{c-1},x_c] = [[x_1,\ldots,x_{c-1}],x_c].$$

2. The Proof of the Main Result

Denote by P_n the set of all elements of L multilinear in $x_1,\ldots,x_n$ and let V_p be the variety of Lie rings determined by (1). Then

$$P_n(V_p) = P_n/(P_n \cap [(L')^p,L])$$

is the additive group of the multilinear polynomials in the relatively free ring of V_p.

LEMMA 1. *Let the finitely generated additive group $P_n(V_p)$ be decomposed into a sum of cyclic groups*

$$(2)\qquad P_n(V_p) = Z^{\oplus r} \oplus Z_{p^{m_1}} \oplus\ldots\oplus Z_{p^{m_s}} \oplus Z_{q_1} \oplus\ldots\oplus Z_{q_t},$$

where p does not divide q_j, $j = 1,\ldots,t$. Then there are elements of order p in $P_n(V_p)$ if and only if

$$\dim_Q((Q \otimes P_n) \cap [(L'_Q)^p,L_Q]) > \dim_{Z_p}((Z_p \otimes P_n) \cap [(L'_{Z_p})^p, L_{Z_p}]).$$

Proof. Clearly, there is p-torsion in $P_n(V_p)$ if and only if $s \neq 0$ in (2), that is when

$$\dim_Q Q \otimes P_n(V_p) = r < r + s = \dim_{Z_p} Z_p \otimes P_n(V_p).$$

For every field K,

$$\dim_K K \otimes P_n(V_p) = \dim_K K \otimes P_n - \dim_K((K \otimes P_n) \cap [(L'_K)^p,L_K]).$$

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This gives the result immediately since $\dim_K K\otimes P_n = (n-1)!$. □

LEMMA 2. *Let*

(3) $$[[x_{\sigma(1)},x_{\sigma(2)}]\dots[x_{\sigma(2p-1)},x_{\sigma(2p)}],x_{\sigma(2p+1)}] ,$$

$$\sigma(1)<\sigma(2),\ \sigma(3)<\sigma(4),\dots,\sigma(2p-1)<\sigma(2p),\ \sigma\in \mathrm{Sym}(2p+1) ,$$

be all multilinear values in $K\langle X\rangle$ *of*

$$[[x_1,x_2]\dots[x_{2p-1},x_{2p}],x_{2p+1}] ,$$

K *being a field. Then the only linear dependence upon (3) is given by*

$$\sum(\mathrm{sign}\ \sigma)[[x_{\sigma(1)},x_{\sigma(2)}]\dots[x_{\sigma(2p-1)},x_{\sigma(2p)}],x_{\sigma(2p+1)}] = 0 .$$

Proof. It suffices to consider the case $K = Q$ only. By the equality $[x_1,x_2] = -[x_2,x_1]$, we assume that

(4) $$\sum a_\sigma[[x_{\sigma(1)},x_{\sigma(2)}]\dots[x_{\sigma(2p-1)},x_{\sigma(2p)}],x_{\sigma(2p+1)}] = 0 ,$$

where $a_\sigma \in Q$, the sum in (4) is over all $\sigma\in \mathrm{Sym}(2p+1)$ and

(5) $$a_\sigma = -a_\tau ,\quad \tau = \sigma(2k-1,2k),\ k=1,\dots,p .$$

Here the multiplication in the symmetric group is from right to left. We rewrite (4) in the form

(6) $$\sum_{i=1}^{p}\ \sum a_\sigma[x_{\sigma(1)},x_{\sigma(2)}]\dots[x_{\sigma(2i-1)},x_{\sigma(2i)},x_{\sigma(2p+1)}]\dots[x_{\sigma(2p-1)},x_{\sigma(2p)}] = 0 .$$

A basis for the vector space spanned by the multilinear products of commutators from $K\langle X\rangle$ is given in [4]. In particular, the polynomials

$$[x_{j_1},x_{j_2}]\dots[x_{j_{2i-1}},x_{j_{2i}},x_{j_{2p+1}}]\dots[x_{j_{2p-1}},x_{j_{2p}}]$$

are linearly independent for

$$j_1<j_2;\ j_3<j_4;\dots;j_{2i-1}<j_{2i},j_{2i-1}<j_{2p+1};\dots;j_{2p-1}<j_{2p};i=1,\ \dots,\ p .$$

It follows from (6) that

$$(a_\varepsilon[x_1,x_2,x_{2p+1}] + a_{(1,2,2p+1)}[x_2,x_{2p+1},x_1] + a_{(2,2p+1)}[x_1,x_{2p+1},x_2])[x_3,x_4]\dots[x_{2p-1},x_{2p}] = 0 ,$$

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where ε is the identical permutation. The only linear dependence on $[x_1,x_2,x_{2p+1}],[x_2,x_{2p+1},x_1]$ and $[x_1,x_{2p+1},x_2]$ is given by the Jacobi identity. Hence

$$a_\varepsilon = a_{(1,2,2p+1)} = -a_{(2,2p+1)}.$$

Similarly we obtain the relations

$$(7) \qquad a_\sigma = a_{\sigma\rho}\ ,\rho = (2k-1,2k,2p+1),\ k=1,\ldots,p\ .$$

The permutations $(2k-1,2k)$, $(2k-1,2k,2p+1)$, $k=1,\ldots,p$, generate the symmetric group $\mathrm{Sym}(2p+1)$. Therefore, we derive from (5) and (7) that $a_\sigma = (\mathrm{sign}\ \sigma)a_\varepsilon$. On the other hand,

$$\sum(\mathrm{sign}\ \sigma)[[x_{\sigma(1)},x_{\sigma(2)}]\ldots[x_{\sigma(2p-1)},x_{\sigma(2p)}],x_{\sigma(2p+1)}]$$
$$= 2^p\sum(\mathrm{sign}\ \sigma)(x_{\sigma(1)}\ldots x_{\sigma(2p)}x_{\sigma(2p+1)}-x_{\sigma(2p+1)}x_{\sigma(1)}\ldots x_{\sigma(2p)}) = 0\ ,$$

because

$$\sum(\mathrm{sign}\ \sigma)x_{\sigma(1)}\ldots x_{\sigma(2p)}x_{\sigma(2p+1)} = \sum(\mathrm{sign}\ \sigma)x_{\sigma(2p+1)}x_{\sigma(1)}\ldots x_{\sigma(2p)}.$$

Consequently, the desired linear dependence does exist. □

LEMMA 3. *The standard polynomial*

$$S_{2p}(x_1,\ldots,x_{2p}) = \sum(\mathrm{sign}\ \sigma)x_{\sigma(1)}\cdots x_{\sigma(2p)}$$
$$= \sum(\mathrm{sign}\ \tau)[x_{\tau(1)},x_{\tau(2)}]\ldots[x_{\tau(2p-1)},x_{\tau(2p)}] \in Z_p\langle X\rangle\ ,$$
$$\sigma,\tau \in \mathrm{Sym}(2p),\ \tau(1)<\tau(2),\ \tau(3)<\tau(4),\ldots,\ \tau(2p-1)<\tau(2p)\ ,$$

belongs to $(L'_{Z_p})^p$.

Proof. Obviously, $S_{2p}(x_1,\ldots,x_{2p})$ is a linear combination of values of the polynomial

$$h(y_1,\ldots,y_p) = \sum y_{\rho(1)}\ldots y_{\rho(p)} \in Z_p\langle Y\rangle\ ,\ \rho \in \mathrm{Sym}(p)\ .$$

Hence, it suffices to establish that $h(y_1,\ldots,y_p)$ is a Lie element. Having in mind that $Z_p\langle Y\rangle$ is a restricted Lie algebra, we obtain that

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$$(y_1+\ldots+y_p)^p = y_1^p+\ldots+y_p^p + f(y_1,\ldots,y_p) ,$$

where $f(y_1,\ldots,y_p) \in L_{Z_p}$. The multilinear component of $f(y_1,\ldots,y_p)$ is a Lie element as well, and equals $h(y_1,\ldots,y_p)$. □

Proof of the Theorem. By [1, Proposition 1, p.115], the relatively free ring of the variety of Lie rings $N_{p-1}A$ is torsion-free. Hence

$$\dim_K((K\otimes P_{2p}) \cap (L_K')^p)$$

does not depend on the choice of the field K. Let

$$u_s(x_1,\ldots,x_{2p}) = [[x_{s_1},x_{s_2}],\ldots,[x_{s_{2p-1}},x_{s_{2p}}]], \ s \in I ,$$

be a basis of $(K \otimes P_{2p}) \cap (L_K')^p$. Therefore

$$R_K = (K \otimes P_{2p+1}) \cap [(L_K')^p , L_K]$$

is spanned by the polynomials

(8) $[u_s(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{2p+1}),x_i], \ s \in I, \ i=1,\ldots,2p+1 .$

Let us assume first that $K = Q$. Then $S_{2p}(x_1,\ldots,x_{2p})$ is not a Lie element in $Q\langle X\rangle$ and by Lemma 2 the polynomials (8) are linearly independent over Q. Now, if $K = Z_p$, then Lemma 3 gives

$$S_{2p}(x_1,\ldots,x_{2p}) = \sum b_s u_s(x_1,\ldots,x_{2p}) \in (L_{Z_p}')^p .$$

In virtue of Lemma 2 we obtain that (8) are linearly dependent over Z_p and

$$\dim_{Z_p} R_{Z_p} < \dim_Q R_Q .$$

The proof of the theorem follows immediately from Lemma 1. □

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