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THE METRIC FUGLEDE PROPERTY AND NORMALITY

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1. Introduction. In [**4**], H. Kamowitz considered the condition, to be satisfied by a bounded operator N on a Hilbert space $\mathscr{H}$, that

$$||NX - XN|| = ||N^*X - XN^*||$$

for all operators X on $\mathscr{H}$. Kamowitz discovered that such an N must be normal and its spectrum must lie on a line or a circle; that is, N must be of the form $\alpha J + \beta$, where α and β are complex numbers and J is either Hermitian or unitary. G. Weiss [**5**] showed that the Hilbert-Schmidt norm behaves differently: N need only be normal in order that

$$||N\ X - XN||_2 = ||N^*X - XN^*||_2$$

for all finite-rank operators X, and in fact this condition is equivalent to normality. Actually, the result in [**5**] removes the restriction that X be finite-rank, that is, if N is normal and X is any bounded operator, then

$$||NX - NX||_2 = ||N^*X - XN^*||_2.$$

It is understood here that if one side of the equation is infinite, so is the other; in particular, $NX - XN$ is a Hilbert-Schmidt operator if and only if $N^*N - XN^*$ is.

A close look at both the Kamowitz and Weiss papers reveals that, for the most part, it is operators X of rank two that carry the proof; other ranks are either unnecessary or can be included easily once the rank-two results are known. One might suspect, therefore, that the facts would be different provided that N is assumed to satisfy the requirement

$$||NX - XN|| = ||N^*X - XN^*||$$

only for rank-one X. We will consider this condition for a large class of norms, the "uniform" norms defined by Gohberg and Krein [**2**]. The main result is that the condition is equivalent to normality. We will also show

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that the "Hilbert-Schmidt" result of [**5**] is unique in the following way: Suppose that $1 \leqq p \leqq \infty$ and

$$\|NX - XN\|_p = \|N^*X - XN^*\|_p \quad \text{for all } X \text{ of rank 1 or 2.}$$

If $p \neq 2$, then N must be normal with spectrum on a circle or a line, whereas if $p = 2$ [**1**] then N need only be normal. We remark that related results have been obtained by Furuta [**1**].

2. The metric Fuglede property. Our notation is as follows: $\mathscr{H}$ will denote a complex Hilbert space, $\mathscr{B}(\mathscr{H})$ the algebra of bounded linear operators on $\mathscr{H}$, and $\mathscr{F}(\mathscr{H})$ the ideal of finite rank operators in $\mathscr{B}(\mathscr{H})$. If f and g are vectors in $\mathscr{H}$, the symbol $f \otimes g$ denotes the rank-one operator defined by $f \otimes g\,(x) = (x, f)g$. Recall the following facts:

(a) $(\alpha f) \otimes (\beta g) = \bar{\alpha}\beta(f \otimes g)$;
(b) $(f \otimes g)^* = g \otimes f$;
(c) $\|f \otimes g\| = \|f\|\,\|g\|$;
(d) $(f \otimes g)(x \otimes y) = (y, f)(x \otimes g)$;
(e) for any operator A,

$$A\,(f \otimes g) = f \otimes Ag \quad \text{and} \quad (f \otimes g)\,A = (A^*f \otimes g).$$

Since we will be dealing with many norms on subsets of $\mathscr{B}(H)$, we will denote an arbitrary norm by $\|\cdot\|_0$ and reserve the symbol $\|\cdot\|$ for the usual operator norm. Of the many norms that can be defined on $\mathscr{F}(\mathscr{H})$ (and sets containing it), the interesting ones from a Hilbert space point of view are those that respect either the geometry of $\mathscr{H}$, or the algebraic structure of $\mathscr{B}(\mathscr{H})$. One useful minimal condition is that a norm $\|\ \|_0$ be a *cross norm*, that is, that

$$\|f \otimes g\|_0 = \|f\|\,\|g\|,$$

or, equivalently, that $\|\cdot\|$ and $\|\cdot\|_0$ agree on rank-one operators. As is indicated by Guichardet [**3**], the "interesting" cross norms are those that dominate the operator norm: $\|A\|_0 \geqq \|A\|$ whenever $\|A\|_0$ is defined. When we speak of a cross norm we shall assume the latter condition. If a cross-norm is defined on some set $\mathscr{J}$ such that whenever $A \in \mathscr{J}$, then $(A^*A)^{1/2} \in \mathscr{J}$ and $UA \in \mathscr{J}$ for all unitary U, and if

$$\|(A^*A)^{1/2}\|_0 = \|UA\|_0 = \|A\|_0,$$

then we refer to $\|\cdot\|_0$ as a *uniform norm* [**2**]. All of the Schatten p-norms are uniform norms.

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Suppose that $\|\cdot\|_0$ is defined on (at least) $\mathscr{F}(\mathscr{H})$ and let k be an integer. We will say that an operator N has the *rank-k metric Fuglede property with respect to* $\|\cdot\|_0$ if

$$\|NX - XN\|_0 = \|N^*X - XN^*\|_0$$

for all X of rank less than or equal to k. We denote the class of all such operators N by $MF(k, \|\cdot\|_0)$, or simply by $MF(k)$ if the norm is fixed by context; note that

$$MF(k, \|\cdot\|_0) \supseteq MF(k + 1, \|\cdot\|_0).$$

The name "Fuglede" is invoked because of the Fuglede theorem concerning normal operators: if N is normal and $NX = XN$, then $N^*X = XN^*$. One might say that the oridinary Fuglede property (equivalent to normality) is that $\|NX - XN\|$ and $\|N^*X - XN^*\|$ are equal for any X, provided one of them is zero. The metric Fuglede property is in one way stronger ($\|NX - XN\|_0$ need not be zero) and in another way weaker (the rank of X is restricted) than the ordinary Fuglede property. The results of [**4**] and [**5**] say that $MF(k, \|\cdot\|_2)$ is the set of normal operators, regardless of k; and $MF(2, \|\cdot\|) = \{\alpha J + \beta$: α, β are complex and J is either Hermitian or unitary$\}$. We now proceed to investigate $MF(k, \|\cdot\|_0)$ more generally. One easy fact is the following:

PROPOSITION 1. *Let $\|\cdot\|_0$ be a uniform norm. If $N = \alpha J + \beta$ for some Hermitian or unitary J, then*

$$\|NX - XN\|_0 = \|N^*X - XN^*\|_0$$

for any X for which either side of the equality is defined; in particular

$$N \in MF(k, \|\cdot\|_0) \quad \textit{for all } k.$$

Proof. If J is unitary,

$$\|JX - XJ\|_0 = \|J^*(JX - XJ)J^*\|_0 = \|XJ^* - J^*X\|_0.$$

Everything else is obvious.

3. The case $k = 1$. In this section we show that when $\|\cdot\|_0$ is a uniform norm, $MF(1, \|\cdot\|_0)$ is precisely the set of normal operators. Actually, the proof that $MF(1, \|\cdot\|_0)$ is a subset of the normal operators requires only that $\|\cdot\|_0$ be a cross norm.

THEOREM 2. *Let $N \in MF(1, \|\cdot\|_0)$, with $\|\cdot\|_0$ a cross-norm. Then N is normal.*

Proof. Choose a sequence of unit vectors $\{f_n\}$ such that $||Nf_n|| \to 0$. (Here $0 \in \partial\, \sigma\,(N)$ is assumed.) For any vector g, we have

$$||N^*\,(f_n \otimes g) - (f_n \otimes g)N^*||_0 = ||f_n \otimes N^*g - Nf_n \otimes g||_0$$

which tends to $||N^*g||$ as $n \to \infty$. On the other hand

$$||N(f_n \otimes g) - (f_n \otimes g)\,N||_0$$
$$\geqq ||\,(N\,(f_n \otimes g) - (f_n \otimes g)N)\,f_n|| = ||Ng - (f_n \otimes g)\,NF_n||$$

which tends to $||Ng||$ as $n \to \infty$. Hence $||Ng|| \leqq ||N^*g||$. By reversing the role of N and N^*, we obtain $||N^*g|| \leqq ||Ng||$.

$$||f_n \otimes Ng - N^*f_n \otimes g||_0 = ||f_n \otimes N^*g - Nf_n \otimes g||_0.$$

Let ϵ be a positive number and choose n so that $||Nf_n||$ and $||N^*f_n||$ are both less than ϵ. Several applications of the triangle inequality show that

$$|\;||f_n \otimes Ng||_0 - ||f_n \otimes N^*g||_0\;| < 2\epsilon\;||g||,$$

which is the same as

$$|\;||Ng|| - ||N^*g||\;| < 2\epsilon\;||g||.$$

The number ϵ can be as small as desired, and we are done.

To prove the converse of Theorem 3 we introduce the further restriction that $||\cdot||_0$ be a uniform norm. We require a technical lemma.

LEMMA 3. *Let f, e, h be three non-zero vectors such that neither e nor h is a scalar multiple of f. The following are equivalent:*

(a) *There exists a unitary operator U such that $Uf = f$ and $Ue = h$;*
(b) $||e|| = ||h||$ *and* $(e, f) = (h, f)$.

Proof. To see that (a) implies (b), observe that $U^*h = e$ and thus

$$(f, e) = (f, U^*h) = (Uf, h) = (f, h).$$

The other implication is only a trifle harder. On the orthogonal complement of $\{f, e, h\}$ we let U be the identity. By the Gram-Schmidt process we obtain an orthonormal set $\{e_1, e_2, e_3\}$ such that

$$f = \alpha e_1$$
$$e = \beta e_1 + \gamma\, e_2$$
$$h = \beta e_1 + \delta\, e_2 + \epsilon e_3.$$

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Because $(e, f) = (h, f)$, the "e_1" components of e and h are the same, and because $||e|| = ||h||$ we know that $|\gamma|^2 = |\delta|^2 + |\epsilon|^2$. It is easy to check that the operator

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta/\gamma & -\bar{\epsilon}/\bar{\gamma} \\ 0 & \epsilon/\gamma & \bar{\delta}/\bar{\gamma} \end{pmatrix} \oplus I$$

satisfies all the requirements.

THEOREM 4. *Let* $||\cdot||_0$ *be a uniform norm and suppose that* N *is normal. Then* $||NX - XN||_0 = ||N^*X - XN^*||_0$ *for all rank-one* X. *Consequently,* $MF(1, ||\cdot||_0)$ *is the set of normal operators.*

Proof. Choose appropriate scalars α, β such that, if we put $M = \alpha N + \beta$, then $(Mf, f) = 0$ and (Mg, g) is real. By Lemma 3, there exist unitary operators W and V such that $Wf = f$, $WMf = M^*f$, $Vg = g$, $VM^*g = Mg$. Thus

$$\begin{aligned} V(M^*(f \otimes g) - (f \otimes g)M^*)W^* &= Wf \otimes VM^*g - WMf \otimes Vg \\ &= f \otimes Mg - M^*f \otimes g = M(f \otimes g) - (f \otimes g)M. \end{aligned}$$

Question. Is Theorem 4 true if the words "uniform norm" are replaced by "cross norm"?

4. The case $k = 2$. In this section we restrict attention to Schatten p-norms and X of rank two. Recall that if K is any compact operator, the eigenvalues of $(K^*K)^{1/2}$ are called the s-numbers of K. Index the s-numbers to form a decreasing sequence: $s_1(K) \geqq s_2(K) \geqq \dots$. The subset $\mathcal{C}_p$ consists of all those compact operators whose s-number sequence lies in the sequence space l^p, $1 \leqq p < \infty$. The sets $\mathcal{C}_p$ are ideals of operators, and a uniform norm $||\cdot||_p$ is defined on each $\mathcal{C}_p$ class:

$$||K||_p = \left(\sum_{j=1}^{\infty} s_j^p(K)\right)^{1/p}.$$

It is sometimes illuminating to think of the set of all compact operators as $\mathcal{C}_\infty$ (with s-numbers tending to 0) with operator norm playing the part of $||\cdot||_\infty$.

In the case of rank-two X, the results quoted in Section 1 are that $MF(2, ||\cdot||_2)$ is the set of all normal operators [**5**], while $MF(2, ||\cdot||)$ is the set of normal operators with spectrum contained in a line or a circle [**4**]. In many respects, $||\cdot||_2$ behaves differently from other p-norms, and that is the situation here. Indeed, when $p \neq 2$, $MF(2, \|\cdot\|_p)$ is the subset of normal operators with spectrum on a line or circle. The easiest way to prove this result is to consider first the situation on a Hilbert space of low dimension. If the dimension is three or less, it is obvious that every normal operator has specrum on a line or circle, so only the case $\dim \mathscr{H} \geqq 4$ is relevant.

LEMMA 5. *Let $\mathscr{H}$ be a four-dimensional Hilbert space.*
(a) *$MF(2, \|\cdot\|_2)$ is the set of normal operators.*
(b) *If $1 \leqq p < \infty$, $p \neq 2$, then $MF(2, \|\cdot\|_p)$ is the set of normal operators whose spectrum lies on a line or a circle.*

The proof of Lemma 5 makes use of the next fact, which is surprising only in that the case $p = 2$ is an exception. The proof shows "why" the exception occurs.

LEMMA 6. *Let a and c be non-negative real numbers and let b be complex, with $ac - |b|^2 > 0$. Suppose that $1 \leqq p < \infty$, $p \neq 2$, and that*

$$\left\|\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}^{1/2}\right\|_p = \left\|\begin{pmatrix} a & 0 \\ 0 & c \end{pmatrix}^{1/2}\right\|_p.$$

Then $b = 0$.

Proof. The p-norm of a 2×2 positive matrix with eigenvalues d_1, d_2 is $(d_1^p + d_2^p)^{1/p}$. Consequently the equality of the norms of the square roots of the matrices above yields the equation

$$(1/2\,(a + c + \sqrt{(a - c)^2 + 4\,|\,b\,|^2}\,))^{p/2} + (1/2(a + c - \sqrt{(a - c)^2 + 4\,|\,b\,|^2}\,))^{p/2} = a^{p/2} + c^{p/2}.$$

It does no harm to assume that $a \geqq c$, and we can then define a positive number ϵ so that

$$\sqrt{(a - c)^2 + 4\,|\,b\,|^2} = (a - c) + 2\epsilon.$$

Since

$$(a - c)^2 + 4\,|\,b\,|^2 = (a + c)^2 - 4\,(ac - |\,b\,|^2),$$

we know that $(a - c) + 2\epsilon \leqq a + c$ and thus $\epsilon \leqq c$. The equation above becomes

$$(a + \epsilon)^{p/2} + (c - \epsilon)^{p/2} = a^{p/2} + c^{p/2}.$$

Consider the function

$$f(x) = (a + x)^{p/2} + (c - x)^{p/2} - a^{p/2} - c^{p/2}, \quad 0 \leqq x \leqq c.$$

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We have $f(0) = 0$ and

$$f'(x) = (p/2)\,[(a + x)^{(p/2)-1} - (c - x)^{(p/2)-1}].$$

If $p > 2$ then f is strictly increasing (recall that $a \geqq c$); if $p < 2$, f is strictly decreasing. In either case, $f(x) = 0$ has only one solution, namely $x = 0$, which means that $\epsilon = 0$.

It is surprising to note that not only does the proof fail when $p = 2$, but every positive matrix provides a counterexample (look at the first equation in the proof).

Proof of Lemma 5. Part (a) is the result in [**5**], so suppose that $p \neq 2$. In view of Proposition 1 we need only prove that if $N \in MF(2, \|\cdot\|_p)$, then $\sigma(N)$ lies on a line or a circle. N is normal by Theorem 2, and we can choose a basis in which N is diagonal, with eigenvalues λ_1, λ_2, λ_3, and λ_4. Any three points lie on a line or circle, so we assume that the four eigenvalues are distinct. Let

$$X = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where r will be chosen later. As before, let $T = NX - XN$ and $\bar{T} = N^*X - XN^*$. Then

$$T = \begin{pmatrix} 0 & 0 & \lambda_1 - \lambda_3 & \lambda_1 - \lambda_4 \\ 0 & 0 & \lambda_2 - \lambda_3 & r(\lambda_2 - \lambda_4) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$T^*T = \begin{pmatrix} |\lambda_1 - \lambda_3|^2 + |\lambda_2 - \lambda_3|^2 & (\bar{\lambda}_1 - \bar{\lambda}_3)(\lambda_1 - \lambda_4) + r(\bar{\lambda}_2 - \bar{\lambda}_3)(\lambda_2 - \lambda_4) & 0 & 0 \\ (\bar{\lambda}_1 - \bar{\lambda}_4)(\lambda_1 - \lambda_3) + \bar{r}(\bar{\lambda}_2 - \bar{\lambda}_4)(\lambda_2 - \lambda_3) & |\lambda_1 - \lambda_4|^2 + |r|^2 |\lambda_2 - \lambda_4|^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

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For any $\mathscr{C}_p$ norm, the norm of $(T^*T)^{1/2}$ is the same as the norm of the square root of the 2×2 upper left corner. Now choose r so that the off-diagonal entries are 0, that is,

$$r = -\frac{(\bar{\lambda}_1 - \bar{\lambda}_3)(\lambda_1 - \lambda_4)}{(\bar{\lambda}_2 - \bar{\lambda}_3)(\lambda_2 - \lambda_4)}.$$

With this choice,

$$\|T\|_p^p = (|\lambda_1 - \lambda_3|^2 + |\lambda_2 - \lambda_3|^2)^{p/2} + (|\lambda_1 - \lambda_4|^2 + |r|^2|\lambda_2 - \lambda_4|^2)^{p/2}.$$

Observe that $\bar{T}^*\bar{T}$ has the same form as T^*T, with λ_j and $\bar{\lambda}_j$ interchanged; thus the diagonal entries of $\bar{T}^*\bar{T}$ and T^*T are the same. An application of Lemma 6 shows that the off-diagonal terms of $\bar{T}^*\bar{T}$ must also be zero, that is,

$$r = -\frac{(\lambda_1 - \lambda_3)(\bar{\lambda}_1 - \bar{\lambda}_4)}{(\lambda_2 - \lambda_3)(\bar{\lambda}_2 - \bar{\lambda}_4)}.$$

Thus the expression $(\lambda_1 - \lambda_3)(\bar{\lambda}_1 - \bar{\lambda}_4)(\lambda_2 - \lambda_3)(\bar{\lambda}_2 - \bar{\lambda}_4)$ must be real. But this quantity is a real multiple of the cross-ratio $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and it follows that the four numbers $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ lie either on a line or a circle.

THEOREM 7. *Let $\mathscr{H}$ be an infinite-dimensional Hilbert space. If $1 \leqq p < \infty$, $p \neq 2$, then $MF(2, \|\cdot\|_p)$ is the set of normal operators whose spectrum lies on a line or a circle, whereas $MF(2, \|\cdot\|_2)$ is the set of all normal operators.*

Proof. The second statement is the result of [**5**]. Let $N \in MF(2, \|\cdot\|_p)$ for $p \neq 2$. By Theorem 2 N is normal, and by the spectral theorem N can be approximated in operator norm by diagonal operators. (Use the version that pictures N as a multiplication operator.) Let ϵ be a positive number and suppose that $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ lie in the spectrum of N. By a judicious choice of the diagonal operator D we can ensure both that $\lambda_j \in \sigma(D)$ for $j = 1, 2, 3, 4$ and that $\|N - D\| < \epsilon$. Let f_1, f_2, f_3, f_4 be orthonormal vectors such that $Df_j = \lambda_j f_j$ for $j = 1, 2, 3, 4$. Let r be defined as in Lemma 7 and let X be the rank-two operator defined by the equations

$$Xf_3 = f_1 + f_2, \quad Xf_4 = f_1 + rf_2, \quad \text{and} \quad Xf = 0 \text{ if } f \perp \{f_3, f_4\}.$$

This X is unitarily equivalent to the direct sum of the matrix X of Lemma 5 with the zero operator. We have

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$$\begin{aligned}
&\|DX - XD\|_p - \|D^*X - XD^*\|_p \\
&\leqq \|NX - XN\|_p + \|(N - D)X - X(N - D)\|_p \\
&- (\|N^*X - XN^*\|_p - \|(N^* - D^*)X - X(N^* - D^*)\|_p) \\
&= \|(N - D)X - X(N - D)\|_p \\
&+ \|(N^* - D^*)X - X(N^* - D^*)\|_p \\
&\leqq 4\epsilon \|X\|.
\end{aligned}$$

Similarly we obtain

$$\|D^*X - XD^*\|_p - \|DX - XD\|_p \leqq 4\epsilon \|X\|,$$

and thus

$$| \|DX - XD\|_p - \|D^*X - XD^*\|_p | \leqq 4\epsilon \|X\|.$$

Let T and $\bar{T}$ be the 4×4 matrices in the proof of Lemma 7, written in some fixed basis $\{e_1, e_2, e_3, e_4\}$. Observe that as ϵ varies, the vectors f_1, f_2, f_3, f_4, must vary, and hence so do X and D. However, regardless of the choice of ϵ we always have that $DX - XD$ is unitarily equivalent to $T \oplus 0$, and $D^*X - XD^*$ is equivalent to $\bar{T} \oplus 0$, where the "0" acts on the space $\{e_1, e_2, e_3, e_4,\}^\perp$. Thus, for all positive ϵ we have

$$| \|T\|_p - \|\bar{T}\|_p | \leqq 4\epsilon \|X\|$$

and since $\|X\|$ does not depend on ϵ, $\|T\|_p = \|\bar{T}\|_p$. Lemma 5 now shows that the complex numbers $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ lie on a line or a circle. The proof is complete.

We close with a question. If N is normal, and X is compact, it is known that $\|XN - XN\|_1$ need not be the same as $\|N^*X - XN^*\|_1$. Is it possible for one norm to be finite while the other is infinite? To be precise, we ask the following:

Question. Does there exist a normal operator N and a compact operator X such that $NX - XN$ is trace class but $N^*X - XN^*$ is not?

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