Proceedings of the Royal Society of Edinburgh, **153**, 307–326, *2023*
DOI:10.1017/prm.2021.89

Periodic solutions of *p*-Laplacian differential equations with jumping nonlinearity across half-eigenvalues

Tengfei Shen and Wenbin Liu
School of Mathematics, China University of Mining and Technology, Xuzhou Jiangsu, 221116, PR China (stfcool@126.com, liuwenbin-xz@163.com)

(Received 23 April 2021; accepted 13 December 2021)

This paper aims to investigate the existence of periodic solutions for p-Laplacian differential equations with jumping nonlinearity under the frame of half-eigenvalue. Based on the continuity theorem, some new results are obtained, which enrich and generalize the previous results.

Keywords: Jumping nonlinearity; Topological degree; Periodic solution; p-Laplacian equation

2020 *Mathematics Subject Classification:* Primary: 34B15, 34C25

1. Introduction

In this paper, we consider the existence of periodic solutions of p-Laplacian differential equations with jumping nonlinearity of variable coefficients as follows.

$$(\phi_p(u'(t)))' + a_+(t)\phi_p(u^+(t)) - a_-(t)\phi_p(u^-(t)) + W_0(t, u(t)) = e(t), \quad \text{and} \tag{1.1}$$

$$(\phi_p(u'(t)))' + a_+(t)\phi_p(u^+(t)) - a_-(t)\phi_p(u^-(t)) + W_1(t, u(t), u'(t)) = e(t), \tag{1.2}$$

where $W_0(t, u) = g_0(u) + g_1(t, u)$, $W_1(t, u, u') = \overline{g}_0(u) + \overline{g}_1(t, u')$; $g_0, \overline{g}_0 \in C(\mathbb{R}, \mathbb{R})$; $g_1, \overline{g}_1 \in C([0, T] \times \mathbb{R}, \mathbb{R})$ and are T-periodic in t; $a_\pm \in C([0, T], (0, \infty))$ and are T-periodic in t; $e \in C([0, T], \mathbb{R})$ and is T-periodic in t; $\phi_p(s) = |s|^{p-2}s$ $(s \neq 0)$, $\phi_p(0) = 0$, $1 < p < \infty$, $1/p + 1/q = 1$; $u^+(t) = \max\{u(t), 0\}$, $u^-(t) = \max\{-u(t), 0\}$.

In several decades ago, Dancer [6] and Fučik [11] presented the definition of the Dancer–Fučik spectrum for second-order linear differential equation with periodic boundary condition, which is a generalization of the common spectrum. Since then, many scholars devoted to investigating the existence of periodic solutions for second-order non-dissipative differential equation under the frame of Dancer–Fučik spectrum when asymptotic behaviours of potential function are given (see [8–10, 13, 16, 20, 21, 27] and references therein). After that, Del Pino, Manásevich, Murúa [7] gave the concept of Dancer–Fučik spectrum for second-order p-Laplacian differential equation with periodic boundary condition, i.e., if there exists a pair of positive constants (α, β) such that

$$\begin{cases}(\phi_p(u'))' + \alpha\phi_p(u^+) - \beta\phi_p(u^-) = 0, \\ u(0) = u(T), u'(0) = u'(T)\end{cases} \tag{1.3}$$

has a nontrivial solutions, then we say that (α, β) lies in the Dancer–Fučik spectrum S_p of the problem (). In the case $p = 2$, the set S_2 is the classical Dancer–Fučik spectrum. It is known that the set S_p has the structure (see [7]) $S_p = \bigcup_{k=0}^{+\infty} C_k$, where

$$C_0 = \{(\alpha, \beta)|\alpha \geqslant 0, \beta \geqslant 0, \alpha = 0 \text{ or } \beta = 0\},$$
$$C_k = \left\{(\alpha, \beta)|\alpha > 0, \beta > 0, C_n = \left\{(a, b)|a > 0, b > 0, \frac{1}{\sqrt[p]{a}} + \frac{1}{\sqrt[p]{b}} = \frac{T}{n\pi_p}\right\}\right.$$

in which $n \geqslant 1$ and π_p is given by

$$\pi_p = 2(p-1)^{\frac{1}{p}} \int_0^1 (1 - t^p)\,\mathrm{d}t = \frac{2\pi(p-1)^{\frac{1}{p}}}{p \sin\frac{\pi}{p}}.$$

The problem (1.3) is non-resonant if and only if $(\alpha, \beta) \notin S_p$, otherwise it is resonant. The Dancer–Fučik spectrum S_p has been used to investigate the existence of solutions of the following p-Laplacian problem.

$$\begin{cases}(\phi_p(u'))' + g(t, u) = 0, \\ u(0) = u(T), u'(0) = u'(T),\end{cases} \tag{1.4}$$

where $g(t, u) \in C([0, T] \times \mathbb{R}, \mathbb{R})$. Assuming that the potential function g satisfies the following asymptotic estimates.

$$p_1 \leqslant \liminf_{u\to+\infty} \frac{g(t, u)}{|u|^{p-2}u} \leqslant \limsup_{u\to+\infty} \frac{g(t, u)}{|u|^{p-2}u} \leqslant p_2 \text{ uniformly for a.e. } t \in [0, T], \tag{1.5}$$

$$q_1 \leqslant \liminf_{u\to-\infty} \frac{g(t, u)}{|u|^{p-2}u} \leqslant \limsup_{u\to-\infty} \frac{g(t, u)}{|u|^{p-2}u} \leqslant q_2, \text{ uniformly for a.e. } t \in [0, T], \tag{1.6}$$

where $p_i, q_i, i = 1.2$ are positive constant satisfying

$$\frac{T}{(n+1)\pi_p} < \frac{1}{\sqrt[p]{p_2}} + \frac{1}{\sqrt[p]{q_2}} \leqslant \frac{1}{\sqrt[p]{p_1}} + \frac{1}{\sqrt[p]{q_1}} < \frac{T}{n\pi_p}, \tag{1.7}$$

then existence results for periodic solutions have been obtained using Leray–Schauder degree theory and the comparison argument, see [7]. The condition (1.7) extends the preceding idea of non-resonance ($(\alpha, \beta) \notin S_p$), in the sense that the asymptotes of the ratio $\frac{g(u)}{|u|^{p-2}u}$ are now forced to lie in the rectangle $[p_1, p_2] \times [q_1, q_2] \subset \mathbb{R}^2$, and this rectangle lies between two consecutive curves of the spectrum S_p, so does not intersect S_p.

A further extension was introduced by Nkashama and Robinson [21] in the case $p = 2$. This extension uses an averaging method to allow the asymptotes of the above ratio to cross the spectrum S_p, so long as suitable averages of this ratio do not. This averaging method was extended to more general p by Chang and Qiao in [**5**]. The non-resonance condition in [**5**] assumes that g satisfies

$$p_1 \leqslant \liminf_{u\to+\infty} \frac{pG(t,u)}{|u|^p} \leqslant \limsup_{u\to+\infty} \frac{pG(t,u)}{|u|^p} \leqslant p_2 \text{ uniformly for a.e. } t\in[0,T], \quad (1.8)$$

$$q_1 \leqslant \liminf_{u\to-\infty} \frac{pG(t,u)}{|u|^p} \leqslant \limsup_{u\to-\infty} \frac{pG(t,u)}{|u|^p} \leqslant q_2, \text{ uniformly for a.e. } t\in[0,T] \quad (1.9)$$

with $([p_1,p_2]\times[q_1,q_2])\cap S_p = \emptyset$, where $G(t,u)=\int_0^u g(t,s)\,\mathrm{d}s$. Since

$$\liminf_{u\to\pm\infty} \frac{g(t,u)}{|u|^{p-2}u} \leqslant \liminf_{u\to\pm\infty} \frac{pG(t,u)}{|u|^p} \leqslant \limsup_{u\to\pm\infty} \frac{pG(t,u)}{|u|^p} \leqslant \limsup_{u\to\pm\infty} \frac{g(t,u)}{|u|^{p-2}u},$$

we see that the former non-resonance condition (1.7), in terms of $\frac{g(t,u)}{|u|^{p-2}u}$, implies the latter condition (1.8) and (1.9), that is, the latter condition is more general. In [**17**], Liu and Li considered the classical oscillation equation with p-Laplacian operator as follows.

$$(\phi_p(u'))' + g(u) = e(t),$$

where g satisfies

$$\lim_{u\to+\infty} \frac{pG(u)}{|u|^p} = \omega, \lim_{u\to-\infty} \frac{pG(u)}{|u|^p} = \nu \quad (1.10)$$

and $(\omega,\nu)\notin S_p$. Clearly, the (1.10) implies the (1.8) and (1.9) when g is autonomous. Under the condition that there exist positive constants b, d_1, d_2 such that $d_1 \leqslant \frac{g(u)}{|u|^{p-2}u} \leqslant d_2$ for all $|u|\geqslant b$, which is different from the assumptions given in [**5**], the existence of periodic solutions was obtained. For more articles concerning this topic, the readers can refer to [**1**, **14**, **15**, **26**] and its references.

Recently, Binding and Rynne [**3**] presented the concept of half-eigenvalue for the following p-Laplacian differential equations with periodic boundary condition, i.e., if λ is constant such that the following problem

$$\begin{cases} (\phi_p(u'))' + a_+(t)\phi_p(u^+) - a_-(t)\phi_p(u^-) + \lambda\phi_p(u) = 0, \\ u(0)=u(T), u'(0)=u'(T) \end{cases} \quad (1.11)$$

possesses a nontrivial solution, then λ is termed half-eigenvalue. Let $\sigma = \bigcup_{m=0}^{+\infty}\sigma_m$, where

$$\sigma_m = \{\lambda\in\mathbb{R}|\ (1.11) \text{ has a solution with 2m zeroes in } [0,T)\}. \quad (1.12)$$

By Prüfer approach, the characteristics of the set σ were also given (see [18, theorem 3.1]). It contains two sequences of half-eigenvalues $\{\lambda_{2m}^{min}\}_{m\in\mathbb{Z}^+}$ and $\{\lambda_{2m}^{max}\}_{m\in\mathbb{Z}^+}$ that satisfy

$$\lambda_0^{min} \leqslant \lambda_0^{max} < \lambda_2^{min} \leqslant \lambda_2^{max} < \cdots < \lambda_{2m}^{min} \leqslant \lambda_{2m}^{max} < \cdots \to +\infty,$$

$\sigma_0 = \{\lambda_0^{min}, \lambda_0^{max}\}$. Moreover, if we regard the half eigenvalues $\lambda_n^{\max,\min}$ as functions of the coefficient pairs (a^+,a^-), i.e., $\lambda_n^{\max,\min}(a^+,a^-)$, and we suppose that

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

$a_\pm \in L^\gamma$, it was shown in Li and Yan [18] that $\lambda_n^{\max,\min}(a^+, a^-)$ depends continuously on (a^+, a^-), with respect to the weak topology on $(L^\gamma)^2$. Furthermore, it should be mentioned that if $a_\pm$ are constants, (1.11) becomes

$$\begin{cases}(\phi_p(u'))' + \widetilde{a}_+\phi_p(u^+) - \widetilde{a}_-\phi_p(u^-) = 0,\\ u(0) = u(T), u'(0) = u'(T),\end{cases} \tag{1.13}$$

where $\widetilde{a}_\pm = a_\pm + \lambda$. Clearly, it reduces to the problem of Dancer–Fučik spectrum.

For the problems of half-eigenvalues with variable coefficients, it can be traced back to [2, 4]. After that, more and more scholars devoted to investigating half-eigenvalues with different boundary value problems. For the aspects of existence of solutions, recently, based on the Landesman–Lazer conditions, Genoud and Rynne [12] made use of the shooting method to investigate the existence of solutions for p-Laplacian differential equations with Dirichlet boundary condition under the frame of half-eigenvalue as follows.

$$\begin{cases}-(\phi_p(u'))' - a_+\phi_p(u^+) + a_-\phi_p(u^-) - \lambda\phi_p(u) = f(t,u),\ t \in (0,1),\\ u(0) = u(1) = 0.\end{cases} \tag{1.14}$$

Rynne [22] made a further study on this type of problem with the dissipation term. For more papers concerning this topic, we refer to Rynne [23–25] and references therein. However, it should be pointed out that there are few works on discussing the existence of periodic solutions for p-Laplacian problems with jumping nonlinearities under the frame of half-eigenvalue.

Motivated by the works mentioned above, our paper aims to investigate the existence of periodic solutions of p-Laplacian differential equations with jump nonlinearity of variable coefficients under the frame of half-eigenvalue. The variable coefficients and p-Laplacian operator bring many difficulties such as the prior estimation and convergence analysis. We make use of some analytical skills and the continuity theorem to overcome these difficulties and obtain some new results, which generalize the existence of periodic solutions under the framework of Dancer–Fučik spectral with constant coefficients to the case of half-eigenvalues with variable coefficients. Moreover, the model is more general and complex.

2. Main results

To begin with, we show some necessary basic knowledge and signs. Let $C([0,T],\mathbb{R})$ with norm $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$ and $L^p([0,T],\mathbb{R})$ with norm $\|u\|_p = \left(\int_0^T |u(t)|^p \mathrm{d}t\right)^{\frac{1}{p}}$. Set $a_{\pm,M} = \max_{t\in[0,T]} a_\pm(t)$, $a_{\pm,L} = \min_{t\in[0,T]} a_\pm(t)$. And define the following Banach space

$$C_T^1 = \{u \in C^1([0,T],\mathbb{R})|\ u(0) = u(T),\ u'(0) = u'(T)\}$$

endowed the norm $\|u\| = \max\{\|u\|_\infty, \|u'\|_\infty\}$.

Next, let us introduce the famous continuity theorem for p-Laplacian equation.

LEMMA 2.1 [19]. *Assume that Ω is an open bounded set in C_T^1 such that the following conditions hold:*

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

(i) For each $\lambda \in (0,1)$, the problem

$$\begin{cases} (\phi_p(u'))' = \lambda f(t,u,u'), \\ u(0) = u(T), u'(0) = u'(T) \end{cases} \tag{2.1}$$

has no solution on $\partial\Omega$.

(ii) The equation

$$F(a) := \frac{1}{T}\int_0^T f(t,a,0)\,\mathrm{d}t = 0 \tag{2.2}$$

has no solution on $\partial\Omega \cap \mathbb{R}$.

(iii) The Brouwer degree

$$deg_B(F, \mathbb{R}\cap\partial\Omega, 0) \neq 0. \tag{2.3}$$

Then, when $\lambda = 1$, the problem (2.1) has a solution in $\overline{\Omega}$.

Remark 2.2. In lemma 2.1, it should be mentioned that for given f, a is being regarded as a constant function for $\partial\Omega \cap \mathbb{R}$ to make sense.

In order to get our main results of problems (1.1) and (1.2), we first consider the existence of periodic solutions for the following auxiliary dissipative problems.

$$\begin{cases} (\phi_p(u'))' + \hbar\varepsilon\phi_p(u') + a_+(t)\phi_p(u^+) - a_-(t)\phi_p(u^-) + W_0(t,u) = e(t),\ t\in[0,T], \\ u(0) = u(T), u'(0) = u'(T) \end{cases} \tag{2.4}$$

and

$$\begin{cases} (\phi_p(u'))' + \hbar\varepsilon\phi_p(u') + a_+(t)\phi_p(u^+) - a_-(t)\phi_p(u^-) \\ \quad +W_1(t,u,u') = e(t),\ t\in[0,T], \\ u(0) = u(T), u'(0) = u'(T), \end{cases} \tag{2.5}$$

where $\varepsilon > 0$ is a constant and $\hbar$ represents a symbol defined by

$$\hbar = \begin{cases} 1, & \int_0^T a_+(s)\phi_p(u^+(s))u'(s)\,\mathrm{d}s \geqslant \int_0^T a_-(s)\phi_p(u^-(s))u'(s)\,\mathrm{d}s, \\ -1, & \int_0^T a_+(s)\phi_p(u^+(s))u'(s)\,\mathrm{d}s < \int_0^T a_-(s)\phi_p(u^-(s))u'(s)\,\mathrm{d}s. \end{cases}$$

Now, we introduce some auxiliary lemmas.

Lemma 2.3. *For $2 \leqslant p < \infty$, assume that the following conditions hold.*

$(H1)$ There exist nonnegative functions $a_i > 0, i = 1.2$, and constant $\theta \in [0, p-1)$ such that for $(t,u) \in [0,T]\times\mathbb{R}$,

$$|g_1(t,u)| \leqslant a_1(t) + a_2(t)|u|^\theta.$$

$(H2)$ There exists constant $a_3 > 0$ such that if $|u| > a_3$, for $t \in [0,T], u \in \mathbb{R}$,

$$sgn\{u\}(a_+(t)\phi_p(u^+) + W_0(t,u)) > \|e\|_\infty$$

or

$$sgn\{u\}(-a_-(t)\phi_p(u^-) + W_0(t,u)) > \|e\|_\infty.$$

Then the problem (2.4) has at least one solution.

Proof. Consider the following homotopy equation with periodic boundary condition

$$(\phi_p(u'))' + \lambda\hbar\varepsilon\phi_p(u') + \lambda a_+(t)\phi_p(u^+) - \lambda a_-(t)\phi_p(u^-) + \lambda W_0(t,u) = \lambda e(t), \quad (2.6)$$

where $\lambda \in (0,1)$.

First, if $u \in C^1_T$ is a solution of (2.6), we claim that there exist positive constants $b_i, i = 1,2$ such that $\|u\|_\infty \leqslant b_1 + b_2\|u'\|_1$. Let t_1 and t_2 represent minimum and maximum points respectively. Thus, we know that

$$\phi_p(u'(t_1)) = 0, (\phi_p(u'(t_1)))' \geqslant 0 \text{ and } \phi_p(u'(t_2)) = 0, (\phi_p(u'(t_2)))' \leqslant 0.$$

Thus, we need to consider the following three cases.

$$(A1): u(t_1) \leqslant 0, u(t_2) \geqslant 0,$$
$$(A2): u(t_1) \geqslant 0, u(t_2) \geqslant 0,$$
$$(A3): u(t_1) \leqslant 0, u(t_2) \leqslant 0.$$

For the case (A1), it is clear that $\|u\|_\infty \leqslant \|u'\|_1$. For the case (A2), from (2.6), we have

$$a_+(t_1)\phi_p(u^+(t_1)) + W_0(t_1, u(t_1))) \leqslant e(t_1) \leqslant \|e\|_\infty$$

and

$$a_+(t_2)\phi_p(u^+(t_2)) + W_0(t_2, u(t_2))) \geqslant e(t_2) \geqslant -\|e\|_\infty.$$

Thus, we know that

$$\{a_+(t)\phi_p(u^+(t)) + W_0(t,u(t)) : t \in [0,T]\} \bigcap [-\|e\|_\infty, \|e\|_\infty] \neq \emptyset,$$

i.e., there exists $t_3 \in [0,T]$ such that

$$|a_+(t_3)\phi_p(u^+(t_3)) + W_0(t_3, u(t_3))| \leqslant \|e\|_\infty.$$

From $(H2)$, we can obtain that $|u(t_3)| \leqslant a_3$. Thus, the claim is proved. By the similar way to the case $(A2)$, the case $(A3)$ is also true. Here, we omit the detail.

Multiplying the both sides of (2.6) by u' and integrating from 0 to T, we have

$$\hbar\varepsilon \int_0^T |u'(s)|^p \, ds + \int_0^T a(s)\phi_p(u^+(s))u'(s)\, ds - \int_0^T b(s)\phi_p(u^-(s))u'(s)\, ds$$
$$+ \int_0^T g_1(s, u(s))u'(s)\, ds = \int_0^T e(s)u'(s)\, ds. \quad (2.7)$$

By the definition of $\hbar$, we get two cases as follows.

$$(i): \ \varepsilon \int_0^T |u'(s)|^p \, ds \leqslant -\int_0^T g_1(s,u(s))u'(s)\, ds + \int_0^T e(s)u'(s)\, ds.$$
$$(ii): \ \varepsilon \int_0^T |u'(s)|^p \, ds \leqslant \int_0^T g_1(s,u(s))u'(s)\, ds - \int_0^T e(s)u'(s)\, ds.$$

Moreover, by $(H1)$ and the basic inequality $(x+y)^r \leqslant 2^r(x^r+y^r), x, y, r > 0$, we have

$$\begin{aligned}
&\int_0^T |g_1(s,u(s))u'(s)|\,\mathrm{d}s \\
&\quad \leqslant \left(\int_0^T |g_1(s,u(s))|^q\,\mathrm{d}s\right)^{\frac{1}{q}} \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\quad \leqslant \left(\int_0^T (a_1(t)+a_2(t)|u(s)|^\theta)^q\,\mathrm{d}s\right)^{\frac{1}{q}} \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\quad \leqslant \left[2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{1}{q}+1}\|a_2\|_\infty \left(\int_0^T |u(s)|^{\theta q}\,\mathrm{d}s\right)^{\frac{1}{q}}\right] \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\quad \leqslant \left(2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_1^\theta T^{\frac{1}{q}}\right) \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\qquad + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_2^\theta T^{\frac{1}{q}} \left(\int_0^T |u'(s)|\,\mathrm{d}s\right)^{\theta} \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\quad \leqslant \left(2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_1^\theta T^{\frac{1}{q}}\right) \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\qquad + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_2^\theta T^{\frac{1+\theta}{q}} \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1+\theta}{p}}.
\end{aligned}$$

Thus, from (i) or (ii), we have

$$\begin{aligned}
&\varepsilon \int_0^T |u'(s)|^p\,\mathrm{d}s \\
&\quad \leqslant \left(2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_1^\theta T^{\frac{1}{q}} + \left(\int_0^T |e(s)|^q\right)^{\frac{1}{q}}\right) \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\
&\qquad + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_2^\theta T^{\frac{1+\theta}{q}} \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1+\theta}{p}}.
\end{aligned}$$

So, we can get

$$\begin{aligned}
\varepsilon \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{q}} &\leqslant 2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_1^\theta T^{\frac{1}{q}} + \left(\int_0^T |e(s)|^q\right)^{\frac{1}{q}} \\
&\quad + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_2^\theta T^{\frac{1+\theta}{q}} \left(\int_0^T |u'(s)|^p\,\mathrm{d}s\right)^{\frac{\theta}{p}}. \qquad (2.8)
\end{aligned}$$

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

Noting that $\theta \in [0, p-1)$, we claim that $\|u'\|_p$ is bounded, i.e, there exists a positive constant D such that $\|u'\|_p \leqslant D$. Otherwise, we can find the sequence $\{u_n'(t)\}_{n\in\mathbb{Z}^+}$ which satisfies (2.8) such that

$$\|u_n'\|_p \to \infty, \text{as } n \to \infty.$$

Moreover, multiplying (2.8) by $\frac{1}{\|u_n'\|_p^\theta}$, since $\theta \in [0, p-1)$, we have

$$\begin{aligned}
&\varepsilon\|u_n'\|_p^{p-1-\theta} \\
&\quad\leqslant \frac{2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_1^\theta T^{\frac{1}{q}} + \left(\int_0^T |e(s)|^q\right)^{\frac{1}{q}}}{\|u_n'\|_p^\theta} \\
&\qquad + 2^{\frac{2}{q}+1+\theta}\|a_2\|_\infty b_2^\theta T^{\frac{1+\theta}{q}},
\end{aligned}$$

which implies a contradiction as $n \to \infty$. Thus, the claim is true. Therefore, we have

$$\|u\|_\infty \leqslant b_1 + b_2\|u'\|_1 \leqslant b_1 + b_2 T^{\frac{1}{q}}\|u'\|_p \leqslant b_1 + b_2 T^{\frac{1}{q}} D := \overline{D}.$$

On the other hand, (2.6) is equivalent to

$$(\phi_p(u'(t))e^{\lambda\hbar\varepsilon t})' = \lambda\left[-a_+(t)\phi_p(u^+) + a_-(t)\phi_p(u^-) - W_0(t,u) + e(t)\right]e^{\lambda\hbar\varepsilon t}. \quad (2.9)$$

Since $u(0) = u(T)$, there exists $t^* \in (0, T)$ such that $u'(t^*) = 0$. Integrating from t^* to t on (2.9), we have

$$\begin{aligned}
phi_p(u') = \lambda e^{-\lambda\hbar\varepsilon t}\int_{t^*}^t &(-a_+(s)\phi_p(u^+(s)) + a_-(s)\phi_p(u^-(s)) \\
&- W_0(s,u(s)) + e(s))e^{\lambda\hbar\varepsilon s}\,\mathrm{d}s.
\end{aligned}$$

Thus, we can obtain

$$|u'(t)|^{p-1} \leqslant e^{2\varepsilon T}(Ta_{+,M}\overline{D}^{p-1} + Ta_{-,M}\overline{D}^{p-1} + lT + T\|e\|_\infty) := K^{p-1}, \quad (2.10)$$

where $l = \max_{|u|\leqslant\overline{D}, 0\leqslant t\leqslant T}|W_0(t,u(t))|$. It implies that $\|u'\|_\infty \leqslant K$.

Let

$$\Omega = \{u(t) \in C_T^1 : \|u\| \leqslant M + 1\},$$

where $M > K + \overline{D}$ such that $a_+(t)\phi_p(M+1) + W_0(t, M+1) - e(t) > 0$ and $a_-(t)\phi_p(-M-1) + W_0(t, -M-1) - e(t) < 0$. Clearly, the problem (2.4) has no solution on $\mathbb{R} \cap \partial\Omega$. Moreover, we can get

$$F(M+1) = \frac{1}{T}\int_0^T -(a_+(s)\phi_p(M+1) + W_0(s, M+1) - e(s))\,\mathrm{d}s < 0,$$
$$F(-M-1) = \frac{1}{T}\int_0^T -(a_-(s)\phi_p(-M-1) + W_0(s, -M-1) - e(s))\,\mathrm{d}s > 0.$$

Thus, $F(a)$ has no solution on $\mathbb{R} \cap \partial\Omega$. Making a homotopy

$$H(u,\lambda) = \lambda u + (1-\lambda)\frac{1}{T}\int_0^T (-a_+(s)\phi_p(u^+(s)) + a_-(s)\phi_p(u^-(s)) - W_0(s, u(s)) + e(s))\,\mathrm{d}s,$$

where $u \in \mathbb{R} \cap \partial\Omega$, we have

$$uH(u,\lambda) = \lambda u^2 + (1-\lambda)\frac{1}{T}u\int_0^T (-a_+(s)\phi_p(u^+(s)) + a_-(s)\phi_p(u^-(s)) - W_0(s, u(s)) + e(s))\,\mathrm{d}s < 0.$$

Based on the homotopic transformation, we have

$$\deg_B(F, \mathbb{R} \cap \partial\Omega, 0) = \deg_B(u, \mathbb{R} \cap \partial\Omega, 0) \neq 0.$$

Thus, the problem (2.4) has at least one solution. □

Lemma 2.4. *For $1 < p < \infty$, assume that the following conditions hold.*

$(H3)$ *There exist functions $a_i > 0, i = 1.2,$ and constant $\theta \in [0, p-1)$ such that for $(t, v) \in [0, T] \times \mathbb{R}$,*

$$|\overline{g}_1(t, v)| \leqslant a_1(t) + a_2(t)|v|^\theta.$$

$(H4)$ *There exist positive constants ϱ, d and $\gamma \in (0, p-1]$ such that for $t \in [0, T]$,*

$$sgn\{u\}\overline{g}_0(u) \geqslant \varrho|u|^\gamma,\ \ |u| \geqslant d.$$

Then the problem (2.5) has at least one solution.

Proof. First, we can obtain the priori estimate that if $u \in C^1_T$ is a solution of the problem (2.5), then we have $|u(t)| \leqslant L$, where L is a positive constant and independent of λ. Otherwise, we can find solutions $\{u_n(t)\}$ of the problem (2.5)

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

corresponding with $\lambda = \lambda_n$, namely, $u_n(t)$ satisfies the following equation

$$\begin{aligned}(\phi_p(u_n'))' + \lambda_n \hbar\varepsilon\phi_p(u_n') + \lambda_n a_+(t)\phi_p(u_n^+) \\ - \lambda_n a_-(t)\phi_p(u_n^-) + \lambda_n W_1(t, u_n, u_n') = \lambda_n e(t)\end{aligned} \tag{2.11}$$

and periodic boundary condition such that

$$\|u_n\|_\infty \to \infty, \text{ as } n \to \infty.$$

For any $u \in C_T^1 \setminus \{0\}$, let $z_n(t) = \frac{u_n(t)}{\|u_n\|_\infty}$, so $\|z_n\|_\infty = 1$. Moreover, multiplying (2.11) by $\frac{u_n'}{\|u_n\|_\infty^p}$ and integrating from 0 to T, we have

$$\begin{aligned}\hbar\varepsilon\int_0^T |z_n'(s)|^p \,\mathrm{d}s + \int_0^T a(s)\phi_p(z_n^+(s))z_n'(s)\,\mathrm{d}s - \int_0^T b(s)\phi_p(z_n^-(s))z_n'(s)\,\mathrm{d}s \\ + \frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T \overline{g}_1(s, u_n'(s))z_n'(s)\,\mathrm{d}s = \frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T e(s)z_n'(s)\,\mathrm{d}s.\end{aligned} \tag{2.12}$$

By definition of $\hbar$, we get two cases as follows.

$$\begin{aligned}(i):\ \varepsilon\int_0^T |z_n'(s)|^p\,\mathrm{d}s &\leqslant -\frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T \overline{g}_1(s, u_n'(s))z_n'(s)\,\mathrm{d}s \\ &\quad + \frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T e(s)z_n'(s)\,\mathrm{d}s. \\ (ii):\ \varepsilon\int_0^T |z_n'(s)|^p\,\mathrm{d}s &\leqslant \frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T \overline{g}_1(s, u_n'(s))z_n'(s)\,\mathrm{d}s \\ &\quad - \frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T e(s)z_n'(s)\,\mathrm{d}s.\end{aligned}$$

By $(H3)$, we can get

$$\begin{aligned}&\frac{1}{\|u_n\|_\infty^{p-1}}\int_0^T \overline{g}_1(s, u_n'(s))z_n'(s)\,\mathrm{d}s \\ &\leqslant \frac{1}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |\overline{g}_1(s, u_n'(s))|^q\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}} \\ &\leqslant \frac{1}{\|u_n\|_\infty^{p-1}}\left(\int_0^T (a_1(t) + a_2(t)|u_n'(s)|^\theta)^q\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}\end{aligned}$$

$$\leqslant \frac{2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}}}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}$$

$$+\frac{2^{\frac{1}{q}+1}\|a_2\|_\infty}{\|u_n\|_\infty^{p-\theta-1}}\left(\int_0^T |z_n'(s)|^{\theta q}\,\mathrm{d}s\right)^{\frac{1}{q}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}$$

$$\leqslant \frac{2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}}}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}$$

$$+\frac{2^{\frac{1}{q}+1}\|a_2\|_\infty T^{\frac{p-\theta-1}{p}}}{\|u_n\|_\infty^{p-\theta-1}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1+\theta}{p}}.$$

Thus, from (i) or (ii), one has

$$\varepsilon\int_0^T |z_n'(s)|^p\,\mathrm{d}s \leqslant \frac{2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}}}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}}$$

$$+\frac{2^{\frac{1}{q}+1}\|a_2\|_\infty T^{\frac{p-\theta-1}{p}}}{\|u_n\|_\infty^{p-\theta-1}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1+\theta}{p}}$$

$$+\frac{1}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |e(s)|^q\right)^{\frac{1}{q}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{p}},$$

which yields that

$$\varepsilon\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{1}{q}} \leqslant \frac{2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}}}{\|u_n\|_\infty^{p-1}}+\frac{2^{\frac{1}{q}+1}\|a_2\|_\infty T^{\frac{p-\theta-1}{p}}}{\|u_n\|_\infty^{p-\theta-1}}\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{\theta}{p}}$$

$$+\frac{1}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |e(s)|^q\right)^{\frac{1}{q}}. \tag{2.13}$$

We claim that $\|z_n'\|_p \to 0$ as $n\to\infty$. If not, we have the following two cases.

$(B1)$ $\|z_n'\|_p \to c$ as $n\to\infty$, where c is a positive constant;

$(B2)$ $\|z_n'\|_p \to \infty$ as $n\to\infty$.

For the case $(B1)$, from (2.13), we have

$$\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{\theta}{p}}\left[\varepsilon\left(\int_0^T |z_n'(s)|^p\,\mathrm{d}s\right)^{\frac{p-\theta-1}{p}}-\frac{2^{\frac{1}{q}+1}\|a_2\|_\infty T^{\frac{p-\theta-1}{p}}}{\|u_n\|_\infty^{p-\theta-1}}\right]$$

$$\leqslant \frac{2^{\frac{1}{q}+1}\|a_1\|_\infty T^{\frac{1}{q}}}{\|u_n\|_\infty^{p-1}}+\frac{1}{\|u_n\|_\infty^{p-1}}\left(\int_0^T |e(s)|^q\right)^{\frac{1}{q}},$$

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

which implies that $\varepsilon c^{p-1} \leqslant 0$ as $n \to \infty$. This is impossible. Clearly, for the case $(B2)$, we can also get a contradiction. Thus, $\|z_n'\|_p \to 0$ as $n \to \infty$. Moreover, for any $t \in [0, T]$, by Hölder inequality, we have

$$|z_n(t) - z_n(0)| \leqslant T^{\frac{1}{q}} \left(\int_0^T |z_n'(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \to 0, \text{ as } n \to \infty.$$

Since $\|z_n\| = 1$, we can obtain

$$\lim_{n\to\infty} z_n(t) = \pm 1, \quad \forall t \in [0, T] \Rightarrow \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty} \|u_n\|_\infty z_n(t) = \pm\infty.$$

Thus, we have

$$\lim_{n\to\infty} a_\pm(t)\phi_p(u_n^\pm(t)) = \pm\infty, \ \forall t \in [0, T]. \tag{2.14}$$

Similarly, we have

$$\varepsilon \left(\int_0^T |u_n'(s)|^p \, \mathrm{d}s \right)^{\frac{1}{q}} \leqslant 2^{\frac{1}{q}+1} \|a_1\|_\infty T^{\frac{1}{q}} + 2^{\frac{1}{q}+1} \|a_2\|_\infty T^{\frac{p-\theta-1}{p}} \left(\int_0^T |u_n'(s)|^p \, \mathrm{d}s \right)^{\frac{\theta}{p}} + \left(\int_0^T |e(s)|^q \right)^{\frac{1}{q}}. \tag{2.15}$$

Clearly, $\|u_n'\|_p$ is bounded, i.e, there exists a positive constant $\overline{L}$ such that $\|u_n'\|_p \leqslant \overline{L}$. Otherwise,

$$\|u_n'\|_p \to \infty, \text{ as } n \to \infty.$$

From (2.15), we have

$$\varepsilon \|u_n\|_p^{\frac{p-\theta-1}{p}} \leqslant \frac{2^{\frac{1}{q}+1} \|a_1\|_\infty T^{\frac{1}{q}} + \left(\int_0^T |e(s)|^q \right)^{\frac{1}{q}}}{\|u_n\|_p^{\frac{\theta}{p}}} + 2^{\frac{1}{q}+1} \|a_2\|_\infty T^{\frac{p-\theta-1}{p}}, \tag{2.16}$$

which implies a contradiction. Thus, $\|u_n'\|_p$ is bounded. Moreover, we can also get

$$\begin{aligned} \left| \int_0^T \phi_p(u_n'(s)) \, \mathrm{d}s \right| &\leqslant \int_0^T |u_n'(s)|^{p-1} \, \mathrm{d}s \leqslant T^{\frac{1}{p}} \left(\int_0^T |u_n'(s)|^p \, \mathrm{d}s \right)^{\frac{1}{q}} \\ &\leqslant T^{\frac{1}{p}} \overline{L}^{p-1}, \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} \left| \int_0^T \overline{g}_1(s, u_n'(s)) \, \mathrm{d}s \right| &\leqslant \|a_1\|_\infty T + \|a_2\|_\infty \int_0^T |u_n'| \, \mathrm{d}s \\ &\leqslant \|a_1\|_\infty T + \|a_2\|_\infty T^{\frac{1}{q}} \left(\int_0^T |u_n'|^p \, \mathrm{d}s \right)^{\frac{1}{p}} \\ &\leqslant \|a_1\|_\infty T + \|a_2\|_\infty T^{\frac{1}{q}} \overline{L}. \end{aligned} \tag{2.18}$$

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

By $(H4)$, we have

$$\begin{aligned}&\text{if } \lim_{n\to\infty} u_n(t)=\infty,\ \overline{g}_0(u_n)\geqslant \varrho|u_n|^\gamma \Rightarrow \overline{g}_0(u_n)\to\infty \text{ as } n\to\infty \text{ or}\\ &\text{if } \lim_{n\to\infty} u_n(t)=-\infty,\ \overline{g}_0(u_n)\leqslant -\varrho|u_n|^\gamma \Rightarrow \overline{g}_0(u_n)\to-\infty \text{ as } n\to\infty,\end{aligned} \tag{2.19}$$

which implies that

$$\lim_{n\to\infty}\left|\int_0^T (e(s)-a_+(s)\phi_p(u_n^+(s))+a_-(s)\phi_p(u_n^-(s))\,\mathrm{d}s\right|=\infty. \tag{2.20}$$

However, integrating from 0 to T on (2.11), we have

$$\begin{aligned}&\hbar\varepsilon\int_0^T \phi_p(u_n'(s))\,\mathrm{d}s+\int_0^T a_+(s)\phi_p(u_n^+(s))\,\mathrm{d}s-\int_0^T a_-(s)\phi_p(u_n^-(s))\,\mathrm{d}s\\ &\quad+\int_0^T W_1(s,u_n(s),u_n'(s))\,\mathrm{d}s=\int_0^T e(s)\,\mathrm{d}s.\end{aligned}$$

This implies a contradiction for n large enough. Therefore, $|u(t)|\leqslant L$. By the similar way to lemma 2.2, we can obtain that there exists a positive constant Θ such that $|u'(t)|\leqslant\Theta$. Moreover, $\deg_B(F,\mathbb{R}\cap\partial\Omega,0)\neq 0$. Thus, the problem (2.5) has at least one solution. □

REMARK 2.5. In view of the continuity theorem, the key step is to obtain a priori bound estimation of solutions. In fact, if $W_1(t,u,u')$ is written as $\overline{g}_0(u)+\overline{g}_1(t,u,u')$, by the method used in lemma 2.3, we can not get (2.13) that make us fail to obtain the prior estimation when $1<p<\infty$.

Now, we state and prove our main results for problems (1.1) and (1.2).

THEOREM 2.6. *For $2\leqslant p<\infty$, assume that $(H1)$ and the following conditions hold.*
$(H5)$ *There exist positive constants δ, ζ, d such that for $t\in[0,T]$,*

$$\delta\leqslant\frac{g_0(u)}{\phi_p(u)}\leqslant\zeta,\ \ |u|\geqslant d.$$

$(H6)$ $\lim_{|u|\to\infty}\frac{pG_0(u)}{|u|^p}=\eta$, *where* $\eta\in(0,\infty)\setminus\bigcup_{k=0}^{\infty}[\lambda_{2k}^{min},\lambda_{2k}^{max}]$.
Then the problem (1.1) has at least one solution.

Proof. In view of lemma 2.3, we claim that if $\{u_n(t)\}$ are solutions of the problem (2.4) corresponding to $\varepsilon=\{\varepsilon_n\}$, where $\varepsilon_n\to0$ as $n\to\infty$, then there exist constants

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

$\widetilde{D},\ \widetilde{K}>0$, independent of ε_n, such that for $t\in[0,T]$,

$$|u_n(t)|\leqslant \widetilde{D},\ |u_n'(t)|\leqslant \widetilde{K}.$$

Otherwise,

$$\|u_n\|_\infty\to\infty \text{ as } n\to\infty.$$

For any $u\in C_T^1\setminus\{0\}$, let $z_n(t)=\frac{u_n(t)}{\|u_n\|_\infty}$, so $\|z_n\|_\infty=1$. Moreover, dividing the problem (2.4) by $\|u_n\|_\infty^{p-1}$, we have

$$\begin{aligned}(\phi_p(z_n'))'+\hbar\varepsilon_n\phi_p(z_n')+a_+(t)\phi_p(z_n^+)-a_-(t)\phi_p(z_n^-)+\frac{g_0(u_n)}{\|u_n\|_\infty^{p-1}}\\ -\frac{g_1(t,u_n(t))}{\|u_n\|_\infty^{p-1}}=\frac{e(t)}{\|u_n\|_\infty^{p-1}},\end{aligned}\tag{2.21}$$

which is equivalent to

$$\begin{aligned}(\phi_p(z_n')e^{\varepsilon_n\hbar t})'+e^{\varepsilon_n\hbar t}a_+(t)\phi_p(z_n^+)-e^{\varepsilon_n\hbar t}a_-(t)\phi_p(z_n^-)+e^{\varepsilon_n\hbar t}\frac{g_0(u_n)}{\|u_n\|_\infty^{p-1}}\\ -e^{\varepsilon_n\hbar t}\frac{g_1(t,u_n(t))}{\|u_n\|_\infty^{p-1}}=e^{\varepsilon_n\hbar t}\frac{e(t)}{\|u_n\|_\infty^{p-1}}.\end{aligned}\tag{2.22}$$

Since $z_n(0)=z_n(T)$, there exists $t_*\in(0,T)$ such that $z_n'(t_*)=0$. Thus, we have

$$\begin{aligned}|\phi_p(z_n')|e^{\varepsilon_n\hbar t}\leqslant \int_{t_*}^t\Bigg(\left|a_+(s)\phi_p(z_n^+)\right|+\left|a_-(t)\phi_p(z_n^-)\right|+\left|\frac{g_0(u_n)}{\|u_n\|_\infty^{p-1}}\right|\\ +\left|\frac{g_1(t,u_n(t))}{\|u_n\|_\infty^{p-1}}\right|+\left|\frac{e(s)}{\|u_n\|_\infty^{p-1}}\right|\Bigg)e^{\varepsilon_n\hbar s}\,\mathrm{d}s.\end{aligned}$$

From $\varepsilon_n\to0$ as $n\to\infty$, there exists a positive constant d_1 such that $e^{\varepsilon_n\hbar t}\leqslant d_1$. Moreover, we have

$$\begin{aligned}\int_{t_*}^t\left|\frac{g_1(t,u_n(t))}{\|u_n\|_\infty^{p-1}}e^{\varepsilon_n\hbar s}\,\mathrm{d}s\right| &\leqslant \frac{d_1}{\|u_n\|_\infty^{p-1}}\int_0^T|g_1(s,u_n(s))|\,\mathrm{d}s\\ &\leqslant \frac{d_1}{\|u_n\|_\infty^{p-1}}\left(\|a_1\|_\infty T+\|a_2\|_\infty\int_0^T|u_n(s)|^\theta\,\mathrm{d}s\right)\\ &\leqslant \frac{d_1T\|a_1\|_\infty}{\|u_n\|_\infty^{p-1}}+\frac{d_1T\|a_2\|_\infty}{\|u_n\|_\infty^{p-\theta-1}},\end{aligned}\tag{2.23}$$

which implies that there exists a constant $M_1>0$ which is independent of n such that

$$\int_{t_*}^t\left|\frac{g_1(t,u_n(t))}{\|u_n\|_\infty^{p-1}}e^{\varepsilon_n\hbar s}\,\mathrm{d}s\right|\leqslant M_1.\tag{2.24}$$

Based on $\|z_n\| = 1$ and $(H5)$, for any $t \in [0,T]$, there exists a constant $M_2 > 0$ which is independent of n such that

$$\int_{t_*}^{t} \left(\left|a_+(s)\phi_p(z_n^+)\right| + \left|a_-(s)\phi_p(z_n^-)\right| + \left|\frac{g_0(u_n)}{\|u_n\|_\infty^{p-1}}\right|\right) e^{\varepsilon_n \hbar s}\, \mathrm{d}s \leqslant M_2,$$

which together with $e \in C([0,T])$ and the monotonicity of $\phi_p(\cdot)$ yield that

$$|\phi_p(z_n'(t))| \leqslant M_3, \text{ i.e., } |z_n'(t)| \leqslant M_3^{q-1}, \tag{2.25}$$

where $M_3 > 0$ is a constant and independent of n. Thus, from the definition of $z_n(t)$, it follows that $z_n(t)$ is uniformly bounded. By the standard argument, we can get that $z_n(t)$ is also equicontinuous. Moreover, from (2.22), we can also find a constant $M_4 > 0$ which is independent of n such that

$$|(\phi_p(z_n'(t)))'| \leqslant M_4.$$

Furthermore, $\forall t_1,\ t_2 \in [0,1]$, assuming that $t_1 \leqslant t_2$, for any $n \in \mathbb{N}$, we have

$$\begin{aligned}|\phi_p(z_n'(t_2)) - \phi_p(z_n'(t_1))| &\leqslant \int_{t_1}^{t_2} |(\phi_p(z_n'(t)))'\, \mathrm{d}s| \\ &\leqslant M_4|t_2 - t_1| \to 0 \text{ uniformly as } t_1 \to t_2,\end{aligned}$$

which together with the monotonicity and uniformly continuity of $\phi_p(\cdot)$ on $[-M_3^{q-1}, M_3^{q-1}]$ yield that $z_n'(t)$ is also uniformly bounded and equicontinuous. By Arzelá–Ascoli theorem, there exists a subsequence of $\{z_n\}$ (without loss of generality, take its subsequence) and $z(t) \in C_T^1$ such that for any $t \in [0,T]$,

$$\lim_{n\to\infty} z_n(t) = z(t),\ \lim_{n\to\infty} z_n'(t) = z'(t). \tag{2.26}$$

Thus, there exists a $\widetilde{t} \in [0,T]$ such that $z(\widetilde{t}) \neq 0$. Multiplying both sides of (2.4) by $\frac{u_n'(t)}{\|u_n\|^p}$ and integrating from $\widetilde{t}$ to t, one has

$$\begin{aligned}&\int_{\widetilde{t}}^{t} (\phi_p(z_n'))' z_n'\, \mathrm{d}s + \varepsilon_n \hbar \int_{\widetilde{t}}^{t} |z_n|^p\, \mathrm{d}s + \int_{\widetilde{t}}^{t} a_+(s)\phi_p(z_n^+) z_n'\, \mathrm{d}s - \int_{\widetilde{t}}^{t} a_-(s)\phi_p(z_n^-) z_n'\, \mathrm{d}s \\ &\quad + \frac{G_0(u_n(t)) - G_0(u_n(\widetilde{t}))}{\|u_n\|_\infty^p} - \int_{\widetilde{t}}^{t} \frac{g_1(s,u_n(s)) z_n'}{\|u_n\|_\infty^{p-1}}\, \mathrm{d}s = \int_{\widetilde{t}}^{t} \frac{e(s) z_n'}{\|u_n\|_\infty^{p-1}}\, \mathrm{d}s,\end{aligned} \tag{2.27}$$

Following $(H4)$, we know that

$$\lim_{|u|\to\infty} \frac{pG_0(u)}{|u|^p} = \eta.$$

For any $\epsilon > 0$, there exists a constant $l_1 > 0$ such that

$$\left|\frac{pG_0(u)}{|u|^p} - \eta\right| \leqslant \epsilon,\ |u| \geqslant l_1, \text{ i.e., } \left|G_0(u) - \frac{\eta|u|^p}{p}\right| \leqslant \epsilon\frac{|u|^p}{p},\ |u| \geqslant l_1.$$

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

By the continuity of G_0, define

$$l_2 := \max_{|u|\leqslant l_l} \left| G_0(u) - \frac{\eta |u|^p}{p} \right|.$$

Thus, we have

$$\left| G_0(u_n) - \frac{\eta |u_n|^p}{p} \right| \leqslant \epsilon \frac{|u_n|^p}{p} + l_2 \Rightarrow \left| \frac{G_0(u_n)}{\|u_n\|_\infty^p} - \frac{\eta |z_n|^p}{p} \right| \leqslant \epsilon + \frac{l_2}{\|u_n\|^p}.$$

Taking limit in the above equality, by arbitrariness of ϵ, we can obtain

$$\lim_{n\to\infty} \frac{G_0(u_n)}{\|u_n\|_\infty^p} = \frac{\eta |z|^p}{p}. \tag{2.28}$$

Moreover, by (2.24) and (2.25), we have

$$\int_{\widetilde{t}}^{t} \frac{g_1(s, u_n(s)) z_n'(s)}{\|u_n\|_\infty^{p-1}} \, \mathrm{d}s, \int_{\widetilde{t}}^{t} \frac{e(s) z_n'}{\|u_n\|_\infty^{p-1}} \, \mathrm{d}s \to 0, \text{ as } n \to \infty. \tag{2.29}$$

Furthermore, we know that

$$\int_{\widetilde{t}}^{t} (\phi_p(z_n'(s)))' z_n'(s) \, \mathrm{d}s = \frac{1}{q} |\phi_p(z_n'(t))|^q - \frac{1}{q} |\phi_p(z_n'(\widetilde{t}))|^q. \tag{2.30}$$

By (2.28)–(2.30), taking limit $n \to \infty$ in (2.27), we have

$$\begin{aligned} \frac{1}{q} |\phi_p(z'(t))|^q &+ \int_{\widetilde{t}}^{t} a_+(s) \phi_p(z^+(s)) z'(s) \, \mathrm{d}s \\ &- \int_{\widetilde{t}}^{t} a_-(s) \phi_p(z^-(s)) z'(s) \, \mathrm{d}s + \frac{\eta |z(t)|^p}{p} = \sigma, \end{aligned} \tag{2.31}$$

where

$$\sigma = \frac{1}{q} |\phi_p(z'(\widetilde{t}))|^q + \frac{\eta |z(\widetilde{t})|^p}{p} > 0.$$

Now, we claim that $z'(t)$ has only limited zero points on $[0, T]$. If not, assume that $z'(t)$ has unlimited zero points $\{\mu_i\}_{i\in\mathbb{N}}$ on $[0, T]$. Without loss of generality, there exists a μ_* such that

$$\lim_{i\to\infty} \mu_i = \mu_*.$$

Following the continuity of $z'(t)$, we have $z'(\mu_*) = 0$ and

$$\int_{\widetilde{t}}^{\mu_*} a_+(s) \phi_p(z^+(s)) z'(s) \, \mathrm{d}s - \int_{\widetilde{t}}^{\mu_*} a_-(s) \phi_p(z^-(s)) z'(s) \, \mathrm{d}s + \frac{\eta |z(\mu_*)|^p}{p} = \sigma. \tag{2.32}$$

By the above equality, we can get $z(\mu_*) \neq 0$. In fact, if $z(\mu_*) = 0$, we have

$$\int_{\widetilde{t}}^{\mu_*} a_+(s) \phi_p(z^+(s)) z'(s) \, \mathrm{d}s - \int_{\widetilde{t}}^{\mu_*} a_-(s) \phi_p(z^-(s)) z'(s) \, \mathrm{d}s = \sigma. \tag{2.33}$$

By choosing a suitable radius r_1 and r_2 such that $\widetilde{t}$ belonging to the left neighbourhood of $(\mu_* - r_1, \mu_*)$ or the right neighbourhood of $(\mu_*, \mu_* + r_2)$, we just need to

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

consider the following four cases.

$$(C1)\ z(t) \leqslant 0\ z'(t) \geqslant 0,\ t \in [\widetilde{t}, \mu_*],$$
$$(C2)\ z(t) \geqslant 0\ z'(t) \leqslant 0,\ t \in [\widetilde{t}, \mu_*],$$
$$(C3)\ z(t) \geqslant 0\ z'(t) \geqslant 0,\ t \in [\mu_*, \widetilde{t}],$$
$$(C4)\ z(t) \leqslant 0\ z'(t) \leqslant 0,\ t \in [\mu_*, \widetilde{t}].$$

For the case $(C1)$, we can obtain

$$\begin{aligned} 0 < \frac{1}{q}|\phi_p(z'(\widetilde{t}))|^q + \frac{\eta|z(\widetilde{t})|^p}{p} = \sigma &\leqslant a_{-,L} \int_{\widetilde{t}}^{\mu_*} \phi_p(z(s))z'(s)\,\mathrm{d}s \\ &= -\frac{a_{-,L}}{p}|z(\widetilde{t})|^p < 0, \end{aligned}$$

which is impossible. For the case $(C2)$, it follows

$$\begin{aligned} 0 < \frac{1}{q}|\phi_p(z'(\widetilde{t}))|^q + \frac{\eta|z(\widetilde{t})|^p}{p} = \sigma &\leqslant a_{+,L} \int_{\widetilde{t}}^{\mu_*} \phi_p(z(s))z'(s)\,\mathrm{d}s \\ &= -\frac{a_{+,L}}{p}|z(\widetilde{t})|^p < 0, \end{aligned}$$

which implies a contradiction. Moreover, by the same way, we can also obtain a contradiction under the case of $(C3)$ or $(C4)$. Thus, $z(\mu_*) \neq 0$. Without loss of generality, assume that $z(\mu_*) > 0$. According to its continuity, there exist constants $\xi, \nu > 0$ such that

$$z(t) > \xi, \quad t \in [\mu_* - \nu, \mu_* + \nu].$$

Choosing n large enough, we can also get

$$z_n(t) > \xi, \quad t \in [\mu_* - \nu, \mu_* + \nu].$$

Since μ_* is a limiting point of $\{\mu_i\}$, so we can find two zero points called $\mu^*,\ \mu^{**}$ of $z'(t)$ in $[\mu_* - \nu, \mu_* + \nu]$. Integrating from μ^* to μ^{**} on (2.27), taking limit $n \to \infty$, we can obtain

$$\lim_{n\to\infty} \left(\int_{\mu^*}^{\mu^{**}} \frac{g_0(u_n(s))}{\|u_n\|_\infty^{p-1}} e^{\varepsilon_n \hbar s}\,\mathrm{d}s + \int_{\mu^*}^{\mu^{**}} e^{\varepsilon_n \hbar s} a_+(s)\phi_p(z_n(s))\,\mathrm{d}s \right) = 0. \qquad (2.34)$$

On the other hand, for $t \in [\mu^*, \mu^{**}]$, choose n large enough such that

$$u_n(t) \geqslant z_n(t)\|u\|_\infty \geqslant \xi\|u\|_\infty \geqslant d.$$

Then, we have

$$\frac{g_0(u_n(t))}{\|u_n\|^{p-1}} = \frac{g_0(u_n(t))}{\phi_p(u_n(t))}\phi_p(z_n(t)) \geqslant \delta\phi_p(\xi).$$

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

Thus, we can obtain

$$\lim_{n\to\infty}\left(\int_{\mu^*}^{\mu^{**}}\frac{g_0(u_n(s))}{\|u_n\|^{p-1}}e^{\varepsilon_n\hbar s}\,\mathrm{d}s+\int_{\mu^*}^{\mu^{**}}e^{\varepsilon_n\hbar s}a_+(s)\phi_p(z(s))\,\mathrm{d}s\right)$$
$$\geqslant (a_{+,L}+\delta)\phi_p(\xi)(\mu^{**}-\mu^*)>0, \tag{2.35}$$

which contradicts to (2.34). Thus, $z'(t)$ has only limited zero points on $[0,T]$. Differentiating (2.31), one has

$$z'(t)\left((\phi_p(z'(t)))'+a(t)\phi_p(z^+(t))-b(t)\phi_p(z^-(t))+\eta\phi_p(z(t))\right)=0.$$

Since $z'(t)$ has only limited zero points on $[0,T]$, we have

$$(\phi_p(z'(t)))'+a(t)\phi_p(z^+(t))-b(t)\phi_p(z^-(t))+\eta\phi_p(z(t))=0, \tag{2.36}$$

which together with $\eta\in(0,\infty)\setminus\bigcup_{k=0}^{\infty}[\lambda_{2k}^{min},\lambda_{2k}^{max}]$ yield $z(t)\equiv 0$ for $t\in[0,T]$. In fact, if there exists a $z\in C_T^1$ such that (2.36) is satisfied on a subset $\Delta\subset[0,T]$ and $z\equiv 0$ on $[0,T]\setminus\Delta$, it must be an eigenvector corresponding to a half-eigenvalue, which implies a contradiction. Therefore, $z(t)\equiv 0$. However, it contradicts to $\|z\|_\infty=1$. Thus, there exists a constant $\widetilde{D}>0$ which is independent of ε_n such that $|u_n(t)|\leqslant\widetilde{D}$.

Since $u_n(0)=u_n(T)$, there exists $t_n\in(0,T)$ such that $u_n'(t_n)=0$. By the same way to lemma 2.2, we can also get that there exists a constant $\widetilde{K}>0$ that is independent of ε_n such that $|u_n'(t)|\leqslant\widetilde{K}$. By Arzelá–Ascoli theorem (without loss of generality, take its subsequence), there exist $u\in C_T^1$ and $t_0\in[0,T]$ such that $\lim_{n\to\infty}t_n=t_0$,

$$\lim_{n\to\infty}u_n(t)=u(t),\quad \lim_{n\to\infty}u_n'(t)=u'(t),\ \forall t\in[0,T]. \tag{2.37}$$

Taking limit $n\to\infty$ in the following equation

$$\phi_p(u_n')e^{\varepsilon_n\hbar t}=\int_{t_n}^{t}\left(-a_+(s)\phi_p(u_n^+)+a_-\phi_p(u_n^-)-W_0(s,u(s))+e(s)\right)e^{\varepsilon_n\hbar s}\,\mathrm{d}s,$$

we have

$$\phi_p(u')=\int_{t_0}^{t}\left(-a_+(s)\phi_p(u^+)+a_-\phi_p(u^-)-W_0(s,u(s))+e(s)\right)\,\mathrm{d}s,$$

which implies that for all $t\in[0,T]$

$$(\phi_p(u'(t)))'+a_+(t)\phi_p(u^+(t))-a_-(t)\phi_p(u^-(t))+W_0(t,u(t))=e(t)$$

and $u(0)=u(T),u'(0)=u'(T)$. Thus, the problem (1.1) has at least one periodic solution. □

Remark 2.7. It should be mentioned that in the separated boundary conditions, typical existence results in this area usually assume that η lies in the gap between half-eigenvalues with consecutive nodal counts, not between half-eigenvalues

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

with the same nodal counts that may make the problem not have a solution (see [**23**, **24**]).

THEOREM 2.8. *For $1 < p < \infty$, if $(H3)$, $(H5)$ and $(H6)$ hold. Then the problem (1.2) has at least one periodic solutions.*

Proof. Based on lemma 2.4, by similar way to theorem 2.6, the conclusion of theorem 2.8 is also true. Here, we omit the detail. □

REMARK 2.9. If $\overline{g}_1 = 0$, $a_{\pm}$ are positive constants and $\widetilde{g}_0(u)$ satisfies $(H5)$ and $\lim\limits_{|u|\to\infty} \frac{pG_0(u)}{|u|^p} = \eta$, letting

$$\widetilde{g}_0(u) = \overline{g}_0(u) + a_+\phi_p(u^+) - a_-\phi_p(u^-),$$

by theorem 2.8, the problem (1.2) has at least one periodic solutions, provided that $(a_+ + \eta, a_- + \eta) \notin S_p$, which is corresponding to theorem 2.3 of [**17**]. Thus, our main results generalize their conclusions.

Acknowledgements
The authors are grateful to the referees for a careful and thorough reading of this paper and also for their valuable suggestions and comments, which improved the former version of this paper.

Supported by the Natural Science Foundation of Jiangsu Province, China (No. BK20190620).

References

1 A. Anane and A. Dakkak. Nonexistence of nontrivial solutions for an asymmetric problem with weights. *Proyecciones Rev. De Matemática* **19** (2000), 43–52.

2 H. Berestycki. On some nonlinear Sturm-Liouville problems. *J. Differ. Equ.* **26** (1977), 375–390.

3 P. Binding and B. Rynne. Oscillation and interlacing for various spectra of the *p*-Laplacian. *Nonlinear Anal.: Theory, Methods Appl.* **71** (2009), 2780–2791.

4 J. Browne. A Prüfer approach to half-linear Sturm-Liouville problems. *Proc. Edinburgh Math. Soc. (Series 2)* **41** (1998), 573–583.

5 X. Chang and Y. Qiao. Existence of periodic solutions for a class of *p*-Laplacian equations. *Boundary Value Prob.* **2013** (2013), 96.

6 E. N. Dancer. Boundary-value problems for weakly nonlinear ordinary differential equations. *Bull. Aust. Math. Soc.* **15** (1976), 321–328.

7 M. Del Pino, R. Manásevich and E. Murúa. Existence and multiplicity of solutions with prescribed period for a second order quasilinear ODE. *Nonlinear Anal.: Theory, Methods Appl.* **18** (1992), 79–92.

8 T. Ding and F. Zanolin. Time-maps for the solvability of periodically perturbed nonlinear Duffing equations. *Nonlinear Anal.: Theory, Methods Appl.* **17** (1991), 635–653.

9 P. Drábek and S. Invernizzi. On the periodic BVP for the forced Duffing equation with jumping nonlinearity. *Nonlinear Anal.: Theory, Methods Appl.* **10** (1986), 643–650.

10 C. Fabry. Landesman-Lazer conditions for periodic boundary value problems with asymmetric nonlinearities. *J. Differ. Equ.* **116** (1995), 405–418.

11 S. Fučik. *Solvability of nonlinear equations and boundary value problems*, Vol. 4 (Dordrecht: D. Reidel Publishing Company, 1980).

12 F. Genoud and B. Rynne. Landesman-Lazer conditions at half-eigenvalues of the *p*-Laplacian. *J. Differ. Equ.* **254** (2013), 3461–3475.

https://doi.org/10.1017/prm.2021.89 Published online by Cambridge University Press

13 P. Gossez and P. Omari. Nonresonance with respect to the Fučik spectrum for periodic solutions of second order ordinary differential equations. *Nonlinear Anal.: Theory, Methods Appl.* **14** (1990), 1079–1104.

14 M. Jiang. A Landesman-Lazer type theorem for periodic solutions of the resonant asymmetric p-Laplacian equation. *Acta Math. Sinica* **21** (2005), 1219–1228.

15 B. Liu. Boundedness of solutions for equations with p-Laplacian and an asymmetric nonlinear term. *J. Differ. Equ.* **207** (2004), 73–92.

16 W. Liu and Y. Li. Existence of 2π-periodic solutions for the non-dissipative Duffing equation under asymptotic behaviors of potential function. *Z. Angew. Math. Phys.* **57** (2005), 1–11.

17 W. Liu and Y. Li. Existence of periodic solutions for p-Laplacian equation under the frame of Fučik spectrum. *Acta. Math. Sin. Engl. Ser.* **27** (2011), 545–554.

18 W. Li and P. Yan. Continuity and continuous differentiability of half-eigenvalues in potentials. *Commun. Contemporary Math.* **12** (2010), 977–996.

19 R. Manásevich and J. Mawhin. Periodic solutions for nonlinear systems with p-Laplacian-like operators. *J. Differ. Equ.* **145** (1998), 367–393.

20 A. Morris and B. Robinson. A Landesman-Lazer condition for the boundary-value problem $-u'' = au^+ - bu^- + g(u)$ with periodic boundary conditions. *Electron. J. Differ. Equ. Conf.* **20** (2013), 103–117.

21 M. Nkashama and S. Robinson. Resonance and nonresonance in terms of average values. *J. Differ. Equ.* **132** (1996), 45–65.

22 B. Rynne. Landesman-Lazer conditions for resonant p-Laplacian problems with jumping nonlinearities. *J. Differ. Equ.* **261** (2016), 5829–5843.

23 B. Rynne, *Some recent results on periodic, jumping nonlinearity problems, in Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology*, Universidad Complutense de Madrid (World Scientific, 2004).

24 B. Rynne. p-Laplacian problem with jumping nonlinearities. *J. Differ. Equ.* **226** (2006), 501–524.

25 B. Rynne. Non-resonance conditions for semilinear Sturm-Liouville problems with jumping non-linearities. *J. Differ. Equ.* **170** (2001), 215–227.

26 X. Yang. Boundedness in nonlinear asymmetric oscillations. *J. Differ. Equ.* **183** (2002), 108–131.

27 M. Zhang. Nonresonance conditions for asymptotically positively homogeneous differential systems: the Fučik spectrum and its generalization. *J. Differ. Equ.* **145** (1998), 332–366.