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INVARIANT FACTORS UNDER RANK ONE PERTURBATIONS

ROBERT C. THOMPSON

Let R be a principal ideal domain, i.e., a commutative ring without zero divisors in which every ideal is principal. The invariant factors of a matrix A with entries in R are the diagonal elements when A is converted to a diagonal form $D = UAV$, where U, V have entries in R and are unimodular (invertible over R), and the diagonal entries $d_1, \ldots, d_n$ of D form a divisibility chain: $d_1|d_2| \ldots |d_n$. Very little has been proved about how invariant factors may change when matrices are added. This is in contrast to the corresponding question for matrix multiplication, where much information is now available [**6**]. The objective of this paper is to begin the analysis of the properties of the invariant factors of matrices under matrix addition, by describing in full how the invariant factors may change when a rank one matrix is added to a given matrix. Other results pertaining to additive properties of invariant factors will appear in a subsequent paper [**7**].

The following point, pertaining indirectly to the results in this paper, is worth mentioning. If A is a real or complex matrix, its singular values are, by definition, the eigenvalues of the nonnegative semidefinite Hermitian matrix $(AA^*)^{1/2}$, where A^* is the complex conjugate transpose of A. It was recently found [**4**] that singular values of real or complex matrices often possess properties astonishingly like the properties of invariant factors of integral matrices. The results to be established below further extend this parallelism, since the corresponding exactly analogous theorems for singular values are already known, and appear in [**5**].

Applications to classical similarity theory of our results for matrices over R will be given below. For some other results pertaining to similarity, see [**1**].

An $n \times n$ matrix over R has rank one if it has the form

$$[x_iy_j]_{1 \leqq i, j \leqq n},$$

where $x_1, \ldots, x_n, y_1, \ldots, y_n$ all belong to R, and where not all x_i and not all y_j are zero.

Conceivably, matrices $[x_iy_j]$ may exist with all $x_iy_j \in R$ and yet some x_i or y_j in a larger ring. This is essentially not the case, however, since it is easy to prove that any such matrix has the form $[(x_i\alpha)(y_j\alpha^{-1})]$ where all x_i, $y_j \in R$ and α may be outside of R. Thus any rank one matrix with entries in R has the

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form xy with x a column vector and y a row vector, both x and y having entries in R.

We are now ready to state our first theorem. The symbol | denotes "divides."

THEOREM 1. *Let A be a fixed $n \times n$ matrix over R, with invariant factors $h_1(A), \ldots, h_n(A)$, numbered such that*

$$h_1(A)|h_2(A)| \ldots |h_n(A).$$

Put

$$(1) \quad B = A + xy$$

where x is a column vector over R, y a row vector over R. Then the achievable invariant factors

$$h_1(B)| \ldots |h_n(B)$$

of B, as x and y range over all vectors with entries in R, are precisely those elements of R for which

$$(2) \quad \begin{array}{l} h_1(A)|h_2(B)|h_3(A)|h_4(B) \ldots, \\ h_1(B)|h_2(A)|h_3(B)|h_4(A) \ldots . \end{array}$$

Proof. We first prove that conditions (2) must hold. Let A, x, y be given, and suppose B satisfies (1). For certain unimodular matrices U, V over R, we have $A = U \operatorname{diag}(h_1(A), \ldots, h_n(A))V$. From this, it follows that B is a submatrix of the $(n+1) \times (n+1)$ matrix

$$C = \begin{pmatrix} \cdot & \cdot \\ \cdot & B \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ x & U \end{pmatrix} \operatorname{diag}(1, h_1(A), \ldots, h_n(A)) \begin{pmatrix} 1 & y \\ 0 & V \end{pmatrix}.$$

We now invoke the following result from [**3** or **4**]: If

$$M = \begin{pmatrix} \cdot & \cdot \\ \cdot & N \end{pmatrix}$$

is $(n+1) \times (n+1)$ and N is $n \times n$, the invariant factors of M and N are related by

$$\begin{array}{l} h_1(M)|h_1(N)|h_3(M), \\ h_2(M)|h_2(N)|h_4(M), \\ \quad \ldots \\ h_{n-1}(M)|h_{n-1}(N)|h_{n+1}(M), \\ h_n(M)|h_n(N). \end{array}$$

Applying this result with $M = C$ and $N = B$, and using $h_1(C) = 1$, $h_i(C) = h_{i-1}(A)$ for $i > 1$, we obtain (2).

For the converse part of the proof we begin by assuming that

$$A = \operatorname{diag}(h_1(A), \ldots, h_n(A)).$$

The proof amounts to giving explicit values of the x_i and y_j for which

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$B = A + xy$ has specified invariant factors $h_1(B)| \ldots |h_n(B)$ satisfying (2). Set $h_0(A) = 1$, and put

$$x_i = \prod_{\substack{j=1 \\ j \text{ even}}}^{i} \frac{h_j(B)}{h_{j-1}(A)}, \quad \text{for odd } i,\ 1 \leqq i \leqq n,$$

$$x_i = 0, \quad \text{for even } i,\ 1 \leqq i < n,$$

$$y_i = \prod_{\substack{j=1 \\ j \text{ odd}}}^{i} \frac{h_j(B)}{h_{j-1}(A)}, \quad \text{for even } i,\ 1 \leqq i \leqq n,$$

$$y_i = 0, \quad \text{for odd } i,\ 1 \leqq i < n,$$

$$x_n = \prod_{\substack{j=1 \\ j \text{ even}}}^{n} \frac{h_j(B)}{h_{j-1}(A)}, \quad \text{when } n \text{ is even.}$$

$$y_n = \prod_{\substack{j=1 \\ j \text{ odd}}}^{n} \frac{h_j(B)}{h_{j-1}(A)}, \quad \text{when } n \text{ is odd.}$$

(If $h_t(A) = 0$ for some t, from (2) we get $h_{t+1}(B) = 0$, and so we set $h_{t+1}(B)/h_t(A) = 1$.) Now let $x = [x_1, x_2, \ldots, x_n]^T$ and $y = [y_1, y_2, \ldots, y_n]$, and put

$$M = \begin{pmatrix} 0 & y \\ x & A \end{pmatrix}.$$

We claim that the invariant factors of M are $1, h_1(B), \ldots, h_n(B)$. If $n = 1$ this is evident since $x_1 = 1$ and $y_1 = h_1(B)$. We argue by induction on n. Since $x_1 = 1$, evidently the first invariant factor of M is 1, and we proceed to split off this invariant factor. Add $-h_1(A)$ times column 1 of M to column 2, so that row 2 becomes $[1, 0, 0, 0, \ldots, 0]$, then add appropriate multiples of row 2 to the subsequent rows to make the initial column $[0, 1, 0, 0, 0, \ldots, 0]^T$. Now interchange rows 1 and 2, multiply column 2 by -1, and transpose the resulting matrix. The matrix so obtained has the form

$$[1] \dotplus h_1(B)M'$$

where M' is one dimension smaller than M and is constructed in the same manner as M using $h_2(A)/h_1(B), \ldots, h_n(A)/h_1(B), h_2(B)/h_1(B), \ldots, h_n(B)/h_1(B)$. Thus M' has $1, h_2(B)/h_1(B), \ldots, h_n(B)/h_1(B)$ as its invariant factors, and therefore M has $1, h_1(B), \ldots, h_n(B)$ as invariant factors.

Now let C be obtained from M by adding row 2 to row 1. After a renaming of the y_i, matrix C has the form

$$C = \begin{pmatrix} 1 & y \\ x & A \end{pmatrix},$$

and has $1, h_1(B), \ldots, h_n(B)$ as its invariant factors. We proceed to reduce C

to its Smith form. Let U and V be unimodular matrices, to be specified in a moment. We have

$$\begin{pmatrix} 1 & 0 \\ -Ux & U \end{pmatrix} C \begin{pmatrix} 1 & -yV \\ 0 & V \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & U(-xy+A)V \end{pmatrix};$$

thus the lower right block $U(-xy + A)V$ has $h_1(B), \ldots, h_n(B)$ as its invariant factors. For appropriate unimodular U and V we therefore achieve $B = U(-xy + A)V$, where B is some fixed matrix having $h_1(B), \ldots, h_n(B)$ as its invariant factors. Multiplying by U^{-1}, V^{-1} and changing notation by renaming $U^{-1}BV^{-1}$ as B and $-x$ as x, we obtain $B = A + xy$. This completes the proof.

As an application of Theorem 1, we shall now establish a result showing how similarity invariants (of matrices with entries in a field) change when added to a rank one matrix. Let $\mathfrak{F}$ be a field, and λ an indeterminate over $\mathfrak{F}$. The similarity invariants of an $n \times n$ matrix A over $\mathfrak{F}$ are the invariant factors of the polynomial matrix $\lambda I - A$. Denote them by $h_1(A), h_2(A), \ldots, h_n(A)$. These are monic polynomials in λ with $h_1(A)|h_2(A)| \ldots |h_n(A)$ and $h_1(A) \ldots h_n(A) = \det(\lambda I - A)$.

THEOREM 2. *Let $n \times n$ matrix A over field $\mathfrak{F}$ have similarity invariants $h_1(A)| \ldots |h_n(A)$. Then: as column n-tuple x and row n-tuple y range over all vectors with entries in $\mathfrak{F}$, the similarity invariants assumed by the matrix*

$$B = A + xy$$

are precisely the monic polynomials $h_1(B)| \ldots |h_n(B)$ for which

(i) *degree*$\{h_1(B) \ldots h_n(B)\} = n$,

(ii) $h_1(B)|h_2(A)|h_3(B)|h_4(A)| \ldots,$
$h_1(A)|h_2(B)|h_3(A)|h_4(B)| \ldots.$

Proof. The necessity of conditions (i) and (ii) is an immediate consequence of Theorem 1, taking $R = \mathfrak{F}[\lambda]$, since we have

$$\lambda I - B = (\lambda I - A) + (-x)y.$$

For the converse assertion, additional argument is required in order to cope with the requirement that x, y have entries in $\mathfrak{F}$. Suppose that polynomials $h_1(B), h_2(B), \ldots$ are given satisfying (i) and (ii).

Let

$$C = \begin{pmatrix} 1 & y \\ x & \operatorname{diag}(h_1(A), \ldots, h_n(A)) \end{pmatrix}$$

be the $(n + 1) \times (n + 1)$ matrix introduced in the proof of Theorem 1. This matrix has entries in $\mathfrak{F}[\lambda]$ and invariant factors $1, h_1(B), \ldots, h_n(B)$. Since $\operatorname{diag}(h_1(A), \ldots, h_n(A))$ is equivalent to $\lambda I - A$, i.e., equals $\lambda I - A$ times unimodular polynomial matrices, we may pre and post-multiply C by unimodular matrices of the form diag $(1, U(\lambda))$ and obtain a new matrix

$$C_1 = \begin{pmatrix} 1 & z_2(\lambda) \\ z_1(\lambda) & \lambda I - A \end{pmatrix},$$

still having $1, h_1(B), \ldots, h_n(B)$ as invariant factors. Now, by the division algorithm for matrices, we have

$$z_1(\lambda) = (\lambda I - A)q_1(\lambda) + c_1,$$
$$z_2(\lambda) = q_2(\lambda)(\lambda I - A) + c_2,$$

where $q_1(\lambda)$, $q_2(\lambda)$ are vectors with polynomial entries and c_1, c_2 are constant vectors. But then

$$\begin{pmatrix} 1 & -q_2(\lambda) \\ 0 & I \end{pmatrix} C_1 \begin{pmatrix} 1 & 0 \\ -q_1(\lambda) & I \end{pmatrix} = \begin{pmatrix} c(\lambda) & c_2 \\ c_1 & \lambda I - A \end{pmatrix}$$

(where $c(\lambda)$ is some polynomial) still has $1, h_1(B), \ldots, h_n(B)$ as its invariant factors. Since c_1, c_2 are constant vectors, and since $h_1(B) \ldots h_n(B)$ has degree n, the polynomial $c(\lambda)$ must be a nonzero constant, and may be taken equal to 1. Thus we have found a matrix

$$\begin{pmatrix} 1 & c_2 \\ c_1 & \lambda I - A \end{pmatrix}$$

with invariant factors $1, h_1(B), \ldots, h_n(B)$, and with constant vectors c_1, c_2 bordering the block $\lambda I - A$.

Let $U(\lambda)$ and $V(\lambda)$ be $n \times n$ unimodular matrices, to be specified in a moment. We have

$$\begin{pmatrix} 1 & 0 \\ -U(\lambda)c_1 & U(\lambda) \end{pmatrix} \begin{pmatrix} 1 & c_2 \\ c_1 & \lambda I - A \end{pmatrix} \begin{pmatrix} 1 & -c_2 V(\lambda) \\ 0 & V(\lambda) \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & U(\lambda)(-c_1c_2 + (\lambda I - A))V(\lambda) \end{pmatrix},$$

and this matrix still has $1, h_1(B), \ldots, h_n(B)$ for its invariant factors. Thus the lower right block has the same invariant factors as $\lambda I - B$, and therefore unimodular $U(\lambda)$, $V(\lambda)$ exist such that

$$(3) \quad \lambda I - B = U(\lambda)[\lambda I - A - c_1c_2]V(\lambda),$$

where B is some fixed $\mathfrak{F}$ matrix with similarity invariants $h_1(B), \ldots, h_n(B)$. Now by [**2**], the polynomial matrices $U(\lambda)$, $V(\lambda)$ in (3) may be replaced by constant matrices, i.e.,

$$\lambda I - B = U[\lambda I - A - c_1c_2]V$$

for constant matrices U, V. Comparing coefficients of λ on each side, we obtain

$V = U^{-1}$, and then obtain

$$B = U(A + c_1c_2)U^{-1}.$$

Renaming $U^{-1}BU$ as B, we finally have an $\mathfrak{F}$ matrix

$$B = A + c_1c_2$$

with the prescribed similarity invariants. This is the desired result since c_1, c_2 have entries in $\mathfrak{F}$.

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University of California,
Santa Barbara, California