ONTARIO MATHEMATICAL MEETINGS - Winter, 1969

The details of the two meetings (time, place and invited address) are given as follows:

Tenth meeting: January 18, University of Waterloo (Abstracts 69.1 to 69.7)
R.M. Redheffer (U.C.L.A.), Some recent results on inequalities.

Eleventh meeting: March 15, University of Toronto (Abstracts 69.8 to 69.15)
G.-C. Rota (M.I.T.), Foundations of combinatorial theory, a progress report.

69.1 F.V. Atkinson (University of Toronto)
Definiteness Properties of Arrays of Operators

Let $G_1, \ldots, G_k$ be complex linear spaces, endowed with conjugate-linear maps to their duals (indicated by asterisks), and let

(1) $$A_{rs}, \quad r = 1, \ldots, k, \quad s = 0, \ldots, k,$$

be hermitian-symmetric endomorphisms of the G_r, respectively, so that the quadratic forms

(2) $$g_r^* A_{rs} g_r, \quad g_r \in G_r$$

are real-valued. We suppose the array (1) of operators to be definite in the sense that the matrix (2), with k rows $(r = 1, \ldots, k)$ and $k+1$ columns $(s = 0, \ldots, k)$ has its maximal rank k, for all choices of $g_1 \in G_1, \ldots, g_k \in G_k$, none being zero. The question is posed of whether we can then augment (2) to a square array by a row of $k+1$ real scalars, so that this array has determinant of fixed sign. The answer is affirmative in the cases $k = 1, 2$ only. The counter-example for $k = 3$ is based on a geometrical interpretation in terms of convex bodies. It is necessary to find three such bodies with the properties that no line meets all three, while through every point a plane can be drawn to meet all three bodies.

69.2 B. Banaschewski (McMaster University)
Essential Extensions and Injectivity for Metric and Banach Spaces

In some categories K (e.g. abelian groups; see also [2, 3, 4, 6]) one has the following situation with respect to injectivity and essential extensions:

I. The following are equivalent for $X \in K$:

(1) X is injective.

(2) For any extension $Y \supseteq X$ there exists a morphism

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$f: Y \to X$ mapping X identically.

(3) X has no proper essential extensions, i.e. no proper extensions $Y \supseteq X$ for which any $f: Y \to Z$ whose restriction to X is an embedding is itself an embedding.

II. For any $X \in K$ and any extension Y of X in K, the following are equivalent:

(1) Y is injective and an essential extension of X.

(2) Y is a maximal essential extension of X.

(3) Y is a minimal injective extension of X.

III. Any $X \in K$ has an injective essential extension $Y \supseteq X$ in K. (Such an extension is essentially unique; it is called the injective hull of X.)

In [1] it is shown that a certain set of conditions on a class E of morphisms in a category K implies that E-injectivity and essential E-extensions have properties analogous to those described by I - III, where E-injectivity is injectivity with respect to the $f \in E$ in place of arbitrary monomorphisms, and an essential E-extension of X is given by $f: X \to Y$ where $f \in E$ and $gf \in E$ implies $g \in E$, for any g. The conditions are as follows:

(E1) E is closed under composition.

(E2) If $f \in E$ is a left inverse of a $g \in E$ then f is an isomorphism; conversely, an isomorphism belongs to E.

(E3) For any $f \in E$ there exists a $g \in K$ such that $gf \in E^*$, i.e. $gf \in E$ and, for all $h \in K$, $hgf \in E$ implies $h \in E$.

(E4) K has pushouts, and these preserve E in the sense that for any pushout diagram

$$\begin{tikzcd} {} \arrow[r, "f"] \arrow[d, "g"'] & {} \arrow[d, "v"] \\ {} \arrow[r, "u"'] & {} \end{tikzcd}$$

$u \in E$ whenever $f \in E$.

(E5) Any well-ordered direct system in E has an upper bound in E.

(E6) For any $X \in K$, the classes of all $f: X \to Y$ in E^* and of all $f: Y \to X$ in E^* are small, and all $f \in E^*$ are monomorphisms.

Here, we are concerned with the category MS of all metric spaces and mappings $f: X \to Y$ such that $d(f(x), f(x')) \leq d(x, x')$ for all $x, x' \in X$ ($d(.,.)$ the metrices in either space), and the analogous category BanS of all Banach spaces (either field of scalars) and linear mappings $f: X \to Y$ with $\|f(x)\| \leq \|x\|$. For each of these it is known that every X has a minimal injective extension $Y \supseteq X$ [5, 7], and that this is essential [8]. The proofs in [5] and [7] are ad hoc, and essential extensions do not occur explicitly. A different approach can be based on the above conditions, in view of:

PROPOSITION. In MS and BanS, the isometric embeddings satisfy (E1) - (E6).

It follows that I - III hold in both MS and BanS.

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69.3 J.B. Miller (Trent University)

Some Formulae for Resolvents

In some cases, continuous linear operators mapping a Banach algebra with unit into itself can be shown to satisfy one or more algebraic identities involving the elements of the algebra. It is then usually possible to deduce from the identities a formula for the resolvent, and some spectral properties

Known cases include averaging operators, Reynolds operators, and Baxter operators, specified respectively by the identities

$$T(Tx.y) = Tx.Ty = T(x.Ty),$$

$$Tx.Ty = T(Tx.y + x.Ty - Tx.Ty),$$

$$Tx.Ty = T(Tx.y + x.Ty - \theta.x.y).$$

Here T is the operator, and x and y range over the Banach algebra; θ is a fixed parameter from the algebra.

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The formulae for the resolvents can be put in forms not involving the algebra elements, other than the unit e, its image $t = Te$, and functions of t.

69.4 M.A. McKiernan (University of Waterloo)
A Less Formal Approach to Kaluza-Klein Formalism

The "action" integrals (a) $\lambda(\tau_1) = \int_{\tau_0}^{\tau_1} \sqrt{g_{ij}\dot{y}^i\dot{y}^j}\, d\tau$ and

(b) $\lambda(\tau_1) = \int_{\tau_0}^{\tau_1} \left\{ \sqrt{h_{ij}\dot{x}^i\dot{x}^j} - B_i\dot{y}^i \right\} d\tau$, corresponding respectively

to gravitational and gravitational-electromagnetic phenomena, are shown to be related under continuous groups of null translations. This relation motivates a modified Kaluza-Klein formalism for which the classical cylindrical metric preserving transformations (c) $y^5 = x^5 + f^5(x^j)$, $y^i = f^i(x^j)$ for $i = 1, 2, 3, 4$ are replaced by (d) $y^5 = x^5$, $y^i = f^i(x^j, x^5)$. The cylindrical metric of V^5 is nevertheless preserved under (d), since it is assumed that V^5 admits a metric of the form $(\dot{y}^5)^2 - g_{ij}(y^k)\dot{y}^i\dot{y}^j$ (corresponding to (a)) and that (d) defines a continuous group of null translations in the V^4 metric defined by g_{ij} when x^5 is considered the group parameter. Application of (d) leads to the cylindrical metric $(\dot{x}^5 + B_i\dot{x}^i)^2 - h_{ij}\dot{x}^i\dot{x}^j$ corresponding to (b). The resulting electromagnetic fields $F_{ij} = B_{i,j} - B_{j,i}$ are then related to the curvatures of the V^4 corresponding to g_{ij} and h_{ij}; in particular it is shown that $B_i B_j R_g^{ij} = -\frac{1}{4}F_{ij}F^{ij}$ and $F^{ij}_{,j} = B_j R_g^{ij}$. When $R_{g\,ij} = 0$ it is shown that F_{ij} is a null electromagnetic field which is generally non-trivial. Some physical and geometric interpretations of the mathematical results are also presented.

69.5 Tomasz Pietrzykowski (University of Waterloo)
A Language for the Computer Assisted Theorem Proving

The paper outlines the main features of a proposed language (called the TPL language) for the computer assisted theorem proving. The TPL is destined to describe formal theories in a form suitable for an eventual computer processing. The class of theories which can be written in the TPL includes the predicate calculus of an arbitrary high (but finite) order. The possible applications of the TPL are: mechanical theorem proving, computer theorem checking, algebraic symbol manipulation.

A theory written in the TPL consists of a sequence of statements. There are the following kinds of statements: letter and separator stt (stt denotes statements) for optional expanding the standard TPL alphabet, type stt for declaring the types of objects of a theory, generality stt for establishing the hierarchy of generality between types, schema stt for providing the rules of creating objects, and constant stt for specifying the constants the theory. The objects of a TPL theory are the modified functional expressions where the function head may be placed arbitrarily (not only on the right most end of the expression). The right parentheses are compulsory, the left are optional.

Proofs are realized by means of the proof statement, which consists of a sequence of substatements of the following kinds: assumption, instance, deduction, conclusion and theorem. The proof procedure of the TPL is very tedious and cannot be expected to be used practically by a human. But there are many ways of defining a reduced proof procedure, where the user will only write certain proof substatements and the rest will be automatically produced by an appropriate mechanical theorem proving procedure.

69.6 V. Dlab (Carleton University)
Lattice Representation of Algebraic Dependence

The linear dependence in vector spaces can be studied in terms of LA-dependence structures (cf. [1]):

A LA-dependence structure is a pair $(S, \mathfrak{J})$ of the fundamental set S and a system $\mathfrak{J}$ of ("independent") subsets $I, I_1, I_2, \ldots$ which (i) is inductive and (ii), defining

$$\mathfrak{c}(I) = I \cup \{x \mid x \in S \ \& \ I \cup \{x\} \notin \mathfrak{J}\} ,$$

satisfies the following implication

$$I_1 \subseteq \mathfrak{c}(I_2) \rightarrow \mathfrak{c}(I_1) \subseteq \mathfrak{c}(I_2) .$$

In order to meet needs for wider applicability the concept of a LA-dependence structure has been generalized to that of a GA-dependence structure (cf. [1]):

A GA-dependence structure is a triple $(S, U, \mathfrak{J})$ of the fundamental set S, "canonic zone" $U \subseteq S$ and a system $\mathfrak{J} \subseteq 2^S$ which satisfies (i), (ii) for $I_1 \subseteq U$ and (iii) which has a maximal $I^* \in \mathfrak{J}$ such that $I^* \subseteq U$.

As in the case of a LA-dependence structure, one can prove the invariance of a certain cardinal attached to $(S, U, \mathfrak{J})$ - the rank of the structure (see [1]).

Now, the LA-dependence structures can be characterized in terms of c-dependence in certain lattices.

A subset I of atoms of a lattice $\mathfrak{L}$ is said to be c-independent if

$$x \nleq \bigvee (I \setminus \{x\}) \text{ for every } x \in I.$$

The characterization can be formulated as follows (comp. [2]):

THEOREM. Given a regular (i.e. $c(\phi) = \phi$) LA-dependence structure $(S, \mathfrak{J})$, there exists a complete algebraic atomic semimodular lattice $\mathfrak{L}$ and an (order-preserving) mapping Φ of the power set 2^S into $\mathfrak{L}$ such that

(a) $\Phi(\phi) = 0$, $\Phi(S) = 1$ and, for every system $\{X_\omega \mid \omega \in \Omega\}$ of subsets of S,

$$\Phi\left(\bigcup_{\omega \in \Omega} X_\omega\right) = \bigvee_{\omega \in \Omega} \Phi(X_\omega);$$

(b) $\{\Phi(\{x\}) \mid x \in S\} \subseteq \mathfrak{a}_{\mathfrak{L}}$ - the subset of all atoms of $\mathfrak{L}$;

(c) $I \in \mathfrak{J}$ if and only if $\{\Phi(\{x\}) \mid x \in I\}$ is c-independent.

A natural extension of the definition of c-dependence
concept of d-dependence.

A subset I of a lattice $\mathfrak{L}$ (with 0) is said to be d-independent if

$$x \wedge \bigvee(F) = 0 \text{ for every } x \in I \text{ and every finite } F \subseteq I \setminus \{x\}.$$

And, using this concept we can derive the following two THEOREMS.

Given a regular $\left\{\begin{array}{l}\text{GA-dependence structure } (S, U, \mathfrak{J}) \\ \text{LA-dependence structure } (S, \mathfrak{J})\end{array}\right\}$,

there exists a complete algebraic $\left\{\begin{array}{l}\text{atomic} \\ \text{semimodular balanced}\end{array}\right\}$ lattice $\mathfrak{L}$

and a one-to-one (order-preserving) mapping Φ of the power-set 2^S into $\mathfrak{L}$ such that (a) holds;

(b') $\{\Phi(\{x\}) \mid x \in S\} \subseteq \mathfrak{U}_{\mathfrak{L}}$ - the subset of all uniform elements of $\mathfrak{L}$;

(c') $I \in \mathfrak{J}$ if and only if $\{\Phi(\{x\}) \mid x \in I\}$ is d-independent.

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69.7 D. Solitar (York University)
On Finitely Generated Subgroups of a Free Group

M. Hall, Jr. [Trans. A.M.S. 67 (1949) 421-432] proved the following theorem:

Let H *be a finitely generated subgroup of a free group* F *and suppose* $\beta_1, \dots \beta_n$ *are in* F *but no* β_i *is in* H. *Then we may construct a subgroup* $\bar{H}$ *of finite index in* F *containing* H *and not containing any* β_i.

His proof actually shows more, viz., that H is a free factor of $\bar{H}$. In particular, taking the set of β_i's to be empty one obtains the following:

If H *is a finitely generated subgroup of a free group* F, *then* H *is a free factor of a subgroup* $\bar{H}$ *of finite index in* F.

We show how a number of results about finitely generated subgroups of a free group follow in a natural way from the above special case of the theorem of M. Hall Jr. In particular, we derive the following: a finitely generated subgroup H is of finite index F if and only if H has a non-trivial intersection with every non-trivial normal subgroup of F (this includes the case, when H contains a non-trivial normal subgroup of F [Proc. A.M.S. 8 (1957) 696-697], and the case, when H contains a non-trivial sub-normal subgroup of F [Canad. J. Math. 12 (1960) 414-425]; a generalization of this for a pair of subgroups H, K; other types of conditions for a finitely generated H to be of finite index in F (first proved by L. Greenberg for discrete groups of motions of the hyperbolic plane (which include free groups); and Howson's result that the intersection of two finitely generated subgroups of F is finitely generated. We also derive a quick way of obtaining the precise index of H in F from inspection of a Nielsen reduced set of generators for H.

69.8 Wai-Mee Ching (Louisianna State University)
Non-isomorphic Non-hyperfinite Factors

A von Neumann algebra is called hyperfinite if it is the weak closure of an increasing sequence of finite-dimensional von Neumann subalgebras; both hyperfinite and non-hyperfinite factors of type II_1 exist. Murray and von Neumann proved that all hyperfinite factors of type II_1 are isomorphic; J.T. Schwartz has shown that there exists a pair of non-isomorphic non-hyperfinite factors of type II_1. We will show the existence of three non-isomorphic non-hyperfinite factors of type II_1.

In order to construct the new factor, we first study the notion of crossed product of a von Neumann algebra with a certain group. Earlier, Nakamura and Takeda, Suzuki, and Turumaru developed the idea of crossed product for a finite von Neumann algebra with the coupling constant equal to one, generalizing Murray-von Neumann's measure construction of factors. We extend the notion of crossed product to a von Neumann algebra with a cyclic separating vector. This extension includes the measure construction and the group construction of factors both due to Murray and von Neumann. We give a systematic construction of the crossed product. We then establish a set of sufficient conditions for a crossed product of a von Neumann algebra to be a factor; and classify the type of a factor obtained by the crossed product.

We introduce the following algebraic property of a von Neumann algebra:

Definition. A von Neumann algebra R is said to have property C, if for each sequence $U_k (k = 1, 2, \dots)$ of unitary operators in R with the property that strong $\lim U_k^* T U_k = T$ for each $T \in R$, there exists a sequence $V_k (k = 1, 2, \ldots)$ of mutually commuting operators in R such that strong $\lim(U_k - V_k) = 0$.

Using the technique of crossed product, a factor of type II_1 is constructed which is the crossed product of a factor of type II_1 with an abelian group of outer automorphisms. We prove that this new factor of type II_1 has the new property C as well as the property Γ of Murray and von Neumann. Finally, we establish the non-isomorphism of three non-hyperfinite factors of type II_1 by showing that neither the hyperfinite factor of type II_1 nor the non-hyperfinite factor of type II_1 of Schwartz has the property C.

69.9 Kevin Clancey (Carleton University)
An Example of a Semi-normal Operator whose Spectrum is not a Spectral Set

Let K be a real Cantor set of positive measure. Consider for $f \in L^2(K)$ the operator

$$Tf(s) = sf(s) + i\left(\frac{1}{\pi i} \int_K \frac{f(t)}{s-t}\, dt\right) \quad s \in K .$$

If the singular integral is interpreted as a Cauchy principal value then T is semi normal and the spectrum of T is the set $K \times [-1, 1]$. The operator T has the following properties: (i) T is hyponormal and non-sub-normal; (ii) the spectrum of T is not a spectral set; (iii) for some polynomial p, the operator $p(T)$ is non-normaloid. The example motivates a construction which proves that every subnormal and non-normal operator is a strong limit of a sequence of hyponormal and non-subnormal operators.

69.10 G. Gasper (University of Toronto)
Linearization of the Product of Jacobi Polynomials, II

Let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of degree n, order (α, β), $\alpha, \beta > -1$, and let $g(k, m, n; \alpha, \beta)$ be defined by

$$R_n^{(\alpha,\beta)}(x)\, R_m^{(\alpha,\beta)}(x) = \Sigma_k\, g(k, m, n; \alpha, \beta)\, R_k^{(\alpha,\beta)}(x),$$

where $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(x) / P_n^{(\alpha,\beta)}(1)$. In [Linearization of the product of Jacobi polynomials I, Can. J. Math. (to appear)] we proved that if $\alpha \geq \beta$ and $\alpha + \beta \geq -1$ then $g(k, m, n; \alpha, \beta) \geq 0$ for all k, m, n. In this paper we prove

THEOREM 1. Let $a = \alpha + \beta + 1$, $b = \alpha - \beta$ and $V = \{(\alpha, \beta): \alpha \geq \beta,\ a(a+5)(a+3)^2 \geq (a^2 - 7a - 24)b^2\}$. If $(\alpha, \beta) \in V$ then $g(k, m, n; \alpha, \beta) \geq 0$ for all k, m, n. However, if $(\alpha, \beta) \notin V$ then there exist positive integers k, m and n such that $g(k, m, n; \alpha, \beta) < 0$.

THEOREM 2. Let $W = \{(\alpha, \beta) : \alpha \geq \beta > -1,\ -a(a+3) < 2b^2\} \cup \{(-\frac{1}{2}, -\frac{1}{2})\}$. If $(\alpha, \beta) \in W$ then $\Sigma_k |g(k, m, n; \alpha, \beta)| \leq G$, where G is independent of m and n. If $\beta > \alpha > -1$ or $-\frac{1}{2} > \alpha > -1$, then $\Sigma_k |g(k, m, n; \alpha, \beta)|$ is unbounded.

COROLLARY 1. Suppose $(\alpha, \beta) \in W$, $f(x) = \Sigma_{n=0}^{\infty} c(n)\ R_n^{(\alpha,\beta)}(x)$, $\Sigma_{n=0}^{\infty} |a(n)| < \infty$, and ϕ is a function holomorphic on an open set containing the range of f. Then $\phi(f(x)) = \Sigma_{n=0}^{\infty} d(n)\ R_n^{(\alpha,\beta)}(x)$ with $\Sigma_{n=0}^{\infty} |d(n)| < \infty$.

69.11 W.A. Coppel (University of Toronto)
The Asymptotic Behaviour of Second Order Linear Differential Equations

A result of F.V. Atkinson (see Coppel, Stability and asymptotic behaviour of differential equations. D.C. Heath, Boston, 1965) is given the following stronger form:

THEOREM. Let $g(x)$ be a continuous real-valued function for $x \geq x_0$ and let the integrals $g_0(x) = \int^{\infty} g(\xi)d\xi$, $g_1(x) = \int_x^{\infty} g(\xi)\cos 2\xi\, d\xi$, $g_2(x) = \int_x^{\infty} [g(\xi)\sin 2\xi - h(\xi)]d\xi$ converge, where $h(x)$ is a continuous non-negative function such that $\int^{\infty} h(x)dx = \infty$. If $\int^{\infty} |gg_j|dx < \infty$ $(j = 0, 1, 2)$ then the equation $y'' + [1 + g(x)]y = 0$ has a fundamental system of solutions y_1, y_2 such that for $x \to \infty$.

$$y_1(x) = r(x)[\cos x + o(1)], \quad y_2(x) = [r(x)]^{-1}[\sin x + o(1)]$$

$$y'_1(x) = r(x)[-\sin x + o(1)], \quad y'_2(x) = [r(x)]^{-1}[\cos x + o(1)]$$

where $r(x) = \exp\{\frac{1}{2}\int_{x_0}^{x} g(\xi)\sin 2\xi\, d\xi\}$.

69.12 E. Stamm (University of Toronto)
Sections of Holomorphic Vector Bundles

Let $E \xrightarrow{p} B$ be a holomorphic vector bundle over the connected Stein manifold B. A section is a holomorphic map $s: B \to E$ such that $p \cdot s = id_B$. Let $M(E)$ be the set of these sections. It is a module over the ring $H(B)$ of holomorphic functions on B.

THEOREM. $M(E)$ is a finitely generated projective $H(B)$-module. It can be generated by $N \times (1+n)$ generators, where N=fibredimension and n=dimension of the basemanifold B.

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COROLLARY: a) It is possible to introduce the notion of stable equivalence classes of holomorphic vector bundles over a Stein manifold B.

b) One has canonical isomorphisms of H(B)-modules

$M(E_1 \otimes E_2) = M(E_1) \otimes M(E_2)$, $M(\Lambda^k E) \cong \Lambda^k M(E)$, etc.

69.13 Tae Ho Choe (McMaster University)
Notes on a Locally Compact Connected Topological Lattice

E. Dyre and A. Shields, [Pacific J. Math. 9] conjectured that if L is a compact connected metrizable distributive topological lattice, then the dimension of L is equal to the breadth of L. L. Anderson [Pacific J. Math. 9] showed that the breadth of the $L \leq$ the codimension of L (in the sense of Cohen [Duke Math. J. 21)]. A locally compact topological lattice L of dimension n is called regular if the subset of L, made up of the points at which L has dimension n, has non-void interior. We shall show that if L is a connected distributive regular topological lattice then the inductive dimension (or codimension) and the breadth of L are the same.

L. Anderson conjectured that if L is a locally compact connected topological lattice, then L is chain-wise connected. We shall prove this conjecture is true. As an immediate corollary of this we can extend a Wallace result [Summa Brazil M. 3] that any compact connected topological lattice L is acyclic, i.e., $H^p(L) = 0$ for all $p > 0$, where $H^*(\ )$ denotes the cohomology group of Alexsander Kolmogrof. Our result is that any locally compact connected topological lattice with 0 and I is acyclic.

69.14 V. Dlab (Carleton University)
A New Characterization of Perfect Rings

J.P. Jans has shown in [4] that if a ring R is right perfect (cf. H. Bass [1]), then a certain torsion (cf. [2]) in the category Mod R of left R-modules is closed under taking direct products. In fact, it can be easily shown that every (hereditary) torsion in Mod R is closed under taking direct products provided that R is right perfect. Moreover, making use of a one-to-one correspondence between torsions in Mod R and certain sets of left ideals of R (see [3]) we can give a characterization of perfect rings along these lines:

A ring R is right perfect if and only if every hereditary torsion in Mod R is fundamental (i.e. derived from "prime" torsions) and closed under taking direct products; then there is a finite number 2^n (n natural) of torsions in Mod R.

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69.15 C. T. Ng (University of Waterloo)

Uniqueness Theorems for a General Class of Functional Equations on Topological Vector Spaces

In a previous paper J. Aczél has shown the following:

THEOREM 1: *If in the interval* $\langle A, B\rangle$ *we have*

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y]$$

and there f, F *are continuous*, F *intern (the value* $F(x, y)$ *lies strictly between* x *and* y) *and* $u \to H(u, v, x, y)$ *or* $v \to H(u, v, x, y)$ *are injective, then the functional equation* (*) *with the initial conditions*

$$f(a) = c, \quad f(b) = d \quad (a, b \in \langle A, B\rangle)$$

has at most one solution.

The above result has been established for functions with real variables. In the sequel we extend the notion of internness to vector spaces and derive results in topological spaces.

Definitions and Notations: For two distinct points x and y of a vector space (v.s.) E over the real field $\underline{R}$, we denote the (open) line segment joining x and y by

$$L(x, y) = \{y + t(x - y) : t \in (0, 1)\} .$$

A mapping F defined on some subset S of $E \times E$ into E is said to be intern if $F(x, y) \in L(x, y)$ whenever $(x, y) \in S$ with $x \neq y$.

We have the following results:

THEOREM 2: *Let* E_1 *be a closed subset of a topological vector space* (t.v.s.) E *and let* $F : E_1 \times E_1 \to E_1$ *be intern, continuous in both variables. Let* N *be a set and* $f_1, f_2 : E_1 \to N$ *be mappings satisfying the functional equation*

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y]$$

where the mapping $H : N \times N \times E_1 \times E_1 \to N$ *is injective either in its first variable or in its second variable. If* f_1 *and* f_2 *are identical on some*

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E_1-neighbourhood V of a point $a \in E_1$, then f_1 and f_2 are identical on the entire domain E_1.

THEOREM 3: Let $F : E_1 \times E_1 \to E_1$ be an intern function defined on a closed subset E_1 of a t.v.s. E over R, and let N be a Hausdorff space. Suppose $f_1, f_2 : E_1 \to N$ are continuous mappings satisfying

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y],$$

where H is a mapping from $N \times N \times E_1 \times E_1$ into N. Then the set

$$S = \{x : x \in E_1, \; f_1(x) = f_2(x)\}$$

is convex.

THEOREM 4: Let E_1 be a closed subset of a t.v.s. E, and let $F : E_1 \times E_1 \to E_1$ be intern, continuous in both variables. Let N be a Hausdorff space. Suppose $f_1, f_2 : E_1 \to N$ are continuous mappings satisfying the functional equation

$$(*) \qquad f[F(x, y)] = H[f(x), f(y), x, y]$$

where the mapping $H : N \times N \times E_1 \times E_1 \to N$ is either injective in its first variable or injective in its second variable. If f_1 and f_2 are identical on some subset A of E_1 whose convex hull ΓA has non-empty interior (interior taken in E_1), then f_1 and f_2 are identical on the entire domain E_1.

COROLLARY: If in Theorem 4, E is locally convex Hausdorff of dimension n and $A = \{a_i : i = 1, 2, \ldots, n+1\}$ is such that $\{a_i - a_1 : i = 2, 3, \ldots, n+1\}$ is linearly independent, then there exists at most one continuous solution of $(*)$ satisfying the $n+1$ initial conditions

$$f(a_i) = b_i \qquad i = 1, 2, \ldots, n+1 .$$