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SUMS OF DISTINCT INTEGRAL SQUARES IN $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ AND $\mathbb{Q}(\sqrt{6})$

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Abstract

In this article, we determine all the totally positive integers of $\mathbb{Q}(\sqrt{m})$ which can be represented as sums of distinct integral squares, where $m = 2, 3, 6$.

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1. Introduction

The research on sums of integral squares has a long history going back to Lagrange's four squares theorem [7], that is, every positive rational integer is a sum of four squares of rational integers. In 1902, Hilbert asked whether every totally positive integer of a real quadratic field can be represented as a sum of four integral squares as a generalization of Lagrange's theorem. In 1928, Götzky [5] answered startlingly that every totally positive integer in the field $\mathbb{Q}(\sqrt{5})$ is representable as a sum of four integral squares and Maass [8] proved that three squares suffice instead of four. Furthermore, Siegel [11] proved that if F is a totally real number field, representability as a sum of integral squares in F holds if and only if F is either $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{5})$. However, Cohn and Pall asserted alternative results for some real quadratic fields $\mathbb{Q}(\sqrt{2})$, $\mathbb{Q}(\sqrt{3})$ and $\mathbb{Q}(\sqrt{6})$ in the consecutive articles [1, 2, 4]: every totally positive integer of $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ is representable as a sum of at most four squares if it is represented as a sum of squares. At most five squares are needed for every totally positive integer of $\mathbb{Q}(\sqrt{6})$. Cohn [3] and Scharlau [10] proved that three squares suffice in the field $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$.

However, one can ask which numbers are representable as a sum of *distinct* integral squares. For the case of rational integers, in 1948, Sprague [12] showed that every positive rational integer bigger than 128 can be represented as a sum of distinct

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integral squares. He also found all the 31 positive integers which cannot be represented as sums of distinct integral squares are as follows: 2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 24, 27, 28, 31, 32, 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 96, 108, 112 and 128. Recently, for totally positive algebraic integers in real quadratic number fields, Park [9] classified all the totally positive integers of $\mathbb{Q}(\sqrt{5})$ which cannot be represented as sums of distinct integral squares.

Let $m = 2, 3, 6$. Our main result is that, in the quadratic ring $\mathbb{Z}[\sqrt{m}]$, all totally positive integers of the form $a + 2b\sqrt{m}$ are sums of distinct squares, except finitely many ones up to equivalence. We will describe the exceptions explicitly.

2. Preliminary

Throughout this article, $m = 2$, 3 or 6. Let F be a real quadratic field $\mathbb{Q}(\sqrt{m})$ with involution '$\bar{\cdot}$' whose fixed field is $\mathbb{Q}$ and let $\mathcal{O} = \mathbb{Z}[\sqrt{m}] = \{a + b\sqrt{m} \mid a, b \in \mathbb{Z}\}$ be the ring of algebraic integers of F. We know immediately that any totally positive algebraic integer of the form $a + b\sqrt{m}$ cannot be represented as a sum of squares if b is odd. Let

$$\mathcal{S} = \mathcal{S}(\sqrt{m}) = \{a + 2b\sqrt{m} \in \mathcal{O} \mid a > 2|b|\sqrt{m}\},$$

which is a subset of totally positive integers in $\mathcal{O}$. Two algebraic integers $\alpha, \beta \in \mathcal{O}$ are called *equivalent* if there exists an integer n such that $\beta = \epsilon^{2n}\alpha$ where $\epsilon = \epsilon_m$ is a fundamental unit in $\mathbb{Q}(\sqrt{m})$. We choose fundamental units $\epsilon_2 = 1 + \sqrt{2}$, $\epsilon_3 = 2 + \sqrt{3}$ and $\epsilon_6 = 5 + 2\sqrt{6}$. If there is no confusion, we will use ϵ instead of ϵ_m. The characterization of $\mathcal{S}$ can be obtained as in Lemma 2.1. The characterization of $\mathcal{S}(\sqrt{6})$ was suggested by Kim [6] and we give a simpler proof here.

LEMMA 2.1. *Every element of $\mathcal{S}(\sqrt{m})$ is equivalent to*

$$\begin{array}{lll} p + q\epsilon_2^2 & \text{or } p + q\bar{\epsilon}_2^2 & \text{if } m = 2, \\ p + q(\epsilon_3 - 1)^2 & \text{or } p + q(\bar{\epsilon}_3 - 1)^2 & \text{if } m = 3, \\ p + q\epsilon_6 & \text{or } p + q\bar{\epsilon}_6 & \text{if } m = 6, \end{array}$$

for some nonnegative integers p and q.

PROOF. For any $a + 2b\sqrt{3} \in \mathcal{S}(\sqrt{3})$, we may assume that $b \geq 0$ without loss of generality. Define rational integer sequences $\{a_n\}$ and $\{b_n\}$ by

$$\begin{aligned} a_{n+1} + 2b_{n+1}\sqrt{3} &= \bar{\epsilon}^2(a_n + 2b_n\sqrt{3}) \\ &= (7a_n - 24b_n) + 2(7b_n - 2a_n)\sqrt{3} \end{aligned}$$

and $a_0 = a$, $b_0 = b$. Since ϵ is totally positive, $a_n + 2b_n\sqrt{3}$ is totally positive for all $n \geq 0$. Then

$$b_{n+1} - b_n = 6b_n - 2a_n < \frac{6a_n}{2\sqrt{3}} - 2a_n < 0.$$

Hence b_n is decreasing. Choose k satisfying $b_k \geq 0$, $b_{k+1} < 0$. If $|b_k| < |b_{k+1}|$, then $a_k > 4b_k$, so

$$a_k + 2b_k\sqrt{3} = (a_k - 4b_k) + b_k(\epsilon - 1)^2.$$

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If $|b_k| \geq |b_{k+1}|$, then $a_{k+1} \geq -4b_{k+1}$. Thus

$$a_{k+1} + 2b_{k+1}\sqrt{3} = (a_{k+1} + 4b_{k+1}) + (-b_{k+1})(\overline{\epsilon} - 1)^2.$$

This means $a + 2b\sqrt{3} \in \mathcal{S}(\sqrt{3})$ is equivalent to $p + q(\epsilon - 1)^2$ or $p + q(\overline{\epsilon} - 1)^2$ for some nonnegative integers p and q. Essentially the same argument shows that any $a + 2b\sqrt{2} \in \mathcal{S}(\sqrt{2})$ is equivalent to $p + q\epsilon_2^2$ or $p + q\overline{\epsilon}_2^2$ for some nonnegative integers p and q.

Now we characterize $\mathcal{S}(\sqrt{6})$. Let

$$T = \{a_\ell\epsilon^\ell + a_{\ell+1}\epsilon^{\ell+1} + \cdots + a_k\epsilon^k \mid \ell, k \in \mathbb{Z}, \ell \leq k \text{ and } a_\ell, a_{\ell+1}, \ldots, a_k \in \mathbb{Z}_{\geq 0}\}.$$

Note that $1 \in T$ and $T \subseteq \mathcal{S}(\sqrt{6})$ since ϵ^t are totally positive integers for all $t \in \mathbb{Z}$. If $\mathcal{S}(\sqrt{6}) \neq T$, then there is an element $\alpha = a + 2b\sqrt{6} \in \mathcal{S}(\sqrt{6}) \setminus T$ such that $b \geq 0$ and $\mathrm{Tr}(\alpha) \leq \mathrm{Tr}(\beta)$ for all $\beta \in \mathcal{S}(\sqrt{6}) \setminus T$. If $\alpha - 1$ is totally positive, then

$$\alpha - 1 \in \mathcal{S}(\sqrt{6}) \setminus T \quad \text{and} \quad \mathrm{Tr}(\alpha - 1) < \mathrm{Tr}(\alpha),$$

which is a contradiction. Suppose $\alpha - 1$ is not totally positive and $b > 0$. Then $a - 1 < 2b\sqrt{6}$. If $b \leq 9$,

$$a > 2b\sqrt{6} = 5b - (5 - 2\sqrt{6})b > 5b - \tfrac{1}{9}b \geq 5b - 1,$$

since $5 - 2\sqrt{6} < \frac{1}{9}$. So $a \geq 5b$ and hence $\alpha = a - 5b + b\epsilon \in T$, which contradicts the assumption. If $b \geq 10$, then

$$a < 2b\sqrt{6} + 1 \leq 5b - (5 - 2\sqrt{6})b + \frac{b}{10} \leq 5b,$$

since $5 - 2\sqrt{6} > \frac{1}{10}$. Since

$$\alpha\overline{\epsilon}^2 = (49a - 240b) + (98b - 20a)\sqrt{6} \in \mathcal{S}(\sqrt{6}) \setminus T$$

and

$$\mathrm{Tr}(\alpha\overline{\epsilon}^2) = 98a - 480b \leq 2a + 96(a - 5b) < 2a = \mathrm{Tr}(\alpha),$$

we obtain a contradiction. Thus we have $\mathcal{S}(\sqrt{6}) = T$. For each $\alpha \in \mathcal{S}(\sqrt{6})$, suppose

$$\alpha = a_\ell\epsilon^\ell + a_{\ell+1}\epsilon^{\ell+1} + \cdots + a_k\epsilon^k$$

for some ℓ, $k \in \mathbb{Z}$ and $a_\ell, a_{\ell+1}, \ldots, a_k \in \mathbb{Z}_{\geq 0}$ such that $\ell \leq k$ and $k - \ell$ is minimal. Therefore, if

$$\alpha = b_r\epsilon^r + b_{r+1}\epsilon^{r+1} + \cdots + b_s\epsilon^s$$

for some r, $s \in \mathbb{Z}$ and $b_r, b_{r+1}, \ldots, b_k \in \mathbb{Z}_{\geq 0}$ such that $r \leq s$, then $k - \ell \leq s - r$ holds. Note that $\epsilon^2 = 10\epsilon - 1$ and hence

$$\epsilon^k = 9\epsilon^{k-1} + 8\epsilon^{k-2} + \cdots + 8\epsilon^{\ell+2} + 9\epsilon^{\ell+1} - \epsilon^\ell.$$

If $k - \ell \geq 2$ and $a_\ell \geq a_k$, then

$$\alpha = (a_\ell - a_k)\epsilon^\ell + (a_{\ell+1} + 9a_k)\epsilon^{\ell+1} + \cdots + (a_{k-1} + 9a_k)\epsilon^{k-1}.$$

This gives a contradiction. When $k - \ell \geq 2$ and $a_\ell \leq a_k$, we get a contradiction with a similar argument. Hence $k - \ell \leq 1$ and $a + 2b\sqrt{6} \in \mathcal{S}(\sqrt{6})$ is equivalent to $p\epsilon^n + q\epsilon^{n+1}$

for some nonnegative integers p, q and $n \in \mathbb{Z}$. Therefore $a + 2b\sqrt{6} \in \mathcal{S}(\sqrt{6})$ is equivalent to $p + q\epsilon$ or $p + q\epsilon^{-1} = p + q\bar{\epsilon}$ for some nonnegative integers p and q. □

Remark 2.2. Let U be the set of all rational integers which can be represented as a sum of distinct squares in $\mathbb{Z}$. By [12], we know that

$$\mathbb{Z}_{\geq 0} \setminus U = \left\{ \begin{matrix} 2, 3, 6, 7, 8, 11, 12, 15, 18, 19, 22, 23, 27, 28, 31, 32, \\ 33, 43, 44, 47, 48, 60, 67, 72, 76, 92, 108, 112, 128 \end{matrix} \right\}.$$

3. $\mathbb{Q}(\sqrt{2})$ case

We characterize the set of totally positive algebraic integers of $\mathbb{Q}(\sqrt{2})$ which can be represented by sums of distinct integral squares.

Theorem 3.1. *The set of totally positive algebraic integers of* $\mathbb{Q}(\sqrt{2})$ *which can be represented as sums of distinct integral squares is* $\mathcal{S}(\sqrt{2})$.

Proof. By Lemma 2.1, every element in $\mathcal{S}(\sqrt{2})$ is equivalent to $p + q\epsilon^2$ or $p + q\bar{\epsilon}^2$ for some nonnegative integers p, q. Because $2 = (\epsilon - 1)^2$ is a square in $\mathbb{Q}(\sqrt{2})$ and every nonnegative integer can be represented in base 2, the result follows immediately. □

4. $\mathbb{Q}(\sqrt{3})$ case

For the $\mathbb{Q}(\sqrt{3})$ case, we start by giving a simple lemma.

Lemma 4.1. *Any nonnegative rational integer except* 2 *and* 6 *can be represented as a sum of distinct integral squares in* $\mathbb{Q}(\sqrt{3})$.

Proof. Since $3 = (\epsilon - 2)^2$,

$$a + 3b \quad \text{with } a, b \in U$$

can be represented as a sum of distinct integral squares in $\mathbb{Q}(\sqrt{3})$. Except for 2, 6, 11 and 18, all integers in $\mathbb{Z}_{\geq 0} \setminus U$ can be represented as $a + 3b$. For example, $128 = 4^2 + 10^2 + 3 \cdot 2^2$. On the other hand,

$$11 = (\epsilon - 2)^2 + (\epsilon - 1)^2 + (\bar{\epsilon} - 1)^2 \quad \text{and} \quad 18 = 1^2 + \epsilon^2 + \bar{\epsilon}^2 + (\epsilon - 2)^2.$$

Thus we get the result. □

Theorem 4.2. *Let* α *be a totally positive algebraic integer of* $\mathbb{Q}(\sqrt{3})$. *Then* α *is a sum of distinct squares if and only if* α *belongs to* $\mathcal{S}(\sqrt{3})$ *and is equivalent to an element in*

$$\mathcal{S}(\sqrt{3}) \setminus \left\{ \begin{matrix} 2, 2 + (\epsilon - 1)^2, 2 + (\bar{\epsilon} - 1)^2, 1 + 2(\epsilon - 1)^2, 1 + 2(\bar{\epsilon} - 1)^2, \\ 6, 6 + (\epsilon - 1)^2, 6 + (\bar{\epsilon} - 1)^2, 2 + 4(\epsilon - 1)^2, 2 + 4(\bar{\epsilon} - 1)^2 \end{matrix} \right\}.$$

Proof. Note that if $\alpha \notin \mathcal{S}(\sqrt{3})$, then α cannot be represented as a sum of squares, so $\alpha \in \mathcal{S}(\sqrt{3})$. By Lemma 2.1, we may assume that $\alpha = p + q(\epsilon - 1)^2$ or $\alpha = p + q(\bar{\epsilon} - 1)^2$ for some nonnegative integers p and q. Since $\overline{p + q(\epsilon - 1)^2} = p + q(\bar{\epsilon} - 1)^2$, we may assume that $\alpha = p + q(\epsilon - 1)^2$ for some nonnegative integers p and q.

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If $p \geq 19$ and $q \geq 19$, then by the proof of Lemma 4.1 we are done. Suppose $0 \leq p \leq 18$ and $q \geq 21$. The cases $p = 0$ or 1 are obvious. So suppose $2 \leq p \leq 18$. Since both of p and $p - 2$ cannot be 2, 6, 11 or 18 and

$$p + q(\epsilon - 1)^2 = (p - 2) + (q - 2)(\epsilon - 1)^2 + \epsilon^2 + (\epsilon - 2)^2,$$

we are done, by the proof of Lemma 4.1. If $p \geq 21$ and $0 \leq q \leq 18$, then we can prove that α can be represented as a sum of distinct integral squares with a similar argument.

Suppose $0 \leq p \leq 20$ and $0 \leq q \leq 20$. Let $E = \{2, 6, 11, 18\}$. The cases $p, q \notin E$ are obvious. If $p \in E$, then

$$\alpha = \begin{cases} (p+1) + (q-2)(\epsilon - 1)^2 + \epsilon^2 & \text{if } p \in E \text{ and } 0 \leq q - 2 \notin E, \\ (p+3) + (q-6)(\epsilon - 1)^2 + (1+\epsilon)^2 & \text{if } p \in E \text{ and } 0 \leq q - 6 \notin E, \\ (p-8) + (\epsilon - 1)^2 + (\bar{\epsilon} - 1)^2 & \text{if } p = 11, 18 \text{ and } q = 0, \\ 4 + \epsilon^2 + (\bar{\epsilon} - 1)^2 & \text{if } p = 11 \text{ and } q = 1, \\ 5 + \epsilon^2 + (\bar{\epsilon} - 1)^2 + (2\epsilon - 3)^2 & \text{if } p = 18 \text{ and } q = 1, \\ (p-2) + (\epsilon - 2)^2 + (\epsilon + 2)^2 & \text{if } p = 6, 11, 18 \text{ and } q = 4, \\ (p-2) + (2\epsilon - 3)^2 + (2\epsilon - 1)^2 & \text{if } p \in E \text{ and } q = 8. \end{cases}$$

Hence these can be represented as sums of distinct integral squares. If $q \in E$, then

$$\alpha = \begin{cases} (p+1) + (q-2)(\epsilon - 1)^2 + \epsilon^2 & \text{if } 0 \leq p + 1 \notin E \text{ and } q \in E, \\ (p-5) + (q-2)(\epsilon - 1)^2 + (2\epsilon - 3)^2 & \text{if } 0 \leq p - 5 \notin E \text{ and } q \in E, \\ 4 + (q-6)(\epsilon - 1)^2 + (2\epsilon - 1)^2 & \text{if } p = 1 \text{ and } q = 6, 11, 18. \end{cases}$$

Hence these can be represented as sums of distinct integral squares.

We can easily show that the remaining six elements,

$$2, 6, 2 + (\epsilon - 1)^2, 6 + (\epsilon - 1)^2, 1 + 2(\epsilon - 1)^2, 2 + 4(\epsilon - 1)^2,$$

cannot be represented by sums of distinct integral squares. For example, let us consider $2 + 4(\epsilon - 1)^2$ whose trace is 36. If $2 + 4(\epsilon - 1)^2 = 18 + 8\sqrt{3}$ can be represented by sums of distinct integral squares, then

$$\begin{aligned} 2 + 4(\epsilon - 1)^2 &= 18 + 8\sqrt{3} = \sum_{i=1}^{k} (a_i + b_i\epsilon)^2 \\ &= \sum_{i=1}^{k} \{(a_i + 2b_i)^2 + 3b_i^2\} + 2\sum_{i=1}^{k} (a_i + 2b_i)b_i\sqrt{3}, \end{aligned}$$

where $a_i, b_i \in \mathbb{Z}$ and $a_i + b_i\epsilon \neq a_j + b_j\epsilon$ if $i \neq j$. Therefore $\sum_{i=1}^{k} b_i^2 \leq 6$ and hence $|b_i| \leq 2$. This implies that $2 + 4(\epsilon - 1)^2 = 18 + 8\sqrt{3}$ must have at least one square summand among

$$1, 3 = (\epsilon - 2)^2, 4, 2\epsilon = (-1 + \epsilon)^2, -1 + 4\epsilon = \epsilon^2,$$
$$9, 6\epsilon = (\epsilon + 1)^2, 12, 5 + 4\epsilon = (-1 + 2\epsilon)^2, 8\epsilon = 2^2(\epsilon - 1)^2, 16.$$

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Then, by considering traces, one can show that there is no solution for the above summation. For $2, 6, 2+(\epsilon-1)^2, 6+(\epsilon-1)^2, 1+2(\epsilon-1)^2$, we can prove that these elements cannot be represented by sums of distinct integral squares with a similar argument. □

5. $\mathbb{Q}(\sqrt{6})$ case

For the case $\mathbb{Q}(\sqrt{6})$, the following remark and lemma are useful.

REMARK 5.1. In $\mathbb{Q}(\sqrt{6})$, the following are some square algebraic integers whose traces are less than 100:

$$\begin{aligned} 2+\epsilon &= (1+\sqrt{6})^2 = \Big(\frac{\epsilon-3}{2}\Big)^2, & 2\epsilon &= (2+\sqrt{6})^2 = \Big(\frac{\epsilon-1}{2}\Big)^2, \\ 15+2\epsilon &= (1+2\sqrt{6})^2 = (\epsilon-4)^2, & 3\epsilon &= (3+\sqrt{6})^2 = \Big(\frac{\epsilon+1}{2}\Big)^2, \\ 2+4\epsilon &= (4+\sqrt{6})^2 = \Big(\frac{\epsilon+3}{2}\Big)^2, & 8+4\epsilon &= (2+2\sqrt{6})^2 = (\epsilon-3)^2, \\ 6+5\epsilon &= (5+\sqrt{6})^2 = \Big(\frac{\epsilon+5}{2}\Big)^2, & 3+6\epsilon &= (3+2\sqrt{6})^2 = (\epsilon-2)^2, \\ -1+10\epsilon &= (5+2\sqrt{6})^2 = \epsilon^2. \end{aligned}$$

LEMMA 5.2. *Any nonnegative rational integer except* 2, 3, 8 *and* 12 *can be represented as a sum of distinct integral squares in* $\mathbb{Q}(\sqrt{6})$.

PROOF. Since $6 = \sqrt{6}^2 = ((\epsilon-5)/2)^2$,

$$a + 6b \quad \text{with } a, b \in U$$

can be represented as a sum of distinct integral squares in $\mathbb{Q}(\sqrt{6})$. All integers in $\mathbb{Z}_{\geq 0} \setminus U$ can be represented as $a+6b$ except 2, 3, 8, 12 and 18. On the other hand, since

$$18 = \Big(\frac{\epsilon-3}{2}\Big)^2 + \Big(\frac{\bar{\epsilon}-3}{2}\Big)^2 + 2^2,$$

this lemma follows. □

THEOREM 5.3. *Let α be a totally positive algebraic integer of $\mathbb{Q}(\sqrt{6})$. Then α is a sum of distinct squares if and only if α belongs to $\mathcal{S}(\sqrt{6})$ and is equivalent to an element in*

$$\mathcal{S}(\sqrt{6}) \setminus \left\{ \begin{array}{ll} a & \text{where } a = 2, 3, 8, 12, \\ b+\epsilon, b+\bar{\epsilon}, \epsilon & \text{where } b = 1, 4, 5, 10, 14, \\ c+2\epsilon, c+2\bar{\epsilon} & \text{where } c = 2, 3, 8, 12, \\ d+4\epsilon, d+4\bar{\epsilon}, 4\epsilon & \text{where } d = 1, 4, 5, 10, \\ e+5\epsilon, e+5\bar{\epsilon} & \text{where } e = 2, 3, \\ f+6\epsilon, f+6\bar{\epsilon} & \text{where } f = 1, 5, \\ \multicolumn{2}{l}{1+7\epsilon, 1+7\bar{\epsilon}, 7\epsilon, 2+8\epsilon, 2+8\bar{\epsilon}, 1+9\epsilon, 1+9\bar{\epsilon}, 9\epsilon,} \\ \multicolumn{2}{l}{2+10\epsilon, 2+10\bar{\epsilon}, 16\epsilon, 19\epsilon, 2+20\epsilon, 2+20\bar{\epsilon}} \end{array} \right\}.$$

PROOF. Note that if $\alpha \notin \mathcal{S}(\sqrt{6})$, then α cannot be represented as a sum of squares, so $\alpha \in \mathcal{S}(\sqrt{6})$. By Lemma 2.1, we may assume that $\alpha = p + q\epsilon$ or $\alpha = p + q\bar{\epsilon}$ for some nonnegative integers p and q. Since $\overline{p + q\epsilon} = p + q\bar{\epsilon}$, we give a proof only for the case $\alpha = p + q\epsilon$ for some nonnegative integers p and q. Let $E = \{2, 3, 8, 12\}$.

First, we assume that q is even and let $q = 2r$ for some nonnegative integer r. If $p \geq 13$ and $r \geq 13$, then

$$\alpha = p + r(2\epsilon).$$

Since 2ϵ is square, we are done. Suppose $0 \leq p \leq 12$ and $r \geq 13$. Since α is decomposed as the following, α can be represented as a sum of distinct integral squares by Remark 5.1 and Lemma 5.2:

$$\alpha = \begin{cases} p + r(2\epsilon) & \text{if } p \notin E, \\ (p-2) + (2+4\epsilon) + (r-2)(2\epsilon) & \text{if } p \in E \text{ and } r \neq 14, \\ (p-2) + (2+\epsilon) + 3^2(3\epsilon) & \text{if } p \in E \text{ and } r = 14. \end{cases}$$

If $p \geq 13$ and $0 \leq r \leq 12$, then α can be represented as a sum of distinct integral squares in the following way, with a similar argument:

$$\alpha = \begin{cases} p + r(2\epsilon) & \text{if } r \notin E, \\ (p-2) + (2+4\epsilon) + (r-2)(2\epsilon) & \text{if } p \neq 14 \text{ and } r \in E, \\ 6 + 2^2(2+\epsilon) + (r-2)(2\epsilon) & \text{if } p = 14 \text{ and } r \in E. \end{cases}$$

Suppose $0 \leq p \leq 12$ and $0 \leq r \leq 12$. The cases $p, r \notin E$ are obvious. If $p \in E$, then

$$\alpha = \begin{cases} (p-2) + (2+4\epsilon) + (r-2)(2\epsilon) & \text{if } p \in E \text{ and } 0 \leq r-2 \notin E, \\ (p-3) + (3+6\epsilon) + 2\epsilon & \text{if } p = 3, 8, 12 \text{ and } r = 4, \\ (p+1) + (10\epsilon - 1) + (r-5)(2\epsilon) & \text{if } p = 3, 8, 12 \text{ and } r = 5, 10. \end{cases}$$

Hence these can be represented as sums of distinct integral squares. If $r \in E$, then

$$\alpha = \begin{cases} 1 + (10\epsilon - 1) + 14\epsilon & \text{if } p = 0 \text{ and } r = 12, \\ (2+4\epsilon) + (10\epsilon - 1) + (r-7)(2\epsilon) & \text{if } p = 1 \text{ and } r = 8, 12, \\ (p-2) + (2+4\epsilon) + (r-2)(2\epsilon) & \text{if } 0 \leq p-2 \notin E \text{ and } r \in E, \\ (p-3) + (3+6\epsilon) + (r-3)(2\epsilon) & \text{if } p = 4, 10 \text{ and } r = 3, 8, 12, \\ (p-1) + (10\epsilon - 1) + (2+4\epsilon) + (r-7)(2\epsilon) & \text{if } p = 5 \text{ and } r = 8, 12. \end{cases}$$

Hence these can be represented as sums of distinct integral squares. So far, by direct calculation, we confirm that all $\alpha = p + 2r\epsilon \in \mathcal{S}(\sqrt{6})$ can be represented as sums of distinct integral squares except 20 elements,

$$2, 3, 8, 12, 2+2\epsilon, 3+2\epsilon, 8+2\epsilon, 12+2\epsilon, 4\epsilon, 1+4\epsilon, 4+4\epsilon,$$
$$5+4\epsilon, 10+4\epsilon, 6\epsilon, 1+6\epsilon, 5+6\epsilon, 2+8\epsilon, 2+10\epsilon, 16\epsilon, 2+20\epsilon.$$

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Second, we assume that q is odd and let $q = 2r + 1$ for some nonnegative integer r. If $p \geq 13$ and $r \geq 14$, then

$$\alpha = p + (r-1)(2\epsilon) + 3\epsilon.$$

Hence we are done since 2ϵ and 3ϵ are square. Similarly, if $0 \leq p \leq 12$ and $r \geq 14$, then

$$\alpha = \begin{cases} p + (r-1)(2\epsilon) + 3\epsilon & \text{if } p \notin E, \\ (p-2) + (2+\epsilon) + r(2\epsilon) & \text{if } p \in E, \end{cases}$$

and if $p \geq 13$ and $0 \leq r \leq 13$, then

$$\alpha = \begin{cases} p + (r-1)(2\epsilon) + 3\epsilon & \text{if } 0 \leq r-1 \notin E, \\ (p-2) + (2+\epsilon) & \text{if } p \neq 14 \text{ and } r = 0, \\ (p-2) + (2+4\epsilon) + 3\epsilon + (r-3)(2\epsilon) & \text{if } p \neq 14 \text{ and } r-1 \in E, \\ (p-8) + 2^2(2+\epsilon) + (2r-3)\epsilon & \text{if } p = 14 \text{ and } r-1 \in E, \end{cases}$$

and if $0 \leq p \leq 12$ and $0 \leq r \leq 2$, then

$$\alpha = \begin{cases} (p-2) + (2+\epsilon) & \text{if } r = 0 \text{ and } 0 \leq p-2 \notin E, \\ p + 3\epsilon & \text{if } r = 1 \text{ and } p \notin E, \\ (p-2) + (2+\epsilon) + 2\epsilon & \text{if } r = 1 \text{ and } p \in E, \\ p + 3\epsilon + 2\epsilon & \text{if } r = 2 \text{ and } p \notin E, \\ 4 + (2+\epsilon) + (2+4\epsilon) & \text{if } r = 2 \text{ and } p = 8, \\ 6 + (6+6\epsilon) & \text{if } r = 2 \text{ and } p = 12. \end{cases}$$

Hence these can be represented as a sum of distinct integral squares. Suppose $0 \leq p \leq 12$ and $3 \leq r \leq 13$. If $p \notin E$ and $0 \leq r-1 \notin E$, then α can be represented as sums of distinct squares since

$$\alpha = p + (r-1)(2\epsilon) + 3\epsilon.$$

If $p \in E$, then

$$\alpha = \begin{cases} (p-2) + (2+4\epsilon) + (3\epsilon) + (r-3)(2\epsilon) & \text{if } p \in E \text{ and } r-3 \notin E, \\ (p+\epsilon) + r(2\epsilon) & \text{if } p = 2 \text{ and } r-3 \in E, \\ (p-6) + (6+5\epsilon) & \text{if } p = 12 \text{ and } r = 2, \\ (p-3) + (3+6\epsilon) + (3\epsilon) + (r-4)(2\epsilon) & \text{if } p = 3, 8, 12 \text{ and } r = 5, \\ (p+1) + (10\epsilon - 1) + (3\epsilon) + (r-6)(2\epsilon) & \text{if } p = 3, 8, 12 \text{ and } r = 6, 11, \end{cases}$$

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and if $r - 1 \in E$, then

$$\alpha = \begin{cases} 3^2(3\epsilon) & \text{if } p = 0 \text{ and } r = 13, \\ (2 + 4\epsilon) + (10\epsilon - 1) + (r - 8)(2\epsilon) + 13\epsilon & \text{if } p = 1 \text{ and } r = 9, 13, \\ (p - 2) + (2 + 4\epsilon) + (3\epsilon) + (r - 3)(2\epsilon) & \text{if } 0 \leq p - 2 \notin E \text{ and } r - 1 \in E, \\ (p - 4) + (2 + 4\epsilon) + (2\epsilon) + (\epsilon + 2) & \text{if } p = 4, 5 \text{ and } r = 3, \\ (p - 3) + (3 + 6\epsilon) + (3\epsilon) + (r - 4)(2\epsilon) & \text{if } p = 4 \text{ and } r = 4, 9, 13, \\ (2 + \epsilon) + (2\epsilon) + (3 + 6\epsilon) + (r - 4)(2\epsilon) & \text{if } p = 5 \text{ and } r = 4, 9, 13, \\ (p - 3) + (3 + 6\epsilon) + (3\epsilon) + (r - 4)(2\epsilon) & \text{if } p = 10 \text{ and } r - 1 \in E. \end{cases}$$

Hence these can be represented as a sum of distinct integral squares. So far, by direct calculation, we confirm that all $\alpha = p + (2r + 1)\epsilon \in \mathcal{S}(\sqrt{6})$ can be represented as sums of distinct integral squares except the following thirteen elements,

$$\epsilon, 1 + \epsilon, 4 + \epsilon, 5 + \epsilon, 10 + \epsilon, 14 + \epsilon, 2 + 5\epsilon, 3 + 5\epsilon, 7\epsilon, 1 + 7\epsilon, 9\epsilon, 1 + 9\epsilon \text{ and } 19\epsilon.$$

It is not hard to verify that the remaining elements cannot be represented as sums of distinct integral squares. This verification can be done in a similar manner to Theorem 4.2. For example, let us consider $3 + 5\epsilon$, which is of trace 56. If $3 + 5\epsilon$ can be represented as a sum of distinct squares, it must have at least a square summand among the following irrational squares of trace less than 56, say, $2 + \epsilon$, 2ϵ, 3ϵ, $2 + 4\epsilon$, and $15 + 2\epsilon$, as shown in Remark 5.1. However, this is impossible. □

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