ON THE CRITICAL PHENOMENA IN THE DYNAMICS OF ASTEROIDS

IVAN I. SHEVCHENKO

Institute of Theoretical Astronomy, Russian Academy of Sciences

Abstract. Two statistical effects in the long-term chaotic asteroidal dynamics are considered, namely the power-law character of the dependence of recurrence times on local Lyapunov times and the power-law decay in the tails of the recurrence distributions. The dependences in both cases are shaped by effects of anomalous transport, due to the presence of the chaos border in phase space, and by statistical selection effects.

In the chaotic asteroidal dynamics, two long-term effects in the statistics of sudden orbital changes are known. The first one consists in the power-law character of the dependence of times of sudden orbital changes on Lyapunov times (Soper *et al.*, 1990; Lecar *et al.* , 1992; Levison and Duncan, 1993; Murison *et al.*, 1994; Ferraz-Mello, 1997), while the second one in the power-law decay in the tails of distributions of such times (Shevchenko and Scholl, 1996; 1997). Both effects are considered here as critical phenomena. The critical motion, i.e. the chaotic motion in the vicinity of the chaos border, does not represent normal diffusive process. It is said that transport is anomalous (Chirikov, 1996). In the following note, particular effects of anomalous transport in the behaviour of the standard map are demonstrated and discussed. They correspond to the considered effects in the asteroidal dynamics and provide better understanding of the latter.

Let us associate the time of a sudden orbital change with a suitably defined recurrence time T_r. By "recurrences" we imply sequential returns of a trajectory to some arbitrary domain or surface in phase space. If phase space is divided, i.e. if chaotic and regular components are both present, longest recurrences of a chaotic trajectory are due to stickings to the chaos border. Sporadic stickings result in intermittent behaviour (Shevchenko, 1998a).

Consider the standard map

$$\begin{aligned} y_{n+1} &= y_n + \frac{K}{2\pi}\sin(2\pi x_n), \\ x_{n+1} &= x_n + y_{n+1}. \end{aligned} \tag{1}$$

Let us choose $K = 2$. In fact, the studied effects can be recovered for any non-zero K not too large, i.e. when the regular component is adequately present.

Integral distribution of recurrence times for a single chaotic orbit is shown in Fig. 1. The quantity $F(T_r)$ is the fraction of recurrences longer than T_r. The recurrences are counted at the line $y = 0 \bmod 1$. Stickings to the island of stability around the integer resonance situated at this line lead to the initial steep short-scale drop in the distribution. Then, on some limited interval, namely at $0.7 < \log T_r < 1.2$, the distribution follows the power law with index equal to -0.56. This is close to the inverse square root law, which is inherent to free diffusion

J. Henrard and S. Ferraz-Mello (eds.), Impact of Modern Dynamics in Astronomy, 383–386.

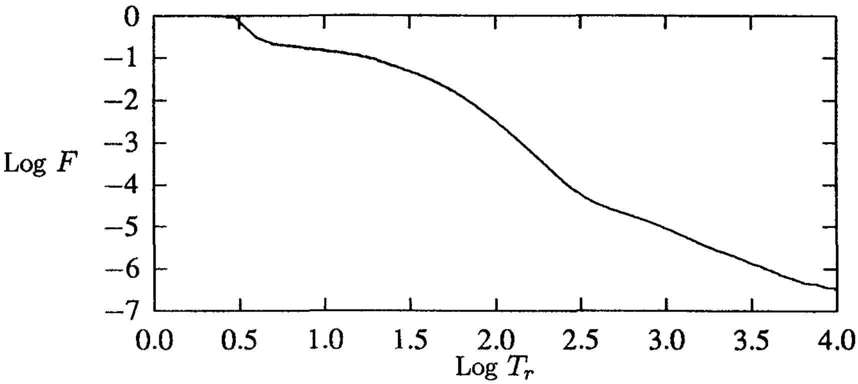

Fig. 1. Integral distribution of recurrence times. $K = 2$; number of iterations $N_{it} = 10^9$. Initial short-scale drop, major exponential decay and subsequent power-law decay are prominent. Logarithms are decimal in Figs. 1—2.

in the central part of a stochastic layer until finite width of the layer becomes important (Chirikov and Shepelyansky, 1981, p. 9). Note that in our case the borders $y = 0 \bmod 1$ of the "layer" are conventional. At $\log T_r = 1.2$–1.3 the dependence becomes exponential, because finite width of the layer starts to be important. Indeed, according to (Chirikov and Shepelyansky, 1981, p. 10), when the time of diffusion across a stochastic layer is finite, the distribution of recurrences decays exponentially due to fluctuations of diffusion. The tail of the distribution in Fig. 1 follows the power law with index $\alpha = -1.48$. Generally, in its over-all shape, the distribution is strikingly close to those presented in Fig. 4 of (Shevchenko and Scholl, 1997) for intervals between eccentricity bursts of model intermittent trajectories in the 3/1 Jovian resonance. According to (Shevchenko and Scholl, 1997), the values of power-law indices in the tails of the latter distributions are close to the value -1.5 theoretically considered and explained by Chirikov (1990, 1996) for critical motion. As our experiment with the standard map indicates, all prominent features of the distributions in the asteroidal case (initial steep short-scale drop, major exponential decay, and subsequent power-law decay) are present in case of the standard map, and have straightforward universal explanations.

In Fig. 2, the dependence "$\log T_L$ — $\log T_r$" is shown. The Lyapunov time T_L is the inverse of the LLCE (largest Lyapunov characteristic exponent). A formal mathematical approach requires the LLCE to be measured on an infinite time scale. In real computations this cannot be achieved. Henceforth the LLCE is measured for a recurrence (for detailed definitions and discussion, see Shevchenko, 1998b). As adopted already, the recurrences are counted at the line $y = 0 \bmod 1$. For convenience of handling large arrays of data, the field of Fig. 2 is partitioned in pixels. The figure represents a kind of a density plot. Each pixel contains no more than one symbol. Recurrences with T_L, T_r in the area covered with dots are relatively frequent. They take place already on the adopted minimum time span

https://doi.org/10.1017/S0252921100072821 Published online by Cambridge University Press

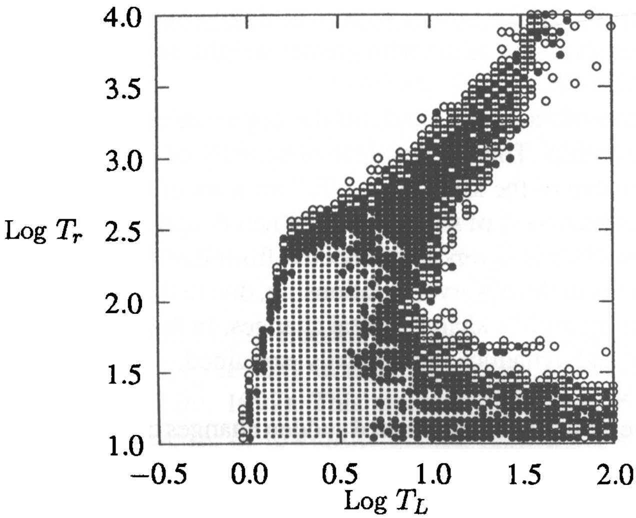

Fig. 2. Statistical dependence "$\log T_L$ — $\log T_r$". $K = 2$. Dots: $N_{it} = 10^6$; dots plus full circles: $N_{it} = 10^8$; dots plus full and open circles: $N_{it} = 10^9$.

of computation $N_{it} = 10^6$. Increasing N_{it} allows one to recover recurrences with less frequent values of the pair T_L, T_r. The area increments corresponding to the increase of N_{it} up to 10^8 and then up to 10^9 are covered respectively with full and open circles. Note that short recurrences (with $T_r < 10$) are not considered on the reason of large statistical fluctuations in evaluations of the LLCE on the short time scale. Such fluctuations are still clearly seen for recurrences with somewhat greater T_r in the form of jumps of T_L to high values.

One can see that recurrences with $\log T_r > 2.5$ are rare if $N_{it} < 10^6$, and the diffusion is normal: the mean $\log T_L$ does not depend on the duration T_r of a recurrence. At $\log T_r > 2.5$ the dependence is a power law, with index $\beta = 1.5$–2. One can graphically demonstrate, e.g. by means of construction of a spectrum of winding numbers (Shevchenko, 1996), that the recurrences with $\log T_r > 2.5$ are due to stickings to the chaos border. The winding number is formally defined for a recurrence. Indeed, at $\log T_r > 2.5$ the broad spectrum degenerates into a sharp peak corresponding to the half-integer resonance.

Theoretical dependence "T_L — T_r" for critical motion was derived in (Shevchenko, 1998b). In particular, on the condition that the LLCE were computed on finite time intervals (corresponding to the recurrences), the dependence was shown to be close to quadratic: $T_r \propto {T_L}^2$. The quadratic relationship is set forth by the fact that the transport near the chaos border is anomalous and by the selection effect following from the limitation of the time of computation of the LLCE from above, since the LLCE correspond to recurrences.

Another important selection effect distorts statistical evaluations of the exponent β of the "T_L — T_r" relationship. The distortion is due to a strong statistical predominance of short recurrence times. Therefore, when calculating mean observed values of β, longer recurrences should be taken with greater weight; see discussion in (Shevchenko, 1998b).

One more selection effect is often present and, on the opposite, enhances appearance of the generic relationship. This is the effect of sparsity of the data set. In order to construct a relationship of the kind "T_L — T_r" for a set of trajectories, one should choose a corresponding array of starting data. When the grid of the data is fine, the presence of narrow chaotic layers disconnected from the main chaotic domain would lead to distortions in the observed relationship, due to the apparition of the chaotic orbits which never exhibit "sharp" orbital changes. In this way, cases of "stable chaos" in the asteroidal dynamics are naturally explained.

Conclusions are as follows.

(1) Two known long-term effects in the statistics of sudden changes of asteroidal orbits, namely the power-law character of the dependence of times of sudden orbital changes on Lyapunov times (Soper *et al.*, 1990; Lecar *et al.* , 1992; Levison and Duncan, 1993; Murison *et al.*, 1994; Ferraz-Mello, 1997) and the power-law decay in the tails of distributions of such times (Shevchenko and Scholl, 1996; 1997), are both plausibly explained as critical phenomena, i.e. effects of anomalous transport near the chaos border. Our experiments with the standard map unambiguously recover similar dependences. (2) The "T_L — T_r" relationship can indeed be used to statistically predict sudden changes in the orbital behaviour of asteroids, if the initial part of the power-law dependence is recovered numerically. (3) When interpreting the observed dependences, it is necessary to take into account selection effects. The main selection effects, in case of the "T_L — T_r" relationship, are: limitations on the time of computation of the LLCE, concentration of data points to the lower time edge, sparsity of statistical data.

It is a pleasure to thank Sylvio Ferraz-Mello for useful remarks on the manuscript.

References

Chirikov, B.V.: 1979, *Phys. Reports*, **52**, 263.

Chirikov, B.V.: 1990, *Patterns in Chaos*, Preprint 90–109, Inst. of Nucl. Phys. Novosibirsk: Novosibirsk; see also: 1991, *Chaos, Solitons and Fractals*, **1**, 79.

Chirikov, B.V.: 1996, *JETP*, **83**, 646 [*Zh. Eksp. Teor. Fiz.*, **110**, 1174].

Chirikov, B.V. and Shepelyansky, D.L.: 1981, *Statistics of Poincaré recurrences and the structure of the stochastic layer of the non-linear resonance*, Preprint 81–69, Inst. of Nucl. Phys. Novosibirsk.

Ferraz-Mello, S.: 1997, *Celest. Mech. & Dyn. Astron.*, **65**, 421.

Lecar, M., Franklin, F. and Murison, M.: 1992, *Astron. J.*, **104**, 1230.

Levison, H. F. and Duncan, M. J.: 1993, *Astrophys. J. Lett.*, **406**, L35.

Murison, M., Lecar, M. and Franklin, F.: 1994, *Astron. J.*, **108**, 2323.

Shevchenko, I. I.: 1996, in J. C. Muzzio et al. (eds.), *Chaos in Gravitational N-Body Systems*, Kluwer: Dordrecht, p. 311.

Shevchenko, I. I.: 1998a, *Physica Scripta*, **57**, 185.

Shevchenko, I. I.: 1998b, *Phys. Letters A*, **241**, 53.

Shevchenko, I. I. and Scholl, H.: 1996, in S. Ferraz-Mello et al. (eds.), *Dynamics, Ephemerides and Astrometry of the Solar System*. Kluwer: Dordrecht, p. 183.

Shevchenko, I. I. and Scholl, H.: 1997, *Celest. Mech. & Dyn. Astron.*, **68**, 163.

Soper, M., Franclin, F. and Lecar, M.: 1990, *Icarus*, **87**, 265.