

## SOME COMPARISON CRITERIA IN OSCILLATION THEORY

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### Abstract

The purpose of this paper is to establish comparison criteria, by which the oscillatory and asymptotic behavior of linear retarded differential equations of arbitrary order is inherited from the oscillation of an associated second order linear ordinary differential equation. These criteria are new even in the case of ordinary differential equations.

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### 1. Introduction

Let us consider the  $n$ th order ( $n > 2$ ) linear retarded differential equation

$$(E) \quad x^{(n)}(t) + p(t)x[g(t)] = 0, \quad t \geq t_0,$$

where  $p$  is a nonnegative continuous function on the interval  $[t_0, \infty)$  and  $g$  is a continuous function on  $[t_0, \infty)$  such that

$$\lim_{t \rightarrow \infty} g(t) = \infty \quad \text{and} \quad g(t) \leq t \quad \text{for every } t \geq t_0.$$

It is supposed that  $p$  is not identically zero on any interval of the form  $[t'_0, \infty)$ ,  $t'_0 \geq t_0$ , and  $g$  is continuously differentiable on  $[t_0, \infty)$  with

$$g'(t) > 0 \quad \text{for every } t \geq t_0.$$

We consider only such solutions  $x(t)$  of (E) which are defined for all large  $t$ . The oscillatory character is considered in the usual sense, that is a continuous

real-valued function defined on an interval of the form  $[\tau, \infty)$  is said to be *oscillatory* if the set of its zeros is unbounded above, and otherwise it is said to be *nonoscillatory*. The differential equation (E) is called oscillatory if all its solutions are oscillatory and (E) is called nonoscillatory if all nontrivial solutions are nonoscillatory. As it is well-known, if  $r$  is a positive continuous function on an interval  $[\tau_0, \infty)$  and  $q$  is a continuous function on  $[\tau_0, \infty)$ , then the second order linear ordinary differential equation  $[r(t)y'(t)]' + q(t)y(t) = 0$  is nonoscillatory if it has at least one nonoscillatory solution.

The oscillatory and asymptotic behavior of the bounded solutions of (E) is well described by the following theorem (see Staikos and Sficas (1975)).

**THEOREM 0.** *Under the condition*

$$(C) \quad \int^{\infty} t^{n-1} p(t) dt = \infty,$$

*for  $n$  even all bounded solutions of (E) are oscillatory while for  $n$  odd every bounded solution of (E) is either oscillatory or tending monotonically to zero at  $\infty$  together with its first  $n - 1$  derivatives.*

It is an interesting problem to establish comparison criteria, by which the oscillatory and asymptotic behavior of all solutions of the differential equation (E) is inherited from the oscillation of an associated second order linear ordinary differential equation. The results of the present paper are concerned with this problem and are new even in the case of the ordinary differential equation

$$(\tilde{E}) \quad x^{(n)}(t) + p(t)x(t) = 0.$$

By our comparison results, known oscillation criteria for second order linear ordinary differential equations can be used to obtain sharp results for the oscillatory and asymptotic behavior of differential equations of the form (E). An application of our results to this direction will be given before closing the paper. Note that there is a voluminous literature on oscillation in second order ordinary equations (see, for example, Swanson (1968), Chapter 2). Some results of the same nature but for the case of ordinary differential equations of the form  $(\tilde{E})$  have been given by Lovelady (1975, 1976) (see also Trench (1981)). Our results are not comparable with the Lovelady's results.

The special case of the retarded differential equation

$$(A) \quad x^{(n)}(t) + p(t)x(\lambda t) = 0,$$

where  $t_0 > 0$  and  $\lambda$  is a constant with  $0 < \lambda < 1$ , is discussed in particular.

2. Lemmas

To obtain our results we need Lemmas 1, 2 and 3 below. Lemma 1 is an adaptation (see Grammatikopoulos, Sficas and Staikos (1979)) of a well-known lemma of Kiguradze (1964), while Lemma 2 is due to Staikos (1976) and partly to the author (1981). Lemma 3 is an extension of a result of Wintner (1951) (see also Swanson (1968), page 63).

LEMMA 1. *Let  $u$  be a positive and  $k$ -times differentiable function on an interval  $[\tau, \infty)$  with its  $k$ th derivative  $u^{(k)}$  nonpositive on  $[\tau, \infty)$  and not identically zero on any interval of the form  $[\tau', \infty)$ ,  $\tau' \geq \tau$ .*

*Then there exist a  $\tau^* \geq \tau$  and an integer  $l$ ,  $0 \leq l \leq k - 1$ , with  $k + l$  odd so that*

$$\begin{cases} (-1)^{l+j} u^{(j)} > 0 & \text{on } [\tau^*, \infty) \quad (j = l, \dots, k - 1), \\ u^{(i)} > 0 & \text{on } [\tau^*, \infty) \quad (i = 1, \dots, l - 1), \text{ when } l > 1. \end{cases}$$

LEMMA 2. *Let  $u$  be as in Lemma 1 and  $\tau^* \geq \tau$  be assigned to  $u$  by Lemma 1. Moreover, let  $\theta$  be a number with  $0 < \theta < 1$ . Then there exists a  $\tau \geq \tau^*/\theta$  such that*

$$(1) \quad u(\theta t) \geq \frac{[\theta(1 - \theta)]^{k-1}}{(k - 1)!} t^{k-1} u^{(k-1)}(t) \quad \text{for all } t \geq \hat{\tau}.$$

*In addition, when  $\lim_{t \rightarrow \infty} u(t) \neq 0$ , for some  $\hat{\tau} \geq \tau$  we have*

$$(2) \quad u(t) \geq \frac{\theta}{(k - 1)!} t^{k-1} u^{(k-1)}(t) \quad \text{for every } t \geq \hat{\tau}.$$

PROOF. The proof given here is that of Staikos (1976). Let  $l$ ,  $0 \leq l \leq k - 1$ , be the integer assigned to the function  $u$  as in Lemma 1. Then, for any  $s$ ,  $t$  with  $\tau^* \leq s \leq t$  we get

$$(3) \quad u^{(l)}(s) \geq \frac{(t - s)^{k-1-l}}{(k - 1 - l)!} u^{(k-1)}(t).$$

This is obvious for  $l = k - 1$  and, when  $l < k - 1$ , it can be derived by applying the Taylor formula. Thus,

$$(4) \quad u^{(l)}(\theta t) \geq \frac{(1 - \theta)^{k-1-l}}{(k - 1 - l)!} t^{k-1-l} u^{(k-1)}(t) \quad \text{for all } t \geq \tau^*/\theta,$$

which proves (1) with  $\hat{\tau} = \tau^*/\theta$  in the case where  $l = 0$ . Hence, we suppose that  $l > 0$  and by using the Taylor formula with integral remainder we obtain

$$u(t) \geq \frac{1}{(l - 1)!} \int_{\tau^*}^t (t - s)^{l-1} u^{(l)}(s) ds, \quad t \geq \tau^*,$$

and then, by (3),

$$(5) \quad u(t) \geq \frac{(t - \tau^*)^{k-1}}{(k-1)!} u^{(k-1)}(t) \quad \text{for } t \geq \tau^*.$$

So, there exists a  $\tau_1 \geq \tau^*$  such that

$$u(t) \geq \frac{(1 - \theta)^{k-1}}{(k-1)!} t^{k-1} u^{(k-1)}(t) \quad \text{for every } t \geq \tau_1,$$

and consequently for all  $t \geq \hat{\tau} = \tau_1/\theta$  we have

$$\begin{aligned} u(\theta t) &\geq \frac{(1 - \theta)^{k-1}}{(k-1)!} (\theta t)^{k-1} u^{(k-1)}(\theta t) \\ &\geq \frac{[\theta(1 - \theta)]^{k-1}}{(k-1)!} t^{k-1} u^{(k-1)}(t), \end{aligned}$$

which proves (1) when  $l > 0$ .

Now, suppose that  $\lim_{t \rightarrow \infty} u(t) \neq 0$ . If  $l > 0$ , then from (5) it follows that for some  $\hat{\tau} \geq \tau^*$  the inequality (2) holds. This fact has also proved by the author (1981). In the case  $l = 0$ , by applying (4) with  $1 - \theta^{1/2(k-1)}$  in place of  $\theta$ , we get

$$u[(1 - \theta^{1/2(k-1)})t] \geq \frac{\theta^{1/2}}{(k-1)!} t^{k-1} u^{(k-1)}(t), \quad t \geq \tau_2,$$

where  $\tau_2 = \tau^*/(1 - \theta^{1/2(k-1)})$ . But, because of the fact that  $\lim_{t \rightarrow \infty} u(t) \neq 0$ , there exists a  $\hat{\tau} \geq \tau_2$  so that

$$\frac{u(t)}{u[(1 - \theta^{1/2(k-1)})t]} \geq \theta^{1/2} \quad \text{for } t \geq \hat{\tau}.$$

So, we get again

$$u(t) \geq \theta^{1/2} u[(1 - \theta^{1/2(k-1)})t] \geq \frac{\theta}{(k-1)!} t^{k-1} u^{(k-1)}(t)$$

for all  $t \geq \hat{\tau}$ .

**LEMMA 3.** *Let  $h$  be a positive continuous function on an interval  $[T, \infty)$ ,  $T \geq t_0$ . If there exists a continuously differentiable function  $w$  on  $[T, \infty)$  such that*

$$p(t) \leq -w'(t) - h(t)w^2(t) \quad \text{for every } t \geq T,$$

*then the second order linear ordinary differential equation*

$$[y'(t)/h(t)]' + p(t)y(t) = 0$$

*is nonoscillatory.*

**PROOF.** The differential equation

$$[y'(t)/h(t)]' + [-w'(t) - h(t)w^2(t)]y(t) = 0$$

is nonoscillatory, since it has the positive solution

$$y_0(t) = \exp\left[\int_T^t h(s)w(s) ds\right], \quad t \geq T.$$

Thus, the lemma follows by the Sturm comparison principle.

### 3. Main results

**THEOREM 1.** *Suppose that for some  $\theta, 0 < \theta < 1$ , the second order ordinary differential equation*

$$(D_1[\theta]) \quad \left\{ \frac{y'(t)}{[g(t)]^{n-2}g'(t)} \right\}' + \frac{\theta}{(n-2)!}p(t)y(t) = 0$$

is oscillatory. Then for the differential equation (E) we have:

(i) *For  $n$  even every nonoscillatory solution  $x$  satisfies*

$$\begin{cases} \lim_{t \rightarrow \infty} x(t) = L_x \text{ monotonically for some } L_x \in R^* - \{0\}, \\ \lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \text{ monotonically} \quad (i = 1, \dots, n-1). \end{cases}$$

(ii) *For  $n$  odd all unbounded solutions are oscillatory. Moreover, if in addition (C) holds, then for the equation (E) we have:*

(I) *For  $n$  even every nonoscillatory solution  $x$  satisfies*

$$\begin{cases} \lim_{t \rightarrow \infty} x(t) = \pm\infty \text{ monotonically,} \\ \lim_{t \rightarrow \infty} x^{(i)}(t) = 0 \text{ monotonically} \quad (i = 1, \dots, n-1). \end{cases}$$

(II) *For  $n$  odd every solution is either oscillatory or tending monotonically to zero at  $\infty$  together with its first  $n-1$  derivatives.*

**PROOF.** Let  $x$  be a positive solution on an interval  $[T_0, \infty), T_0 > \max\{t_0, 0\}$ , of the differential equation (E), which is unbounded if  $n$  is odd. Furthermore, let  $\tau \geq T_0$  be chosen so that

$$(6) \quad g(t) \geq T_0 \quad \text{for every } t \geq \tau.$$

Then from (E) it follows that  $x^{(n)}$  is nonpositive on  $[\tau, \infty)$  and not identically zero on any interval of the form  $[\tau', \infty)$ ,  $\tau' \geq \tau$ . Thus, by Lemma 1, there exist a  $\tau^* \geq \tau$  and an integer  $l$ ,  $0 \leq l \leq n - 1$ , with  $n + l$  odd so that

$$(7) \quad \begin{cases} (-1)^{l+j} x^{(j)} > 0 \text{ on } [\tau^*, \infty) & (j = l, \dots, n - 1), \\ x^{(i)} > 0 \text{ on } [\tau^*, \infty) & (i = 1, \dots, l - 1), \text{ when } l > 1. \end{cases}$$

If  $n$  is even, we always have  $l \geq 1$ . Moreover, when  $n$  is odd,  $x$  is unbounded and consequently we again have that  $l \geq 1$ . Thus,  $x'$  is positive on  $[\tau^*, \infty)$ .

Furthermore, we observe that for  $n$  odd it holds  $l > 1$  and hence  $\lim_{t \rightarrow \infty} x'(t) > 0$ . Also, if  $n$  is even and  $\lim_{t \rightarrow \infty} x'(t) = 0$ , then  $\lim_{t \rightarrow \infty} x(t) = L_x$  for some  $L_x \in R^* - \{0\}$  and  $\lim_{t \rightarrow \infty} x^{(i)} = 0$  ( $i = 1, \dots, n - 1$ ). So, for the case of even  $n$  let us suppose that  $\lim_{t \rightarrow \infty} x'(t) > 0$ .

Now, by Lemma 2, there exists a  $\hat{\tau} \geq \tau^*$  such that

$$x'(t) \geq \frac{\theta}{(n - 2)!} t^{n-2} x^{(n-1)}(t) \quad \text{for all } t \geq \hat{\tau}.$$

Thus, since  $x^{(n-1)}$  is decreasing on  $[\tau, \infty)$ , we have

$$x'[g(t)] \geq \frac{\theta}{(n - 2)!} [g(t)]^{n-2} x^{(n-1)}(t) \quad \text{for every } t \geq T,$$

where  $T \geq \hat{\tau}$  is chosen so that

$$(8) \quad g(t) \geq \hat{\tau} \quad \text{for all } t \geq T.$$

Furthermore, we set

$$w(t) = \frac{x^{(n-1)}(t)}{x[g(t)]}, \quad t \geq T.$$

Then for  $t \geq T$  we obtain

$$\begin{aligned} w'(t) &= \frac{x^{(n)}(t)}{x[g(t)]} - \frac{x^{(n-1)}(t)}{x^2[g(t)]} x'[g(t)] g'(t) \\ &\leq -p(t) - \frac{x^{(n-1)}(t)}{x^2[g(t)]} \cdot \frac{\theta}{(n - 2)!} [g(t)]^{n-2} x^{(n-1)}(t) g'(t). \end{aligned}$$

That is,

$$p(t) \leq -w'(t) - \frac{\theta}{(n - 2)!} [g(t)]^{n-2} g'(t) w^2(t) \quad \text{for every } t \geq T.$$

Hence, Lemma 3 ensures that the equation  $(D_1[\theta])$  is nonoscillatory.

Finally, by applying Theorem 0, we complete the proof of the theorem.

**COROLLARY 1.** *Suppose that for some  $\theta, 0 < \theta < 1$ , the second order differential equation*

$$(\tilde{D}_1[\theta]) \quad \left[ \frac{y'(t)}{t^{n-2}} \right]' + \frac{\theta}{(n-2)!} p(t)y(t) = 0$$

*is oscillatory. Then for the differential equation  $(\tilde{E})$  we have the conclusions (i) and (ii) of Theorem 1 and, when in addition (C) holds, the conclusions (I) and (II) of Theorem 1.*

**THEOREM 2.** *Let  $n$  be even. If the second order ordinary differential equation*

$$(D_2) \quad \left\{ \frac{y'(t)}{[g(t)]^{n-2} g'(t)} \right\}' + \frac{(n-1)^{n-1} (n-2)^{n-2}}{(n-2)! (2n-3)^{2n-3}} p(t)y(t) = 0$$

*is oscillatory, then (E) is also oscillatory.*

**PROOF.** Let  $x$  be a positive solution on an interval  $[T_0, \infty)$ ,  $T_0 > \max\{t_0, 0\}$ , of the equation (E) and let  $\tau \geq T_0$  be such that (6) holds. Then  $x^{(n)}$  is nonpositive on  $[\tau, \infty)$  and not identically zero on any interval of the form  $[\tau', \infty)$ ,  $\tau' \geq \tau$ . So, Lemma 1 ensures the existence of an integer  $l, 0 \leq l \leq n-1$ , with  $n+l$  odd and such that (7) is satisfied for some  $\tau^* \geq \tau$ . Since  $n$  is even we always have  $l \geq 1$  and consequently  $x'$  is positive on  $[\tau^*, \infty)$ .

Now, let  $\theta$  be an arbitrary number with  $0 < \theta < 1$ . By Lemma 2, for some  $\hat{\tau} \geq \tau^*/\theta$  we have

$$x'(\theta t) \geq \frac{[\theta(1-\theta)]^{n-2}}{(n-2)!} t^{n-2} x^{(n-1)}(t) \quad \text{for all } t \geq \hat{\tau}.$$

Thus, if we choose a  $T \geq \hat{\tau}$  so that (8) holds, then we obtain

$$x'[\theta g(t)] \geq \frac{[\theta(1-\theta)]^{n-2}}{(n-2)!} [g(t)]^{n-2} x^{(n-1)}(t) \quad \text{for every } t \geq T.$$

Next, we put

$$w(t) = \frac{x^{(n-1)}(t)}{x[\theta g(t)]}, \quad t \geq T.$$

Then for  $t \geq T$  we get

$$\begin{aligned} w'(t) &= -p(t) \frac{x[g(t)]}{x[\theta g(t)]} - \frac{x^{(n-1)}(t)}{x^2[\theta g(t)]} x'[\theta g(t)] \theta g'(t) \\ &\leq -p(t) - \frac{x^{(n-1)}(t)}{x^2[\theta g(t)]} \cdot \frac{[\theta(1-\theta)]^{n-2}}{(n-2)!} [g(t)]^{n-2} x^{(n-1)}(t) \theta g'(t) \end{aligned}$$

and consequently

$$p(t) \leq -w'(t) - \frac{\theta^{n-1}(1-\theta)^{n-2}}{(n-2)!} [g(t)]^{n-2} g'(t) w^2(t) \quad \text{for every } t \geq T.$$

Thus, by Lemma 3, the differential equation

$$\left\{ \frac{y'(t)}{[g(t)]^{n-2} g'(t)} \right\}' + \frac{\theta^{n-1}(1-\theta)^{n-2}}{(n-2)!} p(t) y(t) = 0$$

is nonoscillatory.

Finally, we observe that the maximum of the function

$$\mu(\theta) = \theta^{n-1}(1-\theta)^{n-2}, \quad \theta \in (0, 1),$$

is equal to

$$\mu\left(\frac{n-1}{2n-3}\right) = \frac{(n-1)^{n-1}(n-2)^{n-2}}{(2n-3)^{2n-3}}.$$

**COROLLARY 2.** *Let  $n$  be even. If the second order differential equation*

$$(\tilde{D}_2) \quad \left[ \frac{y'(t)}{t^{n-2}} \right]' + \frac{(n-1)^{n-1}(n-2)^{n-2}}{(n-2)!(2n-3)^{2n-3}} p(t) y(t) = 0$$

*is oscillatory, then  $(\tilde{E})$  is also oscillatory.*

**THEOREM 3.** *Let  $n$  be even. If the second order ordinary differential equation*

$$(B) \quad \left[ \frac{y'(t)}{t^{n-2}} \right]' + \frac{\lambda^{n-1}(1-\lambda)^{n-2}}{(n-2)!} p(t) y(t) = 0$$

*is oscillatory, then (A) is also oscillatory.*

**PROOF.** Let  $x$  be a positive solution on an interval  $[T_0, \infty)$ ,  $T_0 \geq t_0 > 0$ , of (A). Then  $x^{(n)}$  is nonpositive on  $[\tau, \infty)$ ,  $\tau = T_0/\lambda$ , and not identically zero on any interval of the form  $[\tau', \infty)$ ,  $\tau' \geq \tau$ . Hence, by Lemma 1, (6) holds for some  $\tau^* \geq \tau$  and some integer  $l$ ,  $0 \leq l \leq n-1$ , with  $n+l$  odd. Obviously,  $l \geq 1$  and thus  $x' > 0$  on  $[\tau^*, \infty)$ . Furthermore, Lemma 2 ensures that for some  $T \geq \tau^*/\lambda$  we have

$$x'(\lambda t) \geq \frac{[\lambda(1-\lambda)]^{n-2}}{(n-2)!} t^{n-2} x^{(n-1)}(t) \quad \text{for every } t \geq T.$$

So, if we set

$$w(t) = \frac{x^{(n-1)}(t)}{x(\lambda t)}, \quad t \geq T,$$



then we obtain

$$p(t) \leq -w'(t) - \frac{\lambda^{n-1}(1-\lambda)^{n-2}}{(n-2)!} t^{n-2} w^2(t) \quad \text{for every } t \geq T$$

and hence from Lemma 3 it follows that (B) is nonoscillatory.

REMARK 1. Theorems 1, 2 and 3 and Corollaries 1 and 2 remain valid with the differential equations

$$(D_1[\theta])^* \quad z''(t) + \frac{\theta}{(n-1)!(n-1)} \cdot \frac{p[g^{-1}(t^{1/(n-1)})]}{t^{(n-2)/(n-1)}g'[g^{-1}(t^{1/(n-1)})]} z(t) = 0,$$

$(D_2)^*$

$$z''(t) + \frac{(n-1)^{n-2}(n-2)^{n-2}}{(n-1)!(2n-3)^{2n-3}} \cdot \frac{p[g^{-1}(t^{1/(n-1)})]}{t^{(n-2)/(n-1)}g'[g^{-1}(t^{1/(n-1)})]} z(t) = 0,$$

$(B)^*$

$$z''(t) + \frac{\lambda^{n-1}(1-\lambda)^{n-2}}{(n-1)!(n-1)} \cdot \frac{p(t^{1/(n-1)})}{t^{(n-2)/(n-1)}} z(t) = 0,$$

$(\tilde{D}_1[\theta])^*$

$$z''(t) + \frac{\theta}{(n-1)!(n-1)} \cdot \frac{p(t^{1/(n-1)})}{t^{(n-2)/(n-1)}} z(t) = 0,$$

$(\tilde{D}_2)^*$

$$z''(t) + \frac{(n-1)^{n-2}(n-2)^{n-2}}{(n-1)!(2n-3)^{2n-3}} \cdot \frac{p(t^{1/(n-1)})}{t^{(n-2)/(n-1)}} z(t) = 0,$$

in place of the equations  $(D_1[\theta])$ ,  $(D_2)$ ,  $(B)$ ,  $(\tilde{D}_1[\theta])$  and  $(\tilde{D}_2)$  respectively. Indeed, if  $\rho$  is a continuously differentiable function on  $[t_0, \infty)$  with

$$\lim_{t \rightarrow \infty} \rho(t) = \infty \quad \text{and} \quad \rho'(t) > 0 \quad \text{for every } t \geq t_0$$

and  $K$  is a positive constant, then the substitution

$$z(t) = y[\rho^{-1}(t^{1/(n-1)})]$$

transforms the differential equation

$$\left\{ \frac{y'(t)}{[\rho(t)]^{n-2} \rho'(t)} \right\}' + Kp(t)y(t) = 0$$

into the equation

$$z''(t) + \frac{K}{(n-1)^2} \cdot \frac{p[\rho^{-1}(t^{1/(n-1)})]}{t^{(n-2)/(n-1)}\rho'[\rho^{-1}(t^{1/(n-1)})]} z(t) = 0.$$

REMARK 2. As it is easy to see, Theorems 1 and 2 can be stated, in a more general way, with the differential equations

$$(D_1[\theta; \sigma]) \quad \left\{ \frac{y'(t)}{[\sigma(t)]^{n-2} \sigma'(t)} \right\}' + \frac{\theta}{(n-2)!} p(t)y(t) = 0$$

and

$$(D_2[\sigma]) \quad \left\{ \frac{y'(t)}{[\sigma(t)]^{n-2} \sigma'(t)} \right\}' + \frac{(n-1)^{n-1} (n-2)^{n-2}}{(n-2)! (2n-3)^{2n-3}} p(t)y(t) = 0$$

in place of the equations  $(D_1[\theta])$  and  $(D_2)$  respectively, where  $\sigma$  is a continuously differentiable function on  $[t_0, \infty)$  with

$$\lim_{t \rightarrow \infty} \sigma(t) = \infty \quad \text{and} \quad \sigma(t) \leq g(t), \sigma'(t) > 0 \quad \text{for every } t \geq t_0.$$

In this case the assumption that  $g$  is continuously differentiable with  $g' > 0$  on  $[t_0, \infty)$  is not needed.

REMARK 3. Let  $f$  be a continuous function on  $[t_0, \infty) \times \mathbf{R}$  such that

$$f(t; y)/y \geq p(t) \quad \text{for all } t \geq t_0 \text{ and } y \neq 0.$$

Then Theorems 1, 2 and 3 and Corollaries 1 and 2 hold with the (not necessarily linear) differential equations

$$(E^*) \quad x^{(n)}(t) + f(t; x[g(t)]) = 0,$$

$$(\tilde{E}^*) \quad x^{(n)}(t) + f(t; x(t)) = 0$$

and

$$(A^*) \quad x^{(n)}(t) + f(t; x(\lambda t)) = 0$$

instead of the equations (E),  $(\tilde{E})$  and (A) respectively.

Before closing the paper, we give an application of Theorems 1 and 2. Consider the second order linear ordinary differential equation

$$\left[ \frac{y'(t)}{[g(t)]^{n-2} g'(t)} \right]' + Kp(t)y(t) = 0, \quad t \geq T_0,$$

where  $K$  is a positive constant and  $T_0 \geq t_0$  is chosen so that  $g(t) > 0$  for all  $t \geq T_0$ . By a criterion of Moore (1955), this equation is oscillatory if there exists a  $\beta, 0 < \beta < 1$ , such that

$$\int^\infty \left[ 1 + \int_{T_0}^t [g(s)]^{n-2} g'(s) ds \right]^\beta Kp(t) dt = \infty,$$

that is,

$$\int_{\infty}^{\infty} [g(t)]^{(n-1)\beta} p(t) dt = \infty.$$

The last condition holds for some  $\beta$ ,  $0 < \beta < 1$ , if

$$(*) \quad \int_{\infty}^{\infty} [g(t)]^{n-1-\varepsilon} p(t) dt = \infty \quad \text{for some } \varepsilon, 0 < \varepsilon < 1.$$

Furthermore, we observe that (\*) implies (C). Thus, from Theorems 1 and 2 we can derive the following known (see Sficas (1973)) result: *Under the condition (\*), for  $n$  even all solutions of (E) are oscillatory while for  $n$  odd every solution of (E) is either oscillatory or tending monotonically to zero at  $\infty$  together with its first  $n - 1$  derivatives.*

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