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APPROXIMATION OF FUNCTIONS BY POLYNOMIALS IN $C[-1,1]$

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ABSTRACT. A pointwise estimate for the rate of approximation by polynomials,

$$|f(x) - P_n(x)| \le C(r,\lambda)\omega^r_{\varphi^\lambda}(f, n^{-1}\delta_n(x)^{1-\lambda}),$$

for $0 \le \lambda \le 1$, integer r, $\varphi = \sqrt{1-x^2}$ and $\delta_n(x) = n^{-1} + \varphi(x)$, is achieved here. This formula bridges the gap between the classical estimate mentioned in most texts on approximation and obtained by Timan and others ($\lambda = 0$) and the recently developed estimate by Totik and first author ($\lambda = 1$). Furthermore, a matching converse result and estimates on derivatives of the approximating polynomials and their rate of approximation are derived. These results also cover the range between the classical pointwise results and the modern norm estimates for $C[-1,1]$.

1. **Introduction.** Polynomial approximation of functions in $C[-1,1]$ has been investigated extensively. The direct result was proved by Timan, Dzjadyk, Freud and Brudnyi, (see [5] and [7]), each extending the previous work. They proved that for every function $f \in C[-1,1]$ there exists a sequence of polynomials of degree n, P_n, satisfying

$$(1.1)\qquad |f(x) - P_n(x)| \le C(r)\omega^r\big(f, n^{-1}(n^{-1} + \sqrt{1-x^2})\big) \equiv C(r)\omega^r\big(f, n^{-1}\delta_n(x)\big)$$

where

$$(1.2)\qquad \omega^r(f,t) = \operatorname*{Sup}_{0<h\le t} \|\Delta^r_h f\|_{C[-1,1]}$$

and

$$(1.3)\qquad \Delta^r_h f(x) = \begin{cases} \sum_{k=0}^r (-1)^k \binom{r}{k} f\big(x + (k - \frac{r}{2})h\big) & \text{if } |x \pm \frac{rh}{2}| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Recently, it was shown [4, Chapter 7] that there exists a sequence of polynomials of degree n (the best polynomial approximants to f in $C[-1,1]$ would do) that satisfy

$$(1.4)\qquad \|f - P_n\|_{C[-1,1]} \le C(r)\omega^r_\varphi(f, n^{-1})$$

where

$$(1.5)\qquad \omega^r_\varphi(f,t) = \operatorname*{Sup}_{|h|\le t}\big(\|\Delta^r_{h\varphi} f\|_{C[-1,1]};\ \varphi(x) = \sqrt{1-x^2}\big).$$

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We will show that for every $f \in C[-1,1]$ and $0 \leq \lambda \leq 1$ there exists a sequence of polynomials of degree n satisfying

$$(1.6) \qquad |f(x) - P_n(x)| \leq C(r,\lambda)\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$ and

$$(1.7) \qquad \omega^r_{\varphi^\lambda}(f,t) = \sup_{|h|\leq t}\big(\|\Delta^r_{h\varphi^\lambda}f\|_{C[-1,1]}; \varphi(x) = \sqrt{1-x^2}\big).$$

The inequality (1.6) actually coincides with (1.1) when $\lambda = 0$ and with (1.4) when $\lambda = 1$.

Converse inequalities for both (1.1) and (1.4) of the weak-type variety were proved. We will show that the existence of a sequence of polynomials $P_n(x)$ satisfying

$$(1.8) \qquad |f(x) - P_n(x)| \leq C\Psi\big(n^{-1}\delta_n(x)^{1-\lambda}\big),$$

with $\Psi(t)$ an increasing function satisfying

$$(1.9) \qquad \Psi(\mu t) \leq C(\mu^r + 1)\Psi(t) \qquad \mu,\, t > 0$$

where C is independent of μ and t, implies

$$(1.10) \qquad \omega^r_{\varphi^\lambda}(f,t) \leq C(r,\lambda)t^r \sum_{0<n\leq t^{-1}} (n+1)^{r-1}\Psi(n^{-1}).$$

This result includes results of Timan and others for $\lambda = 0$ and a result of Totik and the first author for $\lambda = 1$. One has to emphasize that while results by Ditzian and Totik in [4] apply to the case $\lambda = 1$ for $1 \leq p \leq \infty$, we have no hope of proving a general result (*i.e.* for $p \neq \infty$ and $0 \leq \lambda < 1$), as in the case $\lambda = 0$ such a result was proved to be impossible (see [6] and [1]).

Results about the approximation of the derivatives and an estimate of the derivatives of the approximating polynomials will be given in Section 6.

2. **The direct estimate.** The direct result already stated in the introduction as (1.6) will be stated and proved here utilizing a crucial lemma which will be proved in the next section.

THEOREM 2.1. *For $0 \leq \lambda \leq 1$, an integer r, and any $f \in C[-1,1]$ we have a sequence of polynomials $P_n(x)$ of degree n satisfying*

$$(2.1) \qquad |f(x) - P_n(x)| \leq C(r,\lambda)\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$.

For the proof we will need the following lemma.

LEMMA 2.2. *For $0 \le \lambda \le 1$, an integer r and a function $f \in C[-1,1]$ we can construct a function $G_n(x)$ such that*

$$|f(x) - G_n(x)| \le M\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big), \tag{2.2}$$

and

$$\delta_n(x)^r |G_n^{(r)}(x)| \le Mn^r\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big), \tag{2.3}$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$.

PROOF OF THEOREM 2.1. Using Lemma 2.2, we approximate $f - G_n$ by the polynomial 0. We now follow [4, Chapter 7] to approximate G_n. We first approximate $G_n^{(r-1)}$ by $P_{n,r-1}$. We need a trigonometric polynomial kernel T_n which satisfies

$$\begin{gathered} T_n(x) \ge 0, \int_{-\pi}^{\pi} T_n(x)\,dx = 1 \text{ and} \\ \int_{-\pi}^{\pi} |x|^{2r+3} T_n(x)\,dx \le Ln^{-2r-3}. \end{gathered} \tag{2.4}$$

(Such polynomials are constructed in [5, p. 57]). We now define $P_{n,r-1}$ by

$$P_{n,r-1}(x) \equiv \int_{-\pi}^{\pi} G_n^{(r-1)}\Big(\cos\big((\arccos x) - t\big)\Big) T_n(t)\,dt.$$

Denoting

$$\Delta_{\frac{1}{t}}(x) = |t|(|t| + \sqrt{1-x^2}) \text{ and } \delta_n(x) = n\Delta_n(x), \tag{2.5}$$

we now have

$$\begin{aligned} I_n(x) &\equiv \Big|\delta_n(x)^{r-1}\Big(G_n^{(r-1)}(x) - P_{n,r-1}(x)\Big)\Big| \\ &= \left|\delta_n(x)^{r-1}\int_{-\pi}^{\pi}\left[\int_{\cos((\arccos x)-t)}^{x} G_n^{(r)}(u)\,du\right]T_n(t)\,dt\right|. \end{aligned}$$

Using straightforward computation, we have

$$\Delta_{\frac{1}{t}}(x) \ge |x(1-\cos t) \mp \sqrt{1-x^2}\sin t| = \Big|x - \cos\big((\arccos x) \pm t\big)\Big|$$

(see [4, p. 80]), and hence

$$I_n(x) \le \int_{-\pi}^{\pi} \frac{1}{|x - \cos((\arccos x) - t)|}\left|\int_{\cos((\arccos x)-t)}^{x} \delta_n(x)^{r-1}\Delta_{\frac{1}{t}}(x) G_n^{(r)}(u)\,du\right| T_n(t)\,dt.$$

We recall (7.2.5) of [4], that is,

$$\delta_n(x) \le 9\max(n^2t^2, 1)\delta_n(u)$$

and

$$\Delta_{\frac{1}{t}}(x) \le 10|t|\max(n^2t^2, 1)\delta_n(u)$$

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for u between x and $\cos\big((\arccos x) - t\big)$, and the elementary estimate

$$\Delta_{\frac{1}{t}}(x) \le |t|(|t| + \sqrt{1-x^2}) \le |t|(n|t|+1)\delta_n(x).$$

These estimates now imply

$$I_n(x) \le \int_{-\pi}^{\pi} \frac{1}{|x-\cos((\arccos x)-t)|} \left| \int_{\cos(\arccos x)-t)}^{x} |t|(n|t|+1)\left(\frac{\delta_n(x)}{\delta_n(u)}\right)^r \right.$$
$$\left. \times |\delta_n(u)^r G_n^{(r)}(u)|\, du \right| T_n(t)\, dt.$$

Using Lemma 2.2 and

$$\omega^r_{\varphi^\lambda}(f,t) \sim K_{r,\varphi^\lambda}(f,t^r) \tag{2.6}$$

where

$$K_{r,\varphi^\lambda}(f,t^r) = \inf_{g\in C^r}(\|f-g\| + t^r\|\varphi^{\lambda r} g^{(r)}\|), \tag{2.7}$$

we have

$$\begin{aligned}
|\delta_n^r(u) G_n^{(r)}(u)| &\le M n^r \omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(u)^{1-\lambda}\big)\\
&\le M_1 n^r K_{r,\varphi^\lambda}\big(f, n^{-r}\delta_n(u)^{r(1-\lambda)}\big)\\
&\le M_1 n^r K_{r,\varphi^\lambda}\left(f, n^{-r}\left(\frac{\delta_n(u)}{\delta_n(x)}\right)^{r(1-\lambda)} \delta_n(x)^{r(1-\lambda)}\right)\\
&\le M_1 n^r \left(\left(\frac{\delta_n(u)}{\delta_n(x)}\right)^{r(1-\lambda)} + 1\right) K_{r,\varphi^\lambda}\big(f, n^{-r}\delta_n(x)^{r(1-\lambda)}\big)\\
&\le M_2 n^r \left(\left(\frac{\delta_n(u)}{\delta_n(x)}\right)^{r(1-\lambda)} + 1\right) \omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).
\end{aligned}$$

We can now write

$$\begin{aligned}
I_n(x) &\le M_2 n^r \omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big) \int_{-\pi}^{\pi} \frac{1}{|x-\cos((\arccos x)-t)|}\\
&\quad \times \left| \int_{\cos((\arccos x)-t)}^{x} |t|(n|t|+1)\left(\left(\frac{\delta_n(x)}{\delta_n(u)}\right)^{\lambda r} + \left(\frac{\delta_n(x)}{\delta_n(u)}\right)^{r}\right) du \right| T_n(t)\, dt\\
&\le M_3 n^r \omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big) \int_{-\pi}^{\pi} |t|(n|t|+1)(n^{2r}t^{2r}+1) T_n(t)\, dt\\
&\le M_4 n^{r-1} \omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).
\end{aligned}$$

We have in fact completed the proof of our theorem for the case $r = 1$. For $r > 1$ we continue the process dealing with $\delta_n(x)^{r-1}\big(G_n^{(r-1)}(x) - P_{n,r-1}(x)\big)$ instead of $\delta_n(x)^r G_n^{(r)}(x)$ and obtaining the estimate for $G_n^{(r-2)}(x)$ using

$$G_{n,r-1}(x) \equiv \int_0^x \big(G_n^{(r-1)}(u) - P_{n,r-1}(u)\big)\, du \equiv G_n^{(r-2)}(x) - \int_0^x P_{n,r-1}(u)\, du + C_n$$

and

$$\left|\delta_n(x)^{r-2}\big(G_{n,r-1}(x) - P_{n,r-2}(x)\big)\right| \le Mn^{r-2}\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

Repeating the process, we have

$$\begin{aligned}|G_n(x) - P_{n+r-1}(x)| &\le C\omega^r_{\varphi^\lambda}\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)\\ &\le C_1\omega^r_{\varphi^\lambda}\big(f, (n+r-1)^{-1}\delta_{n+r-1}(x)^{1-\lambda}\big).\end{aligned}$$

As the result is valid for all n, we have proved our theorem pending the proof of Lemma 2.2. ■

3. **Construction of the auxiliary function G_n.** In this section we will prove Lemma 2.2 using the technique employed in [2] and [4]. For the sake of completeness and as the situation here is simpler and somewhat different we will give the complete proof.

PROOF OF LEMMA 2.2. First we choose $\ell \equiv \ell(n)$ such that $4^{\ell-1} < n^2 \le 4^\ell$, and then we write

$$\begin{aligned}G_n(x) = \Psi_0(x)f_{\frac{1}{n}}(x) + \sum_{k=-\ell+1}^{\ell-1}\Psi_k(x)\big(1-\Psi_{k+\operatorname{sgn}k}(x)\big)f_{\tau_k}(x)\\ + \Psi_\ell(x)f_{\tau_\ell}(x) + \Psi_{-\ell}(x)f_{\tau_{-\ell}}(x)\end{aligned} \tag{3.1}$$

where

$$f_\tau(x) = r^r\int_0^{1/r}\cdots\int_0^{1/r}\sum_{j=1}^r(-1)^{j+1}\binom{r}{j}f\big(x+j\tau(u_1+\cdots+u_r)\big)\,du_1\cdots du_r, \tag{3.2}$$

$$\tau_k \equiv \tau_k(n) = n^{-1}2^{-|k|}\operatorname{sgn}k,$$

$$\Psi(x) \in C^\infty(R),\ \Psi(x) = 1 \text{ for } x \le 1,$$

$$0 \le \Psi(x) \le 1 \text{ for } 1 < x < 3 \text{ and } \Psi(x) = 0 \text{ for } x > 3$$

and

$$\Psi_k(x) = \begin{cases}\Psi\big(4^{-k}(1-x)\big) & k \le -1\\ \Psi\big(4^k(1+x)\big) & k \ge 1\\ 1 - \Psi_1(x) - \Psi_{-1}(x) & k = 0.\end{cases}$$

We will only estimate (2.2) and (2.3) for $x \in I_k \equiv [-1+4^{-k-1}, -1+4^{-k}]$ with $k = 1,\ldots,\ell-1$ and for $I^*_\ell \equiv [-1,-1+4^{-\ell}]$ as the estimates for $[3/4,1]$ are symmetric to that in $[-1,-3/4]$ and since the estimates in $[-3/4,3/4]$ are very simple. (We note that the first term of (3.1), *i.e.* $\Psi_0(x)f_{\frac{1}{n}}(x)$ spoils the symmetry of the expression $G_n(x)$ but this will not negate the above as $\operatorname{supp}\Psi_0(x) \subset [-3/4,3/4]$.) We further observe that $\operatorname{supp}\Psi_k(x)\big(1-\Psi_{k+1}(x)\big) \subset [-1+4^{-k-1}, -1+3\cdot4^{-k}]$ and $\big(\operatorname{supp}\Psi_\ell(x)\big)\cap[-1,1] = [-1,-1+3\cdot4^{-\ell}]$. Therefore, we can write for $x \in I_k = [-1+4^{-k-1},-1+4^{-k}]$ and $1 \le k \le \ell-1$,

$$|f(x) - G_n(x)| \le \max_{j=k,k+1}|f(x) - f_{\tau_j}(x)| \le \operatorname*{Sup}_{0<h<\tau_k}|\vec{\Delta}^r_h f(x)| \tag{3.3}$$

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where

$$\vec{\Delta}_h f(x) \equiv f(x+h) - f(x) \text{ and } \vec{\Delta}_h^r f(x) \equiv \vec{\Delta}_h\left(\vec{\Delta}_h^{r-1} f(x)\right).$$

Since for $x \in I_k$,

$$\operatorname*{Sup}_{0<h\leq\tau_k} h/\varphi\left(x+\frac{r}{2}h\right)^\lambda \leq \operatorname*{Sup}_{0<h\leq\tau_k} h/2^{-(k+1)\lambda} \leq 2^\lambda n^{-1} 2^{-k(1-\lambda)}$$

we have

$$\begin{aligned}(3.4)\qquad |f(x)-G_n(x)| &\leq \operatorname*{Sup}_{0<h\leq\tau_k}\left|\Delta_h^r f\left(x+\frac{r}{2}h\right)\right| \\ &\leq \operatorname*{Sup}_{\xi} \operatorname*{Sup}_{0<\eta\leq 2^\lambda n^{-1}2^{-k(1-\lambda)}} |\Delta^r_{\eta\varphi(\xi)^\lambda} f(\xi)| \\ &\leq \operatorname*{Sup}_{0<\eta\leq 2n^{-1}\delta_n(x)^{1-\lambda}} \|\Delta^r_{\eta\varphi^\lambda} f\| = \omega^r_{\varphi^\lambda}\left(f, 2n^{-1}\delta_n(x)^{1-\lambda}\right).\end{aligned}$$

For $x \in I_\ell^* = [-1, -1+4^{-\ell}]$, we note that when $h \leq \frac{2}{r}$

$$\begin{aligned} h/\varphi\left(x+\frac{r}{2}h\right)^\lambda &\leq h/\varphi\left(-1+\frac{r}{2}h\right)^\lambda \leq h/\left(1-\left(-1+\frac{r}{2}h\right)^2\right)^{\lambda/2} \\ &\leq h/\left(\frac{r}{2}h\right)^{\lambda/2} = \left(\frac{2}{r}\right)^{\lambda/2} h^{1-(\lambda/2)}\end{aligned}$$

and as $0 < h \leq \tau_\ell$ the condition $h \leq 2/r$ means that $n^{-2} < 4/r$. We can now write for $x \in I_\ell^*$

$$\begin{aligned}(3.5)\qquad |f(x)-G_n(x)| &\leq \operatorname*{Sup}_{0<h\leq\tau_\ell} |\vec{\Delta}_h^r f(x)| = \operatorname*{Sup}_{0<h\leq\tau_\ell}\left|\Delta_h^r f\left(x+\frac{r}{2}h\right)\right| \\ &\leq \operatorname*{Sup}_{\xi} \operatorname*{Sup}_{0<\eta\leq(\frac{2}{r})^\lambda\tau_\ell^{1-\lambda/2}} |\Delta^r_{\eta\varphi(\xi)^\lambda} f(\xi)| \\ &= \omega^r_{\varphi^\lambda}\left(f, \left(\frac{2}{r}\right)^\lambda n^{-1}\delta_n(x)^{1-\lambda}\right) \\ &\leq \omega^r_{\varphi^\lambda}\left(f, 2n^{-1}\delta_n(x)^{1-\lambda}\right).\end{aligned}$$

We now use Thorem 4.1.2 of [4] with the step weight φ^λ (which satisfies the conditions on step weights there) to obtain

$$(3.6)\qquad \omega^r_{\varphi^\lambda}(f, 2h) \leq C\omega^r_{\varphi^\lambda}(f, h)$$

where C is independent of f and h. This implies (2.2) for $n \geq n_0$. To obtain (2.2) for all n we increase the constant.

For the estimate (2.3) we first recall that for $\tau > 0$

$$(3.7)\qquad |f_\tau^{(r)}(x)| \leq \sum_{j=1}^r \binom{r}{j}\left(\frac{r}{j}\right)^r \tau^{-r} |\vec{\Delta}^r_{j\tau/r} f(x)|.$$

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Using (3.7) and earlier considerations (see (3.5)) we obtain for $x \in I_\ell^*$

$$|G_n^{(r)}(x)| = |f_{\tau_\ell}^{(r)}(x)| \le \tau_\ell^{-r}\left\{\sum_{j=1}^{r}\binom{r}{j} j^r\right\} \sup_{0<h\le\tau_\ell} |\vec{\Delta}_h^r f(x)|$$
$$\le \tau_\ell^{-r}(2r)^r \omega_{\varphi^\lambda}^r\big(f, 2n^{-1}\delta_n(x)^{1-\lambda}\big).$$

Using (3.6) and $\tau_\ell^{-1} \le 2n^2 \le 6n\delta_n(x)^{-1}$ for $x \in I_\ell^*$ we have

$$|G_n^{(r)}(x)| \le C_1 r^r n^r \delta_n(x)^{-r} \omega_{\varphi^\lambda}^r\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

To prove (2.3) for $x \in I_k$ $1 \le k \le \ell - 1$, we write

$$G_n(x) = f_{\tau_k}(x) + \Psi_{k+1}(x)\big(f_{\tau_{k+1}}(x) - f_{\tau_k}(x)\big). \tag{3.8}$$

For $x \in I_k$ we show using the technique employed in proving (3.4)

$$|f_{\tau_k}^{(r)}(x)| \le \tau_k^{-r}(2r)^r \sup_{0<h\le\tau_k} |\vec{\Delta}_h^r f| \le M\big(n\delta_n(x)^{-1}\big)^r \omega_{\varphi^\lambda}^r\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)$$

and

$$|f_{\tau_{k+1}}^{(r)}(x)| \le M\big(n\delta_n(x)^{-1}\big)^r \omega_{\varphi^\lambda}^r\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)$$

with M independent of k and f.

Since for $x \in I_k$

$$|\Psi_{k+1}^{(m)}(x)| \le M_1 4^{km} \le M_2\big(n\delta_n(x)^{-1}\big)^m$$

we have to show only

$$|f_{\tau_k}^{(s)}(x) - f_{\tau_{k+1}}^{(s)}(x)| \le M_3\big(n\delta_n(x)^{-1}\big)^s \omega_{\varphi^\lambda}^r\big(f, n^{-1}\delta_n(x)^{1-\lambda}\big) \tag{3.9}$$

for $0 \le s \le r$. For $s = 0$ we derive this estimate following (3.4) and using

$$|f_{\tau_k}(x) - f_{\tau_{k+1}}(x)| \le |f_{\tau_k}(x) - f(x)| + |f_{\tau_{k+1}}(x) - f(x)|.$$

For $s = r$ we already obtained the estimate as we may write

$$|f_{\tau_k}^{(r)}(x) - f_{\tau_{k+1}}^{(r)}(x)| \le |f_{\tau_k}^{(r)}(x)| + |f_{\tau_{k+1}}^{(r)}(x)|.$$

For other s (3.9) follows using the estimate

$$\|g^{(s)}\|_{C[a,b]} \le A[(b-a)^{-s}\|g\|_{C[a,b]} + (b-a)^{r-s}\|g^{(r)}\|_{C[a,b]}]$$

(see Lemma 2.1 of [2]) where A is independent of $[a,b]$ and g, $a = -1 + 4^{-k-1}$, $b = -1 + 4^{-k}$ and

$$g(x) = f_{\tau_k}(x) - f_{\tau_{k+1}}(x). \qquad \blacksquare$$

4. **Estimate on derivatives of polynomials.** In the proof of the converse results we need an estimate on the derivatives of polynomials.

THEOREM 4.1. *Suppose for a polynomial of degree n we have the estimate*

$$|P_n(x)| \le M\big(n^{-1}\delta_n(x)\big)^{\beta}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \quad |x|<1 \tag{4.1}$$

where β is a real number and ω satisfies

$$\omega(\mu t) \le C(\mu^s+1)\omega(t), \quad \mu>0,\ t>0. \tag{4.2}$$

Then for $\ell \ge \beta + s(1-\lambda)$

$$|P_n^{(\ell)}(x)| \le M_1\big(n^{-1}\delta_n(x)\big)^{\beta-\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \quad |x|<1 \tag{4.3}$$

where M_1 depends on M, C, ℓ, s, β and λ but not on x, P_n or n.

PROOF. Let us begin by estimating $|P_n'(x)|$. It follows from [5, p. 70 (2)] that for a trigonometric polynomial $T_n(x)$ we have

$$T_n'(t) = 2^{2m}n^{-2m+2}\int_{-\pi}^{\pi} T_n(t+u)H_{n,m}(u)\tilde{K}_{n,m}(u)\,du$$

where $\tilde{K}_{n,m}$ and $H_{n,m}$ are trigonometric polynomials,

$$\tilde{K}_{n,m}(u) = \left(\frac{\sin\frac{nu}{2}}{\sin\frac{u}{2}}\right)^{2m} \text{ and } |H_{n,m}(u)| \le 1.$$

We recall [5, p. 57] that

$$\int \tilde{K}_{n,m}(u)\,du \le c(m)n^{2m-1},$$

and hence we have

$$|T_n'(t)| \le C_1(m)n\int_{-\pi}^{\pi}|T_n(t+u)|S_{nm}(u)\,du$$

with $S_{nm}(u)$ an even positive trigonometric polynomial of degree $(n-1)m$ satisfying

$$\int_{-\pi}^{\pi} S_{n,m}(u)\,du = 1 \text{ and } \int_{-\pi}^{\pi}|u|^{\gamma}S_{n,m}(u)\,du \sim \left(\frac{1}{n}\right)^{\gamma}, \quad 0<\gamma<2m-1. \tag{4.4}$$

We set $T_n(t) = P_n(\cos t)$, and hence

$$\begin{aligned}
|T_n'(t)| &= |P_n'(\cos t)\sin t| \\
&\le C_1(m)n\int_{-\pi}^{\pi}\big|P_n\big(\cos(t+u)\big)\big|S_{nm}(u)\,du \\
&\le MC_1(m)n\int_{-\pi}^{\pi}\left(\frac{1}{n}\Big(\frac{1}{n}+|\sin(t+u)|\Big)\right)^{\beta} \\
&\qquad\times\omega\Big(n^{-1}\big(n^{-1}+|\sin(t+u)|\big)^{1-\lambda}\Big)S_{nm}(u)\,du.
\end{aligned} \tag{4.5}$$

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We have

$$\frac{\frac{1}{n}+|\sin(t+u)|}{\frac{1}{n}+|\sin t|}\leq\frac{\frac{1}{n}+|\sin t|+|\sin u|}{\frac{1}{n}+|\sin t|}\leq 1+n|\sin u|\leq 1+n|u|,$$

and setting $\tau=t+u$ or calculating directly,

$$\frac{\frac{1}{n}+|\sin t|}{\frac{1}{n}+|\sin(t+u)|}\leq 1+n|u|.$$

Therefore, we have

$$\begin{aligned}\omega\Big(n^{-1}\big(n^{-1}+|\sin(t+u)|\big)^{1-\lambda}\Big)&\leq\omega\big(n^{-1}(n^{-1}+|\sin t|)^{1-\lambda}(1+n|u|)^{1-\lambda}\big)\\&\leq C\big(1+(1+n|u|)^{(1-\lambda)s}\big)\omega\big(n^{-1}(n^{-1}+|\sin t|)^{1-\lambda}\big)\end{aligned}$$

and

$$\Big(n^{-1}\big(n^{-1}+|\sin(t+u)|\big)\Big)^{\beta}\leq(1+n|u|)^{|\beta|}\big(n^{-1}(n^{-1}+|\sin t|)\big)^{\beta}.$$

Using the above estimates in (4.5) and $|\sin t|=\sqrt{1-x^2}$, we have for $2m-1>|\beta|+|s|(1-\lambda)$

$$\begin{aligned}|P_n'(x)|\sqrt{1-x^2}&\leq M_1 n\big(n^{-1}\delta_n(x)\big)^{\beta}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big)\\&\quad\times\int_{-\pi}^{\pi}S_{nm}(u)(1+n|u|)^{|\beta|}\big(1+(1+n|u|)^{s}\big)^{1-\lambda}\,du\\&\leq M_2 n\big(n^{-1}\delta_n(x)\big)^{\beta}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big).\end{aligned}$$

We now observe that for $1-x^2\geq\frac{1}{n^2}$ we have $\sqrt{1-x^2}\geq\frac{1}{2}(n^{-1}+\sqrt{1-x^2})$, and hence

$$|P_n'(x)|\leq M_3\big(n^{-1}\delta_n(x)\big)^{\beta-1}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big)\text{ for }1-x^2\geq\frac{1}{n^2}.$$

We set a so that $1-\frac{1}{n^2}=a$ and obtain

$$|P_n'(x)|\leq M_4\big(n^{-1}(n^{-1}+\sqrt{a^2-x^2})\big)^{\beta-1}\omega\big(n^{-1}(n^{-1}+\sqrt{a^2-x^2})^{1-\lambda}\big),\quad |x|\leq a$$

or

$$|P_n'(ax)|\leq M_5\big(n^{-1}\delta_n(x)\big)^{\beta-1}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big),\quad |x|<1,\tag{4.6}$$

and therefore, iterating (4.6), we have

$$|P_n^{(\ell)}(a^{\ell}x)|\leq M_6\big(n^{-1}\delta_n(x)\big)^{\beta-\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big),\quad |x|<1.\tag{4.7}$$

Hence for $|x|\leq a^{\ell}$

$$|P_n^{(\ell)}(x)|\leq M_7\big(n^{-1}\delta_n(x)\big)^{\beta-\ell}\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big).\tag{4.8}$$

Condition (4.2) now implies for $0 < t < t_1$

$$\frac{\omega(t_1)}{(t_1)^s} \le 2C\frac{\omega(t)}{t^s} \tag{4.9}$$

(with C of (4.2)), and therefore, setting

$$t_1 = n^{-1}\delta_n(x)^{1-\lambda} \text{ for } |x| \le a^\ell$$

and $t = n^{-2+\lambda}$, and recalling $\beta - \ell + (1-\lambda)s \le 0$, we have

$$\begin{aligned}|P_n^{(\ell)}(x)| &\le M_8 n^{\ell+(1-\lambda)s-\beta}\delta_n(x)^{\beta-\ell+(1-\lambda)s}\omega(n^{-2+\lambda})\\ &\le M_9 n^{2(\ell-\beta)}\omega(n^{-2+\lambda}), \quad |x| < a^\ell.\end{aligned}$$

Using Lemma 8.4.3 of [4],

$$|P_n^{(\ell)}(x)| \le M_{10} n^{2(\ell-\beta)}\omega(n^{-2+\lambda}) \quad |x| \le 1,$$

which, combined with (4.8), implies (4.3). ∎

5. **The converse result.** In this section we will prove a converse result that will unify the result of [4] for the particular case $p = \infty$ with the classical converse result for polynomial approximation (see [5] and [7]). Of course for $1 \le p < \infty$ only the result in [4] stands.

THEOREM 5.1. *Suppose for $f \in C[-1,1]$ and some given λ, $0 \le \lambda \le 1$ there exists a sequence of polynomials satisfying*

$$|f(x) - P_n(x)| \le M\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big) \tag{5.1}$$

where $\omega(t)$ is an increasing function satisfying (4.2) with $s = r$, that is,

$$\omega(\mu t) \le C(\mu^r + 1)\omega(t)$$

for any $\mu > 0$, $t > 0$, with C independent of μ and t. Then for any $t > 0$

$$\omega^r_{\varphi^\lambda}(f,t) \le Mt^r \sum_{0<n\le t^{-1}} (n+1)^{r-1}\omega(n^{-1}). \tag{5.2}$$

We will also need the following lemma from [5, p. 58].

LEMMA 5.2. *Suppose $0 < u_k \le u_{k+1} \le \cdots \le u_\ell$ such that $2 \le u_k/u_{k-1} \le 4$ and suppose that $\omega(u)$ is an increasing function. Then*

$$\sum_{k=i}^{\ell} u_k^r\omega(u_k^{-1}) \le M \sum_{[\frac{1}{2}u_i]\le n\le u_\ell} (n+1)^{r-1}\omega(n^{-1}). \tag{5.3}$$

PROOF OF THEOREM 5.1. The proof of our theorem, *i.e.*, of (5.2), starts in a way quite similar to that of the cases $\lambda = 0$ [5, p. 73] and $\lambda = 1$ [4, p. 83]. We write for x and h satisfying

$$I(x,h,\lambda,r) \equiv \Big[x - \frac{r}{2}h\varphi^\lambda, x + \frac{r}{2}h\varphi^\lambda\Big] \subset [-1,1] \tag{5.4}$$

$$\begin{aligned} |\Delta^r_{h\varphi^\lambda} f(x)| &\le |\Delta^r_{h\varphi^\lambda}(f - P_{2^\ell})(x)| + |\Delta^r_{h\varphi^\lambda} P_{2^\ell}(x)| \\ &\le 2^r \max_{\xi \in I(x,h,\lambda,r)} |(f - P_{2^\ell})(\xi)| + h^r \varphi(x)^{\lambda r} \max_{\xi \in I(x,h,\lambda,r)} |P^{(r)}_{2^\ell}(\xi)|, \end{aligned} \tag{5.5}$$

and recall that when $I(x,h,\lambda,r) \not\subset [-1,1]$, $\Delta^r_{h\varphi^\lambda} f(x) = 0$. Given x and h for which (5.4) is satisfied, we choose an integer ℓ which is maximal satisfying

$$2^{-\ell}\big(2^{-\ell} + \varphi(x)\big)^{1-\lambda} \ge 3rh. \tag{5.6}$$

We will show that for such an ℓ and $\xi \in I(x,h,\lambda,r)$ (satisfying (5.4)) we have

$$h \le 2^{-\ell}\big(2^{-\ell} + \varphi(\xi)\big)^{1-\lambda} \le 24rh \tag{5.7}$$

and

$$\frac{1}{3r}\big(2^{-\ell} + \varphi(x)\big) \le 2^{-\ell} + \varphi(\xi) \le 2\big(2^{-\ell} + \varphi(x)\big). \tag{5.8}$$

To prove the inequalities on the left of (5.7) we use (5.6) and prove the inequality on the left of (5.8). The latter is evident for $\varphi(x) \le r2^{-\ell+1}$ as $\varphi(\xi) \ge 0$ and $2r + 1 \le 3r$. The inequality $\varphi(x) > r2^{-\ell+1}$ implies $rh\varphi(x)^\lambda \le \frac{1}{2}\varphi(x)^2$. To show this we write, using (5.6),

$$2rh \le \frac{2}{3}2^{-\ell}\big(2^{-\ell} + \varphi(x)\big)^{1-\lambda} \le \frac{2}{3}\frac{1}{2r}\varphi(x)\Big(\frac{1}{2r} + 1\Big)^{1-\lambda}\varphi(x)^{1-\lambda} \le \frac{1}{2}\varphi(x)^{2-\lambda}.$$

The identity $\varphi(\xi)^2 = \varphi(x)^2 + (x - \xi)(\xi + x)$ implies

$$\varphi(\xi)^2 \ge \varphi(x)^2 - rh\varphi^\lambda(x) \ge \varphi(x)^2 - \frac{1}{2}\varphi(x)^2 = \frac{1}{2}\varphi(x)^2,$$

which, using $\sqrt{2} \le 3r$, completes the proof of the left side of (5.8) and therefore of (5.7). To prove the inequality on the right of (5.7), we observe that the choice of ℓ as maximal satisfying (5.6) implies

$$2^{-\ell}\big(2^{-\ell} + \varphi(x)\big)^{1-\lambda} \le 12rh,$$

and we complete the proof of (5.7) when we prove the right hand side of (5.8) (for $\xi \in I(x,h,\lambda,r) \subset [-1,1]$). If $\varphi(\xi)^2 \le \frac{3}{2}rh\varphi(x)^\lambda$, we recall that

$$\varphi(x)^2 \ge \min(|1-x|, |1+x|) \ge \frac{r}{2}h\varphi(x)^\lambda \ge \frac{1}{3}\varphi(\xi)^2 \text{ or } \varphi(\xi) \le \sqrt{3}\varphi(x).$$

If $\varphi(\xi)^2 > \frac{3}{2}rh\varphi(x)^\lambda$, we write

$$\varphi(\xi)^2 \le \varphi(x)^2 + 2|\xi - x| \le \varphi(x)^2 + rh\varphi(x)^\lambda \text{ or } \varphi(\xi)^2 < 3\varphi(x)^2.$$

We now use (4.2), (5.7) and the fact that ω is increasing, and obtain

$$\begin{aligned}(5.9)\qquad \max_{\xi\in I(x,h,\lambda,r)} |f - P_{2^\ell}(\xi)| &\le M \max_{\xi\in I(x,h,\lambda,r)} \omega\big(2^{-\ell}(2^{-\ell}+\sqrt{1-\xi^2})^{1-\lambda}\big)\\ &\le M\omega(24rh) \le M_1\omega(h).\end{aligned}$$

To estimate $|P^{(r)}_{2^\ell}(\xi)|$, we write, recalling $P^{(r)}_1(\xi)=0$,

$$P^{(r)}_{2^\ell}(\xi) = \sum_{k=1}^{\ell}\{P^{(r)}_{2^k}(\xi) - P^{(r)}_{2^{k-1}}(\xi)\}.$$

As

$$\begin{aligned}|P_{2^k}(x) - P_{2^{k-1}}(x)| &\le |P_{2^k}(x) - f(x)| + |P_{2^{k-1}}(x) - f(x)|\\ &\le 2M\omega\big(2^{-k+1}\delta_{2^{-k+1}}(x)^{1-\lambda}\big),\end{aligned}$$

Theorem 4.1 with $\beta=0$, $\ell=s=r$ and $n=2^k$ implies

$$\begin{aligned}|P^{(r)}_{2^k}(\xi) - P^{(r)}_{2^{k-1}}(\xi)| &\le M_2\Big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)\Big)^{-r}\omega\Big(2^{-k+1}\big(2^{-k+1}+\varphi(\xi)\big)^{1-\lambda}\Big)\\ &\le M_2\Big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)\Big)^{-r}\omega\Big(2^{-k+1}\big(2^{-k+1}+\varphi(\xi)\big)^{1-\lambda}\Big).\end{aligned}$$

Therefore, using (5.8) and (4.2), we have

$$\begin{aligned}&h^r\varphi(x)^{\lambda r}|P^{(r)}_{2^\ell}(\xi)|\\ &\quad\le h^r\big(2^{-\ell}+\varphi(x)\big)^{\lambda r}|P^{(r)}_{2^\ell}(\xi)|\\ &\quad\le M_3h^r\big(2^{-\ell}+\varphi(\xi)\big)^{\lambda r}\sum_{k=1}^{\ell}\Big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)\Big)^{-r}\omega\Big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\Big)\\ &\quad\le M_3h^r\sum_{k=1}^{\ell}\Big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\Big)^{-r}\omega\Big(2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\Big).\end{aligned}$$

We now set in Lemma 5.2 $u_k^{-1} = \Big[2^{-k}\big(2^{-k}+\varphi(\xi)\big)^{1-\lambda}\Big]$, and observe that $2\le u_k/u_{k-1}\le 2\cdot 2^{1-\lambda}\le 4$. We further recall, using (5.7), that $u_k^{-1}\le u_\ell^{-1}\le h^{-1}$, and hence we can write

$$\begin{aligned}(5.10)\qquad &h^r\varphi(x)^{\lambda}|P^{(r)}_{2^\ell}(\xi)|\\ &\quad\le M_3h^r\sum_{[2^{-k}(2^{-k}+\varphi(\xi))^{1-\lambda}]\ge h}\big(2^{-k}\delta_{2^{-k}}(\xi)^{1-\lambda}\big)^r\omega\big(2^{-k}\delta_{2^{-k}}(\xi)^{1-\lambda}\big)\\ &\quad\le M_4h^r\sum_{0<n\le h^{-1}}(n+1)^r\omega(n^{-1}).\end{aligned}$$

Combining (5.9) and (5.10), we complete the proof of our theorem. ∎

6. **The derivatives of the approximating polynomial.** In this section we will estimate $f^{(r)} - P_n^{(r)}$ in terms of the estimate of $f - P_n$ in analogy to a classical result of Timan and others (see [5, p. 74, Theorem 5]). The L_p case will also be stated and proved. We will then estimate $P_n^{(r)}$ in terms of the estimate of $|f(x) - P_n(x)|$.

THEOREM 6.1. *Suppose for some $f \in C[-1,1]$ and $0 \le \lambda \le 1$ there exists a sequence of polynomials P_n satisfying*

$$|f(x) - P_n(x)| \le M\omega\big(n^{-1}\delta_n(x)^{1-\lambda}\big), \quad |x| \le 1 \tag{6.1}$$

where $\delta_n(x) = n^{-1} + \sqrt{1-x^2}$, $\omega(t)$ is an increasing function satisfying (4.2), with $s = r$ and $\sum_{k=1}^{\infty} k^{r-1}\omega(k^{-1}) < \infty$. Then f has locally r continuous derivatives and

$$\big|\varphi(x)^{\lambda r}[f^{(r)}(x) - P_n^{(r)}(x)]\big| \le M_1 \sum_{k > n\delta_n^{\lambda-1}} k^{r-1}\omega(k^{-1}). \tag{6.2}$$

PROOF. Following the standard technique, we write

$$f = P_n + \sum_{i=1}^{\infty}(P_{2^i n} - P_{2^{i-1}n}).$$

Furthermore, using Lemma 5.2 with $u_i^{-1} = \frac{1}{2^i n}\big(\frac{1}{2^i n} + \varphi(x)\big)^{1-\lambda}$, and letting ℓ there increase to infinity, the inequality $\sum n^{r-1}\omega(\frac{1}{n}) < \infty$, which implies $\sum n^{-1}\omega(\frac{1}{n}) < \infty$, assures us of the convergence of the series above. Using Theorem 4.1 (and (4.2)), we have

$$\begin{aligned}&|P_{2^i n}^{(r)}(x) - P_{2^{i-1}n}^{(r)}(x)| \\ &\quad \le M_2\Big(2^{-i}n^{-1}\big(2^{-i}n^{-1} + \varphi(x)\big)\Big)^{-r}\omega\Big(2^{-i+1}n^{-1}\big(2^{-i+1}n^{-1} + \varphi(x)\big)^{1-\lambda}\Big).\end{aligned}$$

The convergence of $\sum n^{r-1}\omega(\frac{1}{n})$ now implies local uniform (in $(x_0 - \delta, x_0 + \delta) \subset [-1+\delta, 1-\delta]$) convergence of

$$\sum_{i=1}^{\infty}\big(P_{2^i n}^{(r)}(x) - P_{2^{i-1}n}^{(r)}(x)\big),$$

and therefore, $(f - P_n)^{(r)}$ exists locally. We can now write

$$\begin{aligned}|\varphi^{\lambda r}(f - P_n)^{(r)}(x)| &\le \varphi^{\lambda r}\sum_{i=1}^{\infty}|(P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)})(x)| \\ &\le \varphi^{\lambda r}\sum_{i=1}^{\infty}\big(2^{-i}n^{-1}\delta_{2^i n}(x)\big)^{-r}\omega\big(2^{-i}n^{-1}\delta_{2^i n}(x)^{1-\lambda}\big) \\ &\le \sum_{i=1}^{\infty}\big(2^{-i}n^{-1}\delta_{2^i n}(x)^{1-\lambda}\big)^{-r}\omega\big(2^{-i}n^{-1}\delta_{2^i n}(x)^{1-\lambda}\big) \\ &\le M\sum_{k > n\delta_n^{\lambda-1}} k^{r-1}\omega(1/k),\end{aligned}$$

and this completes the proof of our theorem. ∎

The analogue for L_p was not stated explicitly in [4] or anywhere we know of, so for the sake of completeness, we are stating it here.

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THEOREM 6.2. *Suppose* $f \in L_p[-1,1]$, $1 \le p \le \infty$,

$$(6.3)\qquad E_n(f)_p \equiv \inf \|f - P_n\|_{L_p[-1,1]}$$

(P_n polynomial of degree n) and

$$\sum (n+1)^{r-1} E_n(f)_p < \infty.$$

Then $f^{(r)}$ exists locally in the L_p sense and

$$(6.4)\qquad \|\varphi^r[f^{(r)} - P_n^{(r)}]\|_{L_p[-1,1]} \le M \sum_{k \ge n} (k+1)^{r-1} E_k(f)_p.$$

PROOF. We follow exactly the classical proof using the series $\sum_{i=1}^{\infty}(P_{2^i n} - P_{2^{i-1}n})$ where P_k is the best polynomial approximant to f in $L_p[-1,1]$. Local L_p convergence and the estimate (6.4) itself are guaranteed by the estimate [4, Theorem 8.4.7], *i.e.*,

$$(6.5)\qquad \|\varphi^r P_n^{(r)}\|_{L_p[-1,1]} \le C n^r \|P_n\|_{L_p[-1,1]}$$

for all n-th degree polynomials, P_n, which implies

$$\begin{aligned} \Big\|\varphi^r \sum_{i=1}^{\infty} (P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)})\Big\|_{L_p[-1,1]} &\le \sum_{i=1}^{\infty} \|\varphi^r (P_{2^i n}^{(r)} - P_{2^{i-1}n}^{(r)})\|_p \\ &\le 2C \sum_{i=1}^{\infty} (2^i n)^r E_{2^i n}(f)_p \\ &\le C_1 \sum_{k \ge n} (k+1)^{r-1} E_k(f)_p. \end{aligned}$$ ∎

THEOREM 6.3. *Suppose for some $f \in C[-1,1]$ and some $0 \le \lambda \le 1$ the sequence of polynomials P_n of degree smaller than n satisfies*

$$(6.6)\qquad |f(x) - P_n(x)| \le M \omega_{\varphi^\lambda}^r \big(f, n^{-1}\delta_n(x)^{1-\lambda}\big)$$

for the r-th modulus of smoothness $\omega_{\varphi^\lambda}^r(f,t)$. Then

$$(6.7)\qquad |\varphi(x)^{\lambda r} P_n^{(r)}(x)| \le M_1 n^r \delta_n(x)^{(\lambda-1)r} \omega_{\varphi^\lambda}^r \big(f, n^{-1}\delta_n(x)^{1-\lambda}\big).$$

Obviously, using Theorem 2.1, there exist P_n satisfying (6.6). We stated (6.6) to stress that it is those P_n that satisfy (6.7).

PROOF. Following the proof of Theorem 7.3.1 of [4], we write

$$P_n^{(r)} = P_n^{(r)} - P_1^{(r)} = P_n^{(r)} - P_{2^\ell}^{(r)} + \sum_{k=1}^{\ell} (P_{2^k}^{(r)} - P_{2^{k-1}}^{(r)})$$

where $2^\ell \le n < 2^{\ell+1}$. As

$$\omega^r_{\varphi^\lambda}(f,\mu t) \le C(\lambda,r)(\mu^r+1)\omega^r_{\varphi^\lambda}(f,t)$$

for all f, μ and t, we have, using Theorem 4.1,

$$\begin{aligned}
|\varphi(x)^{\lambda(r+1)}P_n^{(r+1)}(x)| &\le M_2\Big[n^{r+1}\delta_n(x)^{-(r+1)(1-\lambda)}\omega^r_{\varphi^\lambda}\big(f,n^{-1}\delta_n(x)^{1-\lambda}\big)\\
&\quad+\sum_{k=1}^{\ell}2^{k(r+1)}(2^{-k}+\varphi)^{-(r+1)(1-\lambda)}\omega^r_{\varphi^\lambda}\big(f,2^{-k}(2^{-k}+\varphi)^{1-\lambda}\big)\Big]\\
&\le M_3 n^{r+1}\delta_n(x)^{-(r+1)(1-\lambda)}\omega^r_{\varphi^\lambda}\big(f,n^{-1}\delta_n(x)^{1-\lambda}\big).
\end{aligned}\tag{6.8}$$

Recalling [4, Theorem 8.4.8], we observe that it is sufficient to estimate $\varphi^{\lambda r}(\frac{1}{n}+\varphi)^{(1-\lambda)r}P_n^{(r)}(x)$ and $\varphi^\alpha P_n^{(r)}(x)$ only for $x \in [-1+4r^2n^{-2}, 1-4r^2n^{-2}]$. We now write

$$\begin{aligned}
|n^{-r}\varphi^{\lambda r}(n^{-1}+\varphi)^{(1-\lambda)r}P_n^{(r)}(x)| &\le |n^{-r}\varphi^{\lambda r}(n^{-1}+\varphi)^{(1-\lambda)r}P_n^{(r)}(x) - \Delta^r_{\varphi^\lambda(\frac{1}{n}+\varphi)^{1-\lambda}\frac{1}{n}}P_n(x)|\\
&\quad+|\Delta^r_{\varphi^\lambda(\frac{1}{n}+\varphi)^{1-\lambda}\frac{1}{n}}(P_n-f)(x)| + |\Delta^r_{\varphi^\lambda(\frac{1}{n}+\varphi)^{1-\lambda}\frac{1}{n}}f(x)|\\
&\equiv I_1+I_2+I_3.
\end{aligned}$$

Clearly

$$I_3 \le \omega^r_{\varphi^\lambda}\big(f,n^{-1}\delta_n(x)^{1-\lambda}\big).$$

Moreover, for $\xi_j = x+(\frac{r}{2}-j)\varphi(x)^\lambda\big(\frac{1}{n}+\varphi(x)\big)^{1-\lambda}\frac{1}{n}$ and $x\in[-1+4r^2n^{-2}, 1-4r^2n^{-2}]$ we have

$$A^{-1}\varphi(x) \le \varphi(\xi_j) \le A\varphi(x), \quad 0\le j\le r,$$

and hence, using $\omega^r_{\varphi^\lambda}(f,\mu t)\le C(\mu^r+1)\omega^r_{\varphi^\lambda}(f,t)$, we have

$$\begin{aligned}
I_2 &\le 2^r\max_{0\le j\le r}|P_n(\xi_j)-f(\xi_j)| \le M\omega^r_{\varphi^\lambda}\big(f,n^{-1}\delta_n(\xi_j)^{1-\lambda}\big)\\
&\le M_3\omega^r_{\varphi^\lambda}\big(f,n^{-1}\delta_n(x)^{1-\lambda}\big).
\end{aligned}$$

Following Theorem 7.3.1 of [4], we estimate

$$\begin{aligned}
I_1 &\le n^{-r}\varphi^{\lambda r}(x)\big(n^{-1}+\varphi(x)\big)^{(1-\lambda)r}|P_n^{(r)}(x)-P_n^{(r)}(\xi)|\\
&\le n^{-r}\varphi^{\lambda r}(x)\big(n^{-1}+\varphi(x)\big)^{(1-\lambda)r}|x-\xi|\,|P_n^{(r+1)}(\eta)|
\end{aligned}$$

where $\eta,\xi \in \Big(x-\frac{r}{2}\varphi^\lambda(x)\big(\frac{1}{n}+\varphi(x)\big)^{1-\lambda}\frac{1}{n}, x+\frac{r}{2}\varphi^\lambda(x)\big(\frac{1}{n}+\varphi(x)\big)^{1-\lambda}\frac{1}{n}\Big)$. For $x\in[-1+4r^2n^{-2}, 1-4r^2n^{-2}]$, the above estimate and (6.8) yield the desired result. ∎

7. **Remarks.**

REMARK 7.1. In Theorems 2.1, 4.1, 5.1 and 6.1 we needed a condition on the function majorizing $|f(x) - P_n(x)|$ while for the analogues on $L_p[-1,1]$, $1 \le p \le \infty$ no condition was imposed on $E_n(f)$. The technical reason is that in Theorem 8.4.7 of [4] such a condition is not needed and in our Theorem 4.1 it is.

For the behaviour $\omega(t) = t^\alpha$ we may obtain the following corollaries.

COROLLARY 7.2. *For $f \in C[-1,1]$, $\alpha < r$ and $0 \le \lambda \le 1$ the conditions*

$$(7.1) \qquad |f(x) - P_n(x)| \le C\big(n^{-1}(n^{-1} + \sqrt{1-x^2})^{1-\lambda}\big)^\alpha \equiv C\big(n^{-1}\delta_n(x)^{1-\lambda}\big)^\alpha$$

and

$$(7.2) \qquad \omega^r_{\varphi^\lambda}(f,t) \le C_1 t^\alpha$$

are equivalent.

COROLLARY 7.3. *For $f \in C[-1,1]$, $r < \alpha$ and $0 \le \lambda \le 1$ (7.1) implies the existence of $f^{(r)}$ locally, $f^{(r-1)} \in A.C_{loc}$ and*

$$(7.3) \qquad (1-x^2)^{\lambda r/2}|f^{(r)}(x) - P_n^{(r)}(x)| \le C_2\big(n^{-1}(n^{-1} + \sqrt{1-x^2})^{1-\lambda}\big)^{\alpha-r}.$$

COROLLARY 7.4. *For $f \in C[-1,1]$ and $\alpha < r$ the condition (7.1) or*

$$\delta_n(x)^{(\lambda-1)\alpha}|f(x) - P_n(x)| \le Cn^{-\alpha}$$

and

$$(7.4) \qquad |(1-x^2)^{\lambda\alpha}\Delta^r_h f(x)| \le C_1 h^\alpha$$

are equivalent.

Corollary 7.4 is proved in a way similar to the treatment in [3].

REMARK 7.5. For $\lambda = 0$, Theorems 2.1 and 5.1 yield a somewhat different proof of the Timan, Dzjadyk, Freud and Brudnyi result. One should note, examining the proof for $\lambda = 0$, that we do not use theorems on $\omega^r_\varphi(f,t)$. We do, however, use the direction and methods of proofs developed in [4].

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