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RESEARCH ARTICLE

An extension of Venkatesh's converse theorem to the Selberg class

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Abstract
We extend Venkatesh's proof of the converse theorem for classical holomorphic modular forms to arbitrary level and character. The method of proof, via the Petersson trace formula, allows us to treat arbitrary degree 2 gamma factors of Selberg class type.

1. Introduction

Venkatesh, in his thesis [Ven02], gave a new proof of the classical converse theorem for modular forms of level 1, in the context of Langlands' 'Beyond Endoscopy'. The key analytic input is Voronoi summation, or equivalently, the functional equation for additive twists. More precisely, given a modular form $f(z) = \sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz)$ of weight k, level N, and nebentypus character χ, we define the complete additive twist

$$\Lambda_f(s, \alpha) = \Gamma_{\mathbb{C}}\left(s + \tfrac{k-1}{2}\right) \sum_{n=1}^{\infty} \frac{f_n e(n\alpha)}{n^s} \qquad \text{for } \alpha \in \mathbb{Q},$$

where $e(z) = e^{2\pi i z}$ and $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$. Then, for any $q \in N\mathbb{Z}_{>0}$ and any $a, \bar{a} \in \mathbb{Z}$ with $a\bar{a} \equiv 1 \pmod{q}$, $\Lambda_f\left(s, \frac{a}{q}\right)$ and $\Lambda_f\left(s, -\frac{\bar{a}}{q}\right)$ continue to entire functions and satisfy the functional equation [KMV02, (A.10)]

$$\Lambda_f\left(s, \tfrac{a}{q}\right) = i^k \chi(\bar{a}) q^{1-2s} \Lambda_f\left(1 - s, -\tfrac{\bar{a}}{q}\right).$$

Venkatesh proved, conversely, that the modular forms of level $N = 1$ and weight $k \geq 6$ are characterized by the functional equations of their additive twists. The novelty of his proof over earlier approaches to the converse theorem [Wei67, Raz77] is the use of the Petersson trace formula. Roughly, he isolates the contribution of the purported form f on the spectral side of the trace formula using information from additive twists on the geometric side.

In this paper, we generalize Venkatesh's proof to forms of arbitrary level and character. Our precise result is the following.

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Theorem 1.1. *Consider $\{f_n\}_{n\geq 1}$, N, χ, ω, and $\gamma(s)$ with the following properties:*

(1) *$\{f_n\}_{n\geq 1}$ is a sequence of complex numbers such that $\sum_{n=1}^{\infty} f_n n^{-s}$ converges absolutely for $\Re(s) > 1$;*
(2) *N is a natural number, and χ is a Dirichlet character modulo N;*
(3) *ω is a nonzero complex number;*
(4) *$\gamma(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$ for some numbers $Q, \lambda_j \in \mathbb{R}_{>0}$ and $\mu_j \in \mathbb{C}$ with $\Re(\mu_j) > -\frac{1}{2}\lambda_j$ and $\sum_{j=1}^{r} \lambda_j = 1$.*

Given any $\alpha \in \mathbb{Q}$, define the complete twisted L-function

$$\Lambda_f(s,\alpha) = \gamma(s) \sum_{n=1}^{\infty} \frac{f_n e(n\alpha)}{n^s}.$$

Suppose that for every $q \in N\mathbb{Z}_{>0}$ and every pair $a, \bar{a} \in \mathbb{Z}$ with $a\bar{a} \equiv 1 (\mathrm{mod}\ q)$, $\Lambda_f\left(s, \frac{a}{q}\right)$ and $\Lambda_f\left(s, -\frac{\bar{a}}{q}\right)$ continue to entire functions of finite order and satisfy the functional equation

$$\Lambda_f\left(s, \tfrac{a}{q}\right) = \omega\chi(\bar{a}) q^{1-2s} \Lambda_f\left(1-s, -\tfrac{\bar{a}}{q}\right). \tag{1.1}$$

Then there exists $k \in \mathbb{Z}_{>0}$ such that $f(z) = \sum_{n=1}^{\infty} f_n n^{\frac{k-1}{2}} e(nz)$ is a modular form of weight k, level N and nebentypus character χ.

Remarks 1.2.

1. One feature of the argument is that it admits a generalization to gamma factors of Selberg class type. In this way, our result can be viewed as a converse theorem for degree 2 elements of the Selberg class, albeit with infinitely many functional equations. Recently, Kaczorowski and Perelli [KP22] have classified the elements of the Selberg class of conductor 1 without the need for any twists. Very little is known for higher conductor, however, and our result is the first that we are aware of to consider both arbitrary level and degree 2 gamma factor.
2. For $k > 1$, it is enough to assume the analytic properties (analytic continuation, finite order, functional equation) of the finite L-functions $L_f\left(s, \frac{a}{q}\right) = \Lambda_f\left(s, \frac{a}{q}\right)/\gamma(s)$, and in this case we can also conclude that f is cuspidal. When $k = 1$, we need to know that $\Lambda_f\left(s, \frac{a}{q}\right)$ is analytic at $s = 0$, and there are noncuspidal examples satisfying all of the hypotheses of Theorem 1.1.
3. If we suppose that $L_f(s, 1)$ lies in the Selberg class, then we can combine the transformation formula in [KP15, Theorem 2] with the Vandermonde argument in [BK14, Lemma 2.4] to constrain the possible poles of $\Lambda_f\left(s, \frac{a}{q}\right)$. In this way, it is likely possible to prove a result that allows the twisted L-functions $\Lambda_f\left(s, \frac{a}{q}\right)$ to have arbitrary poles inside the critical strip, but we have not pursued this.
4. Using the Bruggeman–Kuznetsov trace formula and the method of [HLN21], it is likely possible to prove a similar converse theorem for Maass forms.

2. Lemmas

We begin with some preparatory lemmas.

Lemma 2.1. *Let $\gamma(s) = \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j)$, where $\lambda_j \in \mathbb{R}_{>0}$, $\mu_j \in \mathbb{C}$. If $\gamma(s)$ has poles at all but finitely many negative integers, then $\sum_{j=1}^{r} \lambda_j \geq 1$.*

Proof. Since the poles of $\Gamma(\lambda_j s + \mu_j)$ are spaced λ_j^{-1} apart, the number of poles of $\gamma(s)$ in $\mathbb{Z} \cap [-T, 0)$ for large $T > 0$ is at most $T \sum_{j=1}^{r} \lambda_j + O(1)$. If all but finitely many negative integers are poles of $\gamma(s)$, then this count is at least $T + O(1)$. The conclusion follows on taking $T \to \infty$. □

Lemma 2.2. *Let $\gamma(s)$ be as in the statement of Theorem 1.1, and suppose that $\gamma(s)$ has poles at all but finitely many negative integers. Then $\gamma(s)$ is of the form $cP(s)H^s\Gamma_{\mathbb{C}}(s)$, where $c, H \in \mathbb{R}_{>0}$ and P is a monic polynomial whose roots are distinct nonpositive integers.*

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Proof. Recall that $\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j)$. By Lemma 2.1, each factor in the product must contribute infinitely many poles in the negative integers. In particular, $\lambda_j, \mu_j \in \mathbb{Q}$ for each j. Let $\lambda_j = a_j/q_j$ in lowest terms. If $a_j > 1$, then we can consider $\tilde{\gamma}(s) = \Gamma(\frac{s-m_j}{q_j}) \prod_{i\neq j} \Gamma(\lambda_i s + \mu_i)$, where m_j is the negative integral pole of $\Gamma(\lambda_j s + \mu_j)$ of smallest absolute value. Then $\tilde{\gamma}(s)$ also has poles at all but finitely many negative integers, contradicting Lemma 2.1.

Hence, we have $a_j = 1$, so $\gamma(s) = Q^s \prod_{j=1}^r \Gamma(\frac{s+\nu_j}{q_j})$, where $\nu_j = \mu_j q_j \in \mathbb{Z}_{\geq 0}$. Let $q = \operatorname{lcm}\{q_1, \ldots, q_j\}$ and write $\Gamma(\frac{s+\nu_j}{q_j}) = \Gamma(\frac{q}{q_j} \cdot \frac{s+\nu_j}{q})$. By the Gauss multiplication formula, we have

$$\gamma(s) = \left(\sqrt{2\pi}\right)^{r-n} Q^s \prod_{j=1}^{r} \left(\frac{q}{q_j}\right)^{\frac{s+\nu_j}{q_j} - \frac{1}{2}} \prod_{i=0}^{q/q_j - 1} \Gamma\left(\frac{s + \nu_j + i q_j}{q}\right),$$

which we can rewrite in the form $cH^s \prod_{j=1}^q \Gamma\left(\frac{s+\nu_j'}{q}\right)$, for some $c, H \in \mathbb{R}_{>0}$ and $\nu_j' \in \mathbb{Z}_{\geq 0}$. Moreover, the ν_j' must run through a complete set of representatives for the residue classes mod q. Replacing each ν_j' by its mod q reduction $\nu_j'' \in \{0, \ldots, q-1\}$ and using the recurrence formula to relate $\Gamma\left(\frac{s+\nu_j'}{q}\right)$ and $\Gamma\left(\frac{s+\nu_j''}{q}\right)$, we get $c' P(s) H^s \prod_{j=1}^q \Gamma\left(\frac{s+\nu_j''}{q}\right)$, with P as described in the statement of the lemma. Finally, applying the Gauss multiplication formula once more and inserting a factor of $2(2\pi)^{-s}$, we arrive at the claimed form. □

The final lemma that we need is a special case of [HLN21, Lemma 4.1].

Lemma 2.3. *Let $r_q(n) = \sum_{\substack{a \bmod q \\ (a,q)=1}} e(na/q)$ be the Ramanujan sum. Then, for $n, N \geq \mathbb{Z}_{>0}$ and $\Re(s) > 1$, we have*

$$\sum_{\substack{q\geq 1 \\ N|q}} \frac{r_q(n)}{q^{2s}} = \begin{cases} \frac{\sigma_{1-2s}(n;N)}{\zeta^{(N)}(2s)} & \text{when } n \neq 0, \\ N^{1-2s} \prod_{p|N} (1-p^{-1}) \frac{\zeta(2s-1)}{\zeta^{(N)}(2s)} & \text{when } n = 0. \end{cases}$$

Here, when $\frac{N}{\prod_{p|N} p} \mid n$,

$$\sigma_s(n;N) = \prod_{\substack{p|n \\ p\nmid N}} \frac{p^{(\operatorname{ord}_p(n)+1)s} - 1}{p^s - 1} \cdot \prod_{p|N} \left(\frac{(1-p^{s-1}) p^{(\operatorname{ord}_p(n)+1)s} - (1-p^{-1}) p^{\operatorname{ord}_p(N)s}}{p^s - 1} \right). \tag{2.1}$$

Otherwise $\sigma_s(n;N) = 0$.

3. Proof of Theorem 1.1

Let $S_k(N,\chi)$ denote the space of holomorphic modular forms of weight k, level N and nebentypus character χ, and let $H_k(N,\chi)$ be an orthonormal basis for $S_k(N,\chi)$. Each $g \in H_k(N,\chi)$ has a Fourier expansion of the form

$$g(z) = \sum_{n=1}^{\infty} \rho_g(n) n^{\frac{k-1}{2}} e(nz)$$

for some coefficients $\rho_g(n) \in \mathbb{C}$. An application of the Petersson formula gives the following formula found in [IK04, Corollary 14.23]. For positive integers n and m, we have

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}} \sum_{g\in H_k(N,\chi)} \rho_g(n)\overline{\rho_g(m)} = \delta_{n=m} + 2\pi i^{-k} \sum_{\substack{q\geq 1 \\ N|q}} \frac{S_\chi(m,n;q)}{q} J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right), \tag{3.1}$$

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where

$$S_\chi(m,n;q)=\sum_{\substack{a \bmod q\\ \gcd(a,q)=1}}\chi(a)e\left(\frac{ma+n\bar{a}}{q}\right)$$

is the twisted Kloosterman sum and $J_{k-1}(y)$ is the classical J-Bessel function.

Fix a choice of data $\{f_n\}_{n\geq 1}$, N, χ, ω, and $\gamma(s)$ satisfying the hypotheses of Theorem 1.1. Let $\epsilon\in\{0,1\}$ be such that $\chi(-1)=(-1)^\epsilon$. For $k\geq 4$, $\Re(s)\in(\frac{1}{2},\frac{k-1}{2})$, $x>0$, and $\sigma_1\in(\frac{1-k}{2},-\Re(s))$, define

$$F_k(s,x)=\frac{1}{2\pi i}\int_{\Re(u)=\sigma_1}\frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})\gamma(1-s-u)}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})\gamma(s+u)}x^u\,du. \tag{3.2}$$

By Stirling's formula, the integral converges absolutely since $\Re(s)>\frac{1}{2}$. Note also that our choice of σ_1 ensures that the contour $\Re(u)=\sigma_1$ separates the poles of $\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})$ and $\gamma(1-s-u)$.

The proof will be split into two cases depending on whether $F_k(1,1)$ is nonzero for some k or not.

Proposition 3.1. *Suppose that* $F_k(1,1)\neq 0$ *for some* $k\geq 4$ *with* $k\equiv\epsilon \pmod 2$. *Then* f *as defined in Theorem 1.1 is in* $S_k(N,\chi)$.

Proof. Fix k as in the hypotheses. For $n\in\mathbb{Z}_{>0}$ and $\Re(s)>1$, define

$$\begin{aligned}K_n(s,f,\chi)&=\zeta^{(N)}(2s)\sum_{m=1}^{\infty}\frac{f_m}{m^s}\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{g\in H_k(N,\chi)}\rho_g(n)\overline{\rho_g(m)}\\&=\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{g\in H_k(N,\chi)}\rho_g(n)L(s,f\times\bar{g}),\end{aligned} \tag{3.3}$$

where

$$L(s,f\times\bar{g})=\zeta^{(N)}(2s)\sum_{m=1}^{\infty}\frac{f_m\overline{\rho_g(m)}}{m^s}\quad\text{and}\quad\zeta^{(N)}(s)=\zeta(s)\prod_{p\mid N}(1-p^{-s}).$$

Note that the series for $L(s,f\times\bar{g})$ converges absolutely for $\Re(s)>1$ thanks to the Ramanujan bound $\rho_g(m)\ll_\varepsilon m^\varepsilon$.

Now consider $s\in\mathbb{C}$ with $\Re(s)\in(\frac{5}{4},\frac{k-1}{2})$. Applying (3.1) to (3.3), we get

$$\begin{aligned}K_n(s,f,\chi)&=\zeta^{(N)}(2s)\sum_{m=1}^{\infty}\frac{f_m}{m^s}\left\{\delta_{n=m}+2\pi i^{-k}\sum_{\substack{q\geq 1\\ N\mid q}}\frac{S_\chi(m,n;q)}{q}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right)\right\}\\&=\zeta^{(N)}(2s)\frac{f_n}{n^s}+2\pi i^{-k}\zeta^{(N)}(2s)\sum_{\substack{q\geq 1\\ N\mid q}}\frac{1}{q}\sum_{m=1}^{\infty}\frac{f_mS_\chi(m,n;q)}{m^s}J_{k-1}\left(\frac{4\pi\sqrt{mn}}{q}\right).\end{aligned}$$

Here the interchange of the sums is justified for $\Re(s)>\frac{5}{4}$ by the estimates

$$J_{k-1}(y)\ll\min\{y^{k-1},y^{-1/2}\}\quad\text{and}\quad S_\chi(m,n;q)\ll_{n,\varepsilon}q^{1/2+\varepsilon}.$$

Let

$$K_{0,n}(s,f,\chi)=K_n(s,f,\chi)-f_nn^{-s}\zeta^{(N)}(2s).$$

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By the Mellin–Barnes type integral representation [GR15, (6.422)], we have

$$2\pi J_{k-1}(4\pi y) = \frac{1}{2\pi i}\int_{\Re(u)=\sigma_0} \frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})} y^{-2u}\, du,$$

for any choice of $\sigma_0 \in (\frac{1-k}{2}, 0)$. Applying this with $\sigma_0 \in (1-\Re(s), 0)$, we may change the order of sum and integral to obtain

$$K_{0,n}(s,f,\chi) = i^{-k}\zeta^{(N)}(2s) \sum_{\substack{q\geq 1\\ N\mid q}} \frac{1}{2\pi i}\int_{\Re(u)=\sigma_0} \frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})} q^{2u-1} \sum_{m=1}^{\infty} \frac{f_m S_\chi(m,n;q)}{m^s}(mn)^{-u}\, du.$$

Opening up the Kloosterman sum, this becomes

$$\sum_{\substack{q\geq 1\\ N\mid q}} \frac{1}{2\pi i}\int_{\Re(u)=\sigma_0} \frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})} q^{2u-1} n^{-u} \sum_{\substack{a \bmod q\\ a\bar{a}\equiv 1 \bmod q}} \chi(a) e(n\bar{a}/q) L_f\left(s+u, \tfrac{a}{q}\right) du.$$

Next we shift the contour to $\Re(u) = \sigma_1 \in (\frac{1-k}{2}, -\Re(s))$, so that $\Re(1-s-u) > 1$, and apply the functional equation

$$\sum_{\substack{a \bmod q\\ a\bar{a}\equiv 1 \bmod q}} \chi(a) e(n\bar{a}/q) L_f\left(s+u, \tfrac{a}{q}\right) = \omega q^{1-2s-2u} \frac{\gamma(1-s-u)}{\gamma(s+u)} \sum_{\substack{a \bmod q\\ a\bar{a}\equiv 1 \bmod q}} e(n\bar{a}/q) L_f\left(1-s-u, -\tfrac{\bar{a}}{q}\right),$$

obtaining

$$K_{0,n}(s,f,\chi) = i^{-k}\omega\zeta^{(N)}(2s) \sum_{\substack{q\geq 1\\ N\mid q}} \frac{1}{q^{2s}} \frac{1}{2\pi i}\int_{\Re(u)=\sigma_1} \frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})} \frac{\gamma(1-s-u)}{\gamma(s+u)} n^{-u}$$
$$\cdot \sum_{\substack{a \bmod q\\ a\bar{a}\equiv 1 \bmod q}} e(n\bar{a}/q) L_f\left(1-s-u, -\tfrac{\bar{a}}{q}\right) du.$$

Note that the contour shift is justified by the fact that $\Lambda_f\left(s, \frac{a}{q}\right)$ has finite order and the estimates

$$L_f\left(1-s-u, -\tfrac{\bar{a}}{q}\right) \ll 1 \quad \text{and} \quad \frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})} \frac{\gamma(1-s-u)}{\gamma(s+u)} \ll |u|^{-2\Re(s)} \quad \text{for } \Re(u) = \sigma_1.$$

The same estimates show that we may swap the order of sum and integral. We also expand $L_f\left(1-s-u, -\frac{\bar{a}}{q}\right)$ as a Dirichlet series, obtaining

$$\sum_{\substack{a \bmod q\\ a\bar{a}\equiv 1 \bmod q}} e(n\bar{a}/q) L_f\left(1-s-u, -\tfrac{\bar{a}}{q}\right) = \sum_{\substack{a \bmod q\\ a\bar{a}\equiv 1 \bmod q}} e(n\bar{a}/q) \sum_{m=1}^{\infty} \frac{f_m e(-m\bar{a}/q)}{m^{1-s-u}} = \sum_{m=1}^{\infty} \frac{f_m r_q(n-m)}{m^{1-s-u}},$$

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where r_q is the Ramanujan sum. An application of Lemma 2.3 leads to the following expression.

$$\begin{aligned} K_n(s,f,\chi) &= f_n n^{-s}\zeta^{(N)}(2s) \\ &+ i^{-k}\omega f_n n^{s-1}\zeta(2s-1)N^{1-2s}\prod_{p\mid N}(1-p^{-1})\cdot\int_{\Re(u)=\sigma_1}\frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})}\frac{\gamma(1-s-u)}{\gamma(s+u)}\,du \\ &+ i^{-k}\omega\sum_{\substack{m\geq 1\\ m\neq n}}\frac{f_m\sigma_{1-2s}(n-m;N)}{m^{1-s}}\int_{\Re(u)=\sigma_1}\frac{\Gamma_{\mathbb{C}}(u+\frac{k-1}{2})}{\Gamma_{\mathbb{C}}(-u+\frac{k+1}{2})}\frac{\gamma(1-s-u)}{\gamma(s+u)}\left(\frac{m}{n}\right)^u du. \end{aligned} \tag{3.4}$$

It is straightforward to see that $\sigma_{1-2s}(r;N) \ll_{N,\varepsilon} |r|^{\varepsilon}$, uniformly for $r \neq 0$ and $\Re(s) \geq \frac{1}{2}$. Thus, for a fixed σ_1, both integrals and the sum over m converge absolutely for $\frac{1}{2} < \Re(s) < -\sigma_1$. This establishes the meromorphic continuation of $K_n(s,f,\chi)$ to that region, and hence also of $\sum_{g\in H_k(N,\chi)}\rho_g(n)L(s,f\times\bar{g})$, in view of (3.3).

Since the g forms a basis for $S_k(N,\chi)$, we can choose a finite set $\{n_i : i = 1,\ldots,d\}$, where $d = \dim S_k(N,\chi)$, such that the vectors $(\rho_g(n_1),\ldots,\rho_g(n_d))$ are linearly independent. Taking a suitable linear combination of (3.4) for $n = n_i$, we deduce the meromorphic continuation of $L(s,f\times\bar{g})$ to $\Re(s) > \frac{1}{2}$ for each individual g. The only possible pole is at $s = 1$, and taking residues we see that

$$\frac{\Gamma(k-1)}{(4\pi)^{k-1}}\sum_{g\in H_k(N,\chi)}\rho_g(n)\operatorname{Res}_{s=1}L(s,f\times\bar{g}) = \tfrac{1}{2}i^{-k}\omega N^{-1}\prod_{p\mid N}(1-p^{-1})\cdot F_k(1,1)f_n.$$

Since $F_k(1,1)\neq 0$, we see that there exist $x_g\in\mathbb{C}$ such that $f_n = \sum_{g\in H_k(N,\chi)}x_g\rho_g(n)$ for any positive integer n. Since $S_k(N,\chi)$ is a vector space, we have proved the claim. □

We have taken care of the case where the integral (3.2) is nonzero at $s = 1$, $x = 1$ for some $k \geq 4$. If this is not the case, we use the following proposition.

Proposition 3.2. *If $F_k(1,1) = 0$ for all $k \geq 4$ with $k \equiv \epsilon \pmod 2$, then $\gamma(s)$ is of the form $cH^s\Gamma_{\mathbb{C}}(s+\frac{\ell-1}{2})$, where $c, H \in \mathbb{R}_{>0}$, and $\ell \in \{1,2,3\}$ with $\ell \equiv \epsilon \pmod 2$.*

Proof. We replace u by $u/2$ in the definition of $F_k(1,1)$ and shift the contour to $\Re(u) = -\frac{5}{2}$, which is permissible for all $k \geq 4$. Our hypothesis then implies that

$$\frac{1}{2\pi i}\int_{\Re(u)=-\frac{5}{2}}\frac{\Gamma_{\mathbb{C}}(\frac{k-1+u}{2})\gamma(-u/2)}{2\Gamma_{\mathbb{C}}(\frac{k+1-u}{2})\gamma(1+u/2)}\,du = 0 \quad \text{for all } k \geq 4 \text{ with } k \equiv \epsilon \pmod 2. \tag{3.5}$$

For $n \geq 0$, define

$$f_n(y) = \frac{\mathbf{1}_{(0,1)}(y)}{\sqrt{1-y^2}}\begin{cases}\cos(n\arcsin y) & \text{if } 2\mid n,\\ \sin(n\arcsin y) & \text{if } 2\nmid n.\end{cases}$$

Using the formulas in [GR15, §3.631], we see that f_n has Mellin transform

$$\widetilde{f_n}(s) = \int_0^\infty f_n(y)y^{s-1}\,dy = \frac{(-1)^{\lfloor n/2\rfloor}\Gamma_{\mathbb{C}}(s)}{2^s\Gamma_{\mathbb{C}}(\frac{s+n+1}{2})\Gamma_{\mathbb{C}}(\frac{s-n+1}{2})} \quad \text{for } \Re(s) > 0.$$

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For $k \equiv \epsilon (\mathrm{mod}\ 2)$, we have

$$\begin{aligned}\frac{\Gamma_{\mathbb{C}}(\frac{k-1+u}{2})\gamma(-u/2)}{2\Gamma_{\mathbb{C}}(\frac{k+1-u}{2})\gamma(1+u/2)} &= \frac{\Gamma_{\mathbb{C}}(1-u)}{2^{1-u}\Gamma_{\mathbb{C}}(\frac{k+1-u}{2})\Gamma_{\mathbb{C}}(\frac{3-k-u}{2})} \cdot \frac{2^{-u}\Gamma_{\mathbb{C}}(\frac{3-k-u}{2})\Gamma_{\mathbb{C}}(\frac{k-1+u}{2})}{\Gamma_{\mathbb{C}}(1-u)} \frac{\gamma(-u/2)}{\gamma(1+u/2)} \\ &= \frac{(-1)^{\lfloor (k-1)/2 \rfloor} \widetilde{f}_{k-1}(1-u)}{2^{u-1}\Gamma_{\mathbb{C}}(1-u)\sin(\frac{\pi}{2}(u+k-1))} \frac{\gamma(-u/2)}{\gamma(1+u/2)} \\ &= \frac{\widetilde{f}_{k-1}(1-u)}{2^{u-1}\Gamma_{\mathbb{C}}(1-u)\sin(\frac{\pi}{2}(u+1-\epsilon))} \frac{\gamma(-u/2)}{\gamma(1+u/2)} \\ &= \widetilde{f}_{k-1}(1-u) \frac{\sqrt{2}\Gamma_{\mathbb{C}}(\frac{1-\epsilon+u}{2})}{\Gamma_{\mathbb{C}}(\frac{2-\epsilon-u}{2})} \frac{\gamma(-u/2)}{\gamma(1+u/2)}.\end{aligned}$$

Thus, (3.5) can be written as

$$\frac{1}{2\pi i} \int_{\Re(u)=-\frac{5}{2}} \widetilde{f}_{k-1}(1-u)\widetilde{g}(u)\, du = 0, \quad \text{where} \quad \widetilde{g}(u) = \frac{\sqrt{2}\Gamma_{\mathbb{C}}(\frac{1-\epsilon+u}{2})}{\Gamma_{\mathbb{C}}(\frac{2-\epsilon-u}{2})} \frac{\gamma(-u/2)}{\gamma(1+u/2)}.$$

Applying the inverse Mellin transform, we define

$$g(y) = \frac{1}{2\pi i} \int_{\Re(u)=-\frac{5}{2}} \widetilde{g}(u) y^{-u}\, du \quad \text{for } y > 0.$$

By Stirling's formula, we have $\widetilde{g}(u) \ll |u|^{-3/2}$ for $\Re(u) = -\frac{5}{2}$, and it follows that $g(y)$ extends continuously to $[0, \infty)$.

An application of Fubini's theorem shows that for any measurable functions f, g on $(0, \infty)$ satisfying $\int_0^\infty |f(y)| y^{-\sigma}\, dy < \infty$ and $\int_{\mathbb{R}} |\widetilde{g}(\sigma + it)|\, dt < \infty$, we have

$$\frac{1}{2\pi i} \int_{\Re(u)=\sigma} \widetilde{f}(1-u)\widetilde{g}(u)\, du = \int_0^\infty f(y)g(y)\, dy.$$

It is easy to see this is satisfied by the functions f_n and g above with $\sigma = -\frac{5}{2}$. Hence, (3.5) becomes

$$\int_0^1 f_{k-1}(y)g(y)\, dy = 0 \quad \text{for all } k \geq 4 \text{ with } k \equiv \epsilon \ (\mathrm{mod}\ 2).$$

Suppose $\epsilon = 1$. Then we have

$$\int_{-\pi}^{\pi} \cos(\tfrac{k-1}{2}\theta) g(|\sin(\theta/2)|)\, d\theta = 0 \quad \text{for every odd } k \geq 5.$$

Thus, the function $h(\theta) = g(|\sin(\theta/2)|)$ has Fourier series of the form $a + b\cos\theta$. Since h is continuous, we have $h(\theta) = a + b\cos\theta$ for all θ, and thus $g(y) = a + b - 2by^2$ for $y \in [0, 1]$.

Since $g(y) = O(y^{5/2})$, we must have $a = b = 0$. Computing the Mellin transform again, we see that $\widetilde{g}(u) = \int_1^\infty g(y)\, y^{u-1}\, dy$, so $\widetilde{g}(u)$ is analytic for $\Re(u) < -\frac{5}{2}$. Since $\frac{\gamma(-u/2)}{\Gamma_{\mathbb{C}}(\frac{1-u}{2})}$ is analytic and nonvanishing for $\Re(u) < -\frac{5}{2}$, it follows that $\frac{\Gamma_{\mathbb{C}}(\frac{u}{2})}{\gamma(1+u/2)}$ is analytic for $\Re(u) < -\frac{5}{2}$. This means that $\gamma(s)$ has a pole at each negative integer.

Applying Lemma 2.2, we have $\gamma(s) = cP(s)H^s\Gamma_{\mathbb{C}}(s)$, where P is a monic polynomial whose roots are distinct nonpositive integers. Since $\gamma(s)$ has poles at all negative integers, either $P(s) = 1$ or $P(s) = s$. Thus, $\gamma(s)$ is of the form $c'H^s\Gamma_{\mathbb{C}}(s + \frac{\ell-1}{2})$ for some $\ell \in \{1, 3\}$, as required.

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Now suppose that $\epsilon = 0$. Then

$$\int_{-\pi}^{\pi} \frac{\sin(\frac{k-1}{2}\theta)}{\sin(\theta/2)} |\sin(\theta/2)| g(|\sin(\theta/2)|)\, d\theta = 0 \quad \text{for every even } k \geq 4. \tag{3.6}$$

Note that $\frac{\sin(\frac{k-1}{2}\theta)}{\sin(\theta/2)} = W_{\frac{k-2}{2}}(\cos\theta)$, where W_n is a polynomial of degree n (the 'Chebyshev polynomial of the fourth kind', see [Mas93]). Writing $v = \cos\theta = 1 - 2\sin^2(\theta/2)$, (3.6) becomes

$$0 = \int_{-1}^{1} W_{\frac{k-2}{2}}(v) h(v) \sqrt{\frac{1-v}{1+v}}\, dv, \quad \text{where } h(v) = \frac{g\left(\sqrt{\frac{1-v}{2}}\right)}{\sqrt{1-v}} \text{ for } v \in (-1,1).$$

Since the W_n are an orthogonal family with respect to the measure $\sqrt{\frac{1-v}{1+v}}\, dv$, there exists a constant a such that $(h(v) - a)\sqrt{\frac{1-v}{1+v}}$ is continuous and absolutely integrable on $(-1,1)$, and orthogonal to all polynomials. It follows that $h(v) = a$ for all $v \in (-1,1)$, and thus $g(y) = a\sqrt{2}y$ for $y \in (0,1)$. Since $g(y) = O(y^{5/2})$, we must have $a = 0$, and thus $\widetilde{g}(u) = \int_1^\infty g(y)\, y^{u-1}\, dy$.

As before, we conclude that $\frac{\Gamma_{\mathbb{C}}(\frac{1+u}{2})}{\gamma(1+u/2)}$ is analytic for $\Re(u) < -\frac{5}{2}$. Defining $\tilde{\gamma}(s) = \gamma(s + \frac{1}{2})$, we see that $\tilde{\gamma}(s)$ satisfies the hypotheses imposed on $\gamma(s)$ in Theorem 1.1 and has a pole at every negative integer. Appealing again to Lemma 2.2, we find that $\tilde{\gamma}(s)$ is of the form $cP(s)H^s\Gamma_{\mathbb{C}}(s)$, where either $P(s) = 1$ or $P(s) = s$. Thus, $\gamma(s) = c'H^sP(s - \frac{1}{2})\Gamma_{\mathbb{C}}(s - \frac{1}{2})$. Because of the hypothesis $\Re(\mu_j) > -\frac{1}{2}\lambda_j$ in Theorem 1.1, $\gamma(s)$ cannot have a pole at $\frac{1}{2}$, and therefore $P(s) = s$. Hence, $\gamma(s) = c''H^s\Gamma_{\mathbb{C}}(s + \frac{1}{2})$, as required. □

Now we can complete the proof of Theorem 1.1. In view of Propositions 3.1 and 3.2, we may assume that $\gamma(s) = cH^s\Gamma_{\mathbb{C}}(s + \frac{\ell-1}{2})$, where $c, H \in \mathbb{R}_{>0}$ and $\ell \in \{1,2,3\}$ with $\ell \equiv \epsilon \pmod 2$. In this case, we fall back on a more traditional proof of the converse theorem as in [Raz77], but we must first address the fact that our gamma factor differs from the expected one by the exponential factor H^s.

Suppose first that $H > 1$. Equation (3.5) becomes

$$\frac{1}{2\pi i}\int_{\Re(u)=-\frac{5}{2}} H^{-u} \frac{\Gamma_{\mathbb{C}}(\frac{k-1+u}{2})\Gamma_{\mathbb{C}}(\frac{-u}{2} + \frac{\ell-1}{2})}{\Gamma_{\mathbb{C}}(\frac{k+1-u}{2})\Gamma_{\mathbb{C}}(1 + \frac{u}{2} + \frac{\ell-1}{2})}\, du = 0 \quad \text{for all } k \geq 4 \text{ with } k \equiv \epsilon \pmod 2.$$

Since $H > 1$, we can shift the contour to the right as the integrand vanishes in the limit as $\Re(u) \to \infty$.

Suppose $\ell = 2$. Then when $k = 4$, we pick up poles at $u = 1, 3$ and derive that $0 = H^{-1} - H^{-3}$. Thus, $H = 1$, giving a contradiction. Similarly, when $\ell = 3$, we take $k = 5$. We have poles at $u = 2, 4$, getting $0 = H^{-2} - H^{-4}$, which results in the same contradiction. When $\ell = 1$, looking at $k = 5$, we have poles at $u = 0, 2, 4$. Thus, $0 = 1 - 4H^{-2} + 3H^{-4}$, which implies that $H^{-2} = \frac{1}{3}$. Now looking at $k = 7$, we get an extra pole at $u = 6$, so that $0 = \frac{2}{3} - 6H^{-2} + 12H^{-4} - \frac{20}{3}H^{-6}$, which is not satisfied for $H^{-2} = \frac{1}{3}$.

Hence, $H \leq 1$. Let $f(z) = \sum_{n=1}^\infty f_n n^{\frac{\ell-1}{2}} e(nz)$. Applying Hecke's argument [Miy06, Theorem 4.3.5] to our Voronoi formulas (1.1), we get the modularity relation

$$f\left(\frac{\frac{-1}{H^2 z} + a}{q}\right) = \omega\chi(\bar{a})(-iHz)^\ell f\left(\frac{z - \bar{a}}{q}\right) \tag{3.7}$$

for all $q \in N\mathbb{Z}_{>0}$ and $a, \bar{a} \in \mathbb{Z}$ with $a\bar{a} \equiv 1 \pmod q$.

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The ℓ-slash operator is defined for matrices $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of positive determinant by

$$(f|M)(z) = (\det M)^{\ell/2}(cz+d)^{-\ell} f\left(\frac{az+b}{cz+d}\right).$$

In this notation, (3.7) becomes

$$f\left|\begin{pmatrix} aH^2 & -1 \\ qH^2 & 0 \end{pmatrix}\right. = i^{-\ell}\omega\chi(\bar{a}) f\left|\begin{pmatrix} 1 & -\bar{a} \\ 0 & q \end{pmatrix}\right.,$$

or equivalently,

$$f\left|\begin{pmatrix} aH & \frac{a\bar{a}H^2-1}{qH} \\ qH & \bar{a}H \end{pmatrix}\right. = i^{-\ell}\omega\chi(\bar{a}) f. \tag{3.8}$$

We may assume that f is not identically 0 since the conclusion is trivial otherwise. Applying (1.1) twice, we have

$$\Lambda_f\left(s, \tfrac{a}{q}\right) = \omega\chi(\bar{a})q^{1-2s}\Lambda_f\left(1-s, -\tfrac{\bar{a}}{q}\right) = \omega^2\chi(-1)\Lambda_f\left(s, \tfrac{a}{q}\right),$$

and thus $(i^{-\ell}\omega)^2 = (-1)^\ell\chi(-1) = 1$.

Suppose $H < 1$. Taking $a = \bar{a} = 1$, the matrix above has absolute trace $2H < 2$, and so is elliptic. Unless $2H = \zeta + \zeta^{-1}$ for some root of unity ζ, this matrix has infinite order, and then a generalization of Weil's lemma [BBB+18, Lemma 4.2] implies that $f = 0$.

Hence, $2H$ must be an algebraic integer. For prime p and $i = 1, 2$, let $M_{i,p}$ be the matrix $\begin{pmatrix} H & \frac{H-H^{-1}}{q_i} \\ q_iH & H \end{pmatrix}$, where $q_1 = pN$, $q_2 = (p+1)N$. We compute that

$$\operatorname{tr}(M_{1,p}M_{2,p}) = (2H)^2 - 2 - \left(4-(2H)^2\right)\left(4p(p+1)\right)^{-1}.$$

Since $\lim_{p\to\infty}\operatorname{tr}(M_{1,p}M_{2,p}) = (2H)^2 - 2$, we have $|\operatorname{tr}(M_{1,p}M_{2,p})| < 2$ for all sufficiently large primes p, and again by Weil's lemma, it follows that $\operatorname{tr}(M_{1,p}M_{2,p})$ is an algebraic integer. Let K be a number field containing $2H$. Clearly, $\operatorname{tr}(M_{1,p}M_{2,p}) \in K$ for each p. Taking norms,

$$N_{K/\mathbb{Q}}\left(\operatorname{tr}(M_{1,p}M_{2,p}) + 2 - (2H)^2\right) = \left(-4p(p+1)\right)^{-[K:\mathbb{Q}]}N_{K/\mathbb{Q}}\left(4-(2H)^2\right).$$

Taking a sufficiently large prime p, this is not an integer, giving a contradiction.

Hence, $H = 1$. By (3.8), it follows that

$$f|M = i^{-\ell}\omega\chi(M)f \quad \text{for } M = \begin{pmatrix} a & b \\ q & \bar{a} \end{pmatrix} \in \Gamma_0(N) \text{ with } q > 0,$$

where we define $\chi(M) = \chi(\bar{a})$. Taking $M_1, M_2 \in \Gamma_0(N)$ such that M_1, M_2 and M_1M_2 are all of this form, we have

$$i^{-\ell}\omega\chi(M_1M_2)f = f|M_1M_2 = (f|M_1)|M_2 = (i^{-\ell}\omega)^2\chi(M_1)\chi(M_2)f.$$

Since $\chi(M_1M_2) = \chi(M_1)\chi(M_2)$, we have must have $\omega = i^\ell$. Finally, since the M of the above form, together with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$,generate $\Gamma_0(N)$, we conclude that $f \in M_\ell(N, \chi)$, as required.

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