LIPSCHITZ CONDITIONS AND LACUNARITY

Dedicated to the memory of Hanna Neumann

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Introduction

We consider 2π-periodic functions on the line and give simple and complete characterizations, in terms of Fourier coefficients, of functions which belong to various Lipschitz classes and whose Fourier series are lacunary. Such characterisations seem to be missing from the literature, though there are various well-known partial characterisations valid for functions with arbitrary spectra; cf. the remarks following Theorem 1. The results given below form complements to and sharpenings of some of the standard results valid for the special case of lacunary series.[1]

In general, we use the notation and terminology of [1]. In particular, we define $\omega_p f$ as in [1], 8.5. If $0 < \alpha < 1$, we write Λ_α for the set of continuous periodic f such that $\omega_\infty f(a) = 0(|a|^\alpha)$ for real a, or equivalently for real a satisfying $|a| \leqq \pi$. Similarly, if $1 \leqq p < \infty$, Λ_α^p denotes the set of periodic $f \in L^p = L^p(0, 2\pi)$ such that $\omega_p f(a) = 0(|a|^\alpha)$. If E is a subset of Z, and if F is a set of periodic integrable functions (or of equivalence classes of such functions), we write F_E for the set of E-spectral elements of F, i.e., the set of $f \in F$ such that

$$\hat{f}(n) = (2\pi)^{-1} \int_0^{2\pi} f(x) e^{-inx} dx$$

is zero for every $n \in Z \cap E'$.

1. Sidon-spectral functions in Λ_α

THEOREM 1. *Let E be a Sidon subset of Z and let $f \in C_E$. Then*
(i) *if* $0 < \alpha < 1$, $f \in \Lambda_\alpha$ *if and only if*

$$\sup_{R>0} R^\alpha \sum_{|n| \geqq R} |\hat{f}(n)| < \infty \; ; \tag{1}$$

(1) In connection with Hadamard lacunarity, see [4], pp. 110–111, Exercises 1 and 2.

(ii) $f \in \Lambda_1$ *if and only if*

$$\Sigma_{n \in Z} |n\hat{f}(n)| < \infty. \tag{2}$$

PROOF. (a) Since E is a Sidon set, $g \in C_E$ implies ([1]), Section 15.1) that

$$\Sigma_{n \in Z} |\hat{g}(n)| \leqq A \cdot \max |g|,$$

where A is independent of g. If $f \in C_E \cap \Lambda_\alpha$, we can apply this with $g : x \mapsto f(x+a) - f(x)$ to conclude that

$$\Sigma_{n \in Z} |\hat{f}(n)| |e^{ina} - 1| \leqq A_1 |a|^\alpha \tag{3}$$

where A_1 is independent of a. In applying (3), we consider separately the cases $0 < \alpha < 1$ and $\alpha = 1$.

In case $0 < \alpha < 1$, we note that

$$|e^{ina} - 1| \geqq 2^{\frac{1}{2}}$$

whenever $a = \pi/2R$ and $0 < R \leqq |n| < 2R$, and so infer from (3) that

$$\Sigma_{R \leqq |n| < 2R} |\hat{f}(n)| \leqq A_2 R^{-\alpha},$$

where A_2 is independent of R. Hence

$$\begin{aligned} \Sigma_{|n| \geqq R} |\hat{f}(n)| &= \Sigma_{k=0}^{\infty} \Sigma_{2^k R \leqq |n| < 2^{k+1} R} |\hat{f}(n)| \\ &\leqq A_2 \Sigma_{k=0}^{\infty} (2^k R)^{-\alpha} \\ &= (1 - 2^{-\alpha})^{-1} A_2 R^{-\alpha}, \end{aligned}$$

showing that (1) holds.

If $\alpha = 1$, (3) gives for any finite subset F of Z and $a \neq 0$:

$$\Sigma_{n \in F} |\hat{f}(n)| |a^{-1}(e^{ina} - 1)| \leqq A_3,$$

where A_3 is independent of a and F. Letting $a \neq 0$ tend to zero, this entails

$$\Sigma_{n \in F} |n\hat{f}(n)| \leqq A_3.$$

Since A_3 is independent of F, this implies (2).

(b) If $0 < \alpha < 1$, $f \in C$ and (1) holds, then, if A_4 denotes the left hand side of (1),

$$\begin{aligned} |f(x+a) - f(x)| &\leqq \Sigma_{n \in Z} |\hat{f}(n)| |e^{ina} - 1| = 2\Sigma_{n \neq 0} |\hat{f}(n)| |\sin \tfrac{1}{2} na| \\ &\leqq \Sigma_{1 \leqq |n| < 2^{j+1}} |\hat{f}(n)| |na| + 2\Sigma_{|n| \geqq 2^{j+1}} |\hat{f}(n)| \\ &\leqq |a| \Sigma_{k=0}^{j} \Sigma_{2^k \leqq |n| < 2^{k+1}} |n\hat{f}(n)| + 2A_4 2^{-(j+1)\alpha} \\ &\leqq |a| \Sigma_{k=0}^{j} A_4 2^{k+1} 2^{-k\alpha} + 2A_4 2^{-(j+1)\alpha} \\ &\leqq 2(2^{1-\alpha} - 1)^{-1} |a| A_4 2^{(j+1)(1-\alpha)} + 2A_4 2^{-(j+1)\alpha}. \end{aligned} \tag{4}$$

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Supposing, as we clearly may, that $|a| \leqq \pi$, choose the nonnegative integer j so that

$$\pi 2^{-j-1} < |a| \leqq \pi 2^{-j}; \tag{5}$$

then (4) yields

$$|f(x+a) - f(x)| \leqq A_5 |a|^\alpha$$

where A_5 is independent of x and a, showing that $f \in \Lambda_\alpha$.

Again, if $\alpha = 1$, $f \in C$ and (2) holds, it is evident that

$$\begin{aligned} |f(x+a) - f(x)| &= \Sigma_{n \in Z} |\hat{f}(n)| |e^{ina} - 1| \\ &\leqq \Sigma_{n \in Z} |\hat{f}(n)| |na| \\ &= |a| (\Sigma_{n \in Z} |n\hat{f}(n)|), \end{aligned}$$

which indicates that $f \in \Lambda_1$.

Remarks. (i) Part (b) of the proof of Theorem 1 does not use the fact that f is E-spectral: it gives sufficient conditions for an arbitrary $f \in C$ to belong to Λ_α. For results in the reverse direction applying to arbitrary $f \in \Lambda_\alpha$, see [1], 10.6.2; [2], Chapter VI; [3], Vol. 1, pp. 215–217.

(ii) If $0 < \alpha < 1$, the relation $f \in \Lambda_{\alpha, E}$ does not imply that

$$\Sigma_{n \in Z} |n|^\alpha |\hat{f}(n)| < \infty;$$

a counterexample is ([2], Vol. I, p. 47)

$$f(x) = \Sigma_{k=1}^{\infty} 2^{-k\alpha} \cos(2^k x).$$

However, it follows readily from Theorem 1 (i) that $f \in \Lambda_{\alpha, E}$ implies

$$\Sigma_{n \in Z} \lambda(|n|) |n|^\alpha |\hat{f}(n) < \infty$$

whenever E is a Sidon subset of Z and $(\lambda(m))_{m=0}^{\infty}$ is positive, decreasing and such that

$$\Sigma_{k=0}^{\infty} \lambda(2^k) < \infty.$$

(iii) For a general $f \in \Lambda_\alpha$ (or even a general $f \in \Lambda_\alpha^2$) it is true that

$$\Sigma_{n \in Z} |n|^\beta |\hat{f}(n)| < \infty$$

for every $\beta < \alpha - \frac{1}{2}$ (see [2], Vol. I, p. 251, Example 9(i)). The conclusion is false for $\alpha = 1$ and $\beta = \frac{1}{2}$, since otherwise we should have

$$\Sigma_{n \neq 0} |n|^{-\frac{1}{2}} |\hat{g}(n)| < \infty$$

for every $g \in C$, which is known to be false (loc. cit., p. 225, Theorem (10.1); alternatively, [1], Exercise 14.14).

2. Functions in Λ_α^p with lacunary Fourier series

We begin by noting that the Parseval formula shows that

$$\omega_2 f(a)^2 = 4 \sum_{n \in Z} |\sin \tfrac{1}{2}na|^2 |\hat{f}(n)|^2 \tag{6}$$

for any $f \in L^2$. From this we deduce the following criterion.

LEMMA. *In order that $f \in \Lambda_\alpha^2$, it is necessary and sufficient that $f \in L^1$ and that*

$$\sup_{R>0} R^{2\alpha} \sum_{|n| \geqq R} |\hat{f}(n)|^2 < \infty, \text{ if } 0 < \alpha < 1, \tag{7}$$

$$\sum_{n \in Z} |n\hat{f}(n)|^2 < \infty, \text{ if } \alpha = 1. \tag{8}$$

PROOF. (a) The case $0 < \alpha < 1$. Suppose first that (7) holds. Then it is clear that $f \in L^2$. Also, if A_6 denotes the supremum on the left of (7), (6) yields

$$\begin{aligned} \omega_2 f(a)^2 &= 4 \sum_{|n| \geqq 1} |\sin \tfrac{1}{2}na|^2 |\hat{f}(n)|^2 \\ &\leqq \sum_{k=0}^{j} \sum_{2^k \leqq |n| < 2^{k+1}} n^2 |a|^2 |\hat{f}(n)|^2 + 4 \sum_{|n| \geqq 2^{j+1}} |\hat{f}(n)|^2 \\ &\leqq a^2 \sum_{k=0}^{j} 2^{2k+2} A_6 2^{-2k\alpha} + 4A_6 2^{-2(j+1)\alpha} \\ &\leqq 4a^2 A_6 (2^{2(1-\alpha)} - 1)^{-1} 2^{2(j+1)(1-\alpha)} + 4A_6 2^{-2(j+1)\alpha} . \end{aligned}$$

Choosing the nonnegative integer j as in (5), we infer that

$$\omega_2 f(a) \leqq A_7 |a|^\alpha$$

for $|a| \leqq \pi$, where A_7 is independent of a, showing that $f \in \Lambda_\alpha^2$.

Conversely, suppose that $f \in \Lambda_\alpha^2$. Then $f \in L^2$ and, by (6),

$$\sum_{n \in Z} |\sin \tfrac{1}{2}na|^2 |\hat{f}(n)|^2 \leqq A_8 |a|^{2\alpha},$$

where A_8 is independent of a. On taking $a = \pi/2R$, it follows that

$$\sum_{R \leqq |n| < 2R} |\hat{f}(n)|^2 \leqq A_9 R^{-2\alpha},$$

where A_9 is independent of R. Replacing R by $2^k R$ and then summing over all nonnegative integers k, it appears that

$$\sum_{|n| \geqq R} |\hat{f}(n)|^2 \leqq A_9 (1 - 2^{-2\alpha})^{-1} R^{-2\alpha},$$

so that (7) is satisfied.

(b) The case $\alpha = 1$. If (8) holds, then $f \in L^2$ and (6) shows that

$$\begin{aligned} \omega_2 f(a)^2 &= 4 \sum_{n \in Z} |\sin \tfrac{1}{2}na|^2 |\hat{f}(n)|^2 \\ &\leqq \sum_{n \in Z} n^2 a^2 |\hat{f}(n)|^2 \end{aligned}$$

showing that $f \in \Lambda_1^2$. Conversely, if $f \in \Lambda_1^2$, then $f \in L^2$ and (6) yields

$$\sum_{n \in Z} |\sin \tfrac{1}{2}na|^2 |\hat{f}(n)|^2 \leqq A_{10} a^2,$$

where A_{10} is independent of a. Hence, for any $R > 0$,

$$\sum_{|n| \leqq R} \left| a^{-1} \sin \tfrac{1}{2} na \right|^2 \left| \hat{f}(n) \right|^2 \leqq A_{10}.$$

Letting a tend to zero,

$$\sum_{|n| \leqq R} n^2 \left| \hat{f}(n) \right|^2 \leqq 4 A_{10}.$$

Since A_{10} is independent of R, (8) follows.

THEOREM 2. *Let E be a subset of Z.*

(i) *Suppose that E is of type $\Lambda(2)$ and that $1 \leqq p \leqq 2$. Then $\Lambda^p_{\alpha\, E} = \Lambda^2_{\alpha\, E}$, so that $f \in \Lambda^p_{\alpha\, E}$ if and only if $f \in L^1_E$ and (7) or (8) holds according as $0 < \alpha < 1$ or $\alpha = 1$.*

(ii) *Suppose that E is of type $\Lambda(p)$ and that $2 < p < \infty$. Then $\Lambda^p_{\alpha, E} = \Lambda^2_{\alpha\, E}$, so that $f \in \Lambda^p_{\alpha\, E}$ if and only if $f \in L^1_E$ and (7) or (8) holds according as $0 < \alpha < 1$ or $\alpha = 1$.*

PROOF. It will suffice to prove (i), the proof of (2) being very similar. Also, in view of the lemma, it suffices to show that $\Lambda^p_{\alpha\, E} = \Lambda^2_{\alpha\, E}$. The inclusion $\Lambda^2_{\alpha\, E} \subseteq \Lambda^p_{\alpha\, E}$ is trivial, since $p \leqq 2$. The reverse inclusion holds because E is of type $\Lambda(2)$ and $p \leqq 2$, so that $L^2_E = L^p_E$ and the norms $\| \cdot \|_2$ and $\| \cdot \|_p$ are equivalent on L^2_E ([1], Section 15.5).

COROLLARY. *If $E \subseteq Z$ is of type $\Lambda(q)$ for every $q > 0$ (in particular, if E is Sidon), then $\Lambda^p_{\alpha, E} = \Lambda^2_{\alpha, E}$ for $1 \leqq p < \infty$ and $0 < \alpha \leqq 1$.*

In view of this corollary, it seems natural to consider conditions under which every $f \in \Lambda^2_{\alpha, E}$ is equal a.e. to an element of Λ_α.

THEOREM 3. *Let $0 < \alpha < 1$ and let E be a subset of Z such that*

$$B = \sup_{R > 0} \operatorname{card} \{ n \in E : R \leqq |n| < 2R \} < \infty. \tag{9}$$

If $f \in \Lambda^2_{\alpha, E}$, then f is equal a.e. to a function in $\Lambda_{\alpha, E}$.

PROOF. By the lemma

$$\sum_{|n| \geqq R} |\hat{f}(n)|^2 \leqq A_{11}^2 R^{-2\alpha}$$

for every $R > 0$, where A_{11} is independent of R. Hence, by (9) and the Cauchy-Schwarz inequality

$$\begin{aligned} \sum_{R \leqq |n| < 2R} \left| \hat{f}(n) \right| &= \sum_{R \leqq |n| < 2R, n \in E} \left| \hat{f}(n) \right| \\ &\leqq B^{\frac{1}{2}} \Big(\sum_{R \leqq |n| < 2R} \left| \hat{f}(n) \right|^2 \Big)^{\frac{1}{2}} \\ &\leqq B^{\frac{1}{2}} A_{11} R^{-\alpha} \end{aligned}$$

Replacing R by $2^k R$ and summing over nonnegative integers k, it follows that

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$$\sum_{|n| \geqq R} |\hat{f}(n)| \leqq (1-2^{-\alpha})^{-1} B^{\frac{1}{2}} A_{11} R^{-\alpha}. \tag{10}$$

Let $g \in C_E$ be the function

$$x \mapsto \sum_{n \in Z} \hat{f}(n) e^{inx}$$

Then $\hat{g}(n) = \hat{f}(n)$ for every $n \in Z$, so that $f = g$ a.e.; and (10) shows (see Remark (i) following Theorem 1) that $g \in \Lambda_\alpha$. Hence $g \in \Lambda_{\alpha,E}$ and the proof is complete.

When $\alpha = 1$, the situation is different, as the following theorem shows.

THEOREM 4. *Let E be an infinite Sidon subset of Z. There exist functions f which belong to $\Lambda^p_{E,1}$ for every $p \in [1, \infty)$ and which are equal a.e. to no function in Λ_1.*

PROOF. Enumerate E as $n_0, n_1, n_2, \cdots$, where $n_k \neq 0$ for $k \in \{1, 2, \cdots\}$. Take complex numbers c_k such that

$$\sum_{k=1}^{\infty} |c_k|^2 < \infty \tag{11}$$

and

$$\sum_{k=1}^{\infty} |c_k| = \infty. \tag{12}$$

Let f be the L^2-sum of the series

$$\sum_{k=1}^{\infty} c_k n_k^{-1} \exp(in_k x).$$

By (11) and the lemma, $f \in \Lambda^2_{E,1}$. By Theorem 2 and the fact ([1], 15.5.3) that E is of type $\Lambda(p)$ for every $p < \infty$, f belongs to $\Lambda^p_{E,1}$ for every $p \in [1, \infty)$. Were f to be equal a.e. to some $g \in \Lambda_1$, then $\hat{f}(n)$ would agree with $\hat{g}(n)$ for every $n \in Z$ and so Theorem 1 would entail that

$$\sum_{n \in Z} |n\hat{f}(n)| < \infty;$$

but this contradicts (12) and completes the proof.

REMARK. An infinite Sidon subset E of Z may or may not satisfy (9); see [1], Exercise 15.3.

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