TRIGONOMETRIC SUMS OVER PRIMES II

by GLYN HARMAN†

(Received 14 April, 1981)

1. Introduction. We write $e(x)$ for $e^{2\pi ix}$, $\|x\|$ for the distance of x from the nearest integer and use $A \ll B$ to mean $|A| < c\,|B|$, where c is a positive constant depending at most on k and ε. The letter p always denotes a prime number; P_2 represents a number with precisely two prime factors. We continue the investigations started in **[6]** and will make many references to the analysis there. Here we prove the following theorems.

THEOREM 1. *Let k be an integer $\geqq 3$, and $\varepsilon > 0$. Suppose*

$$N^{1-1/k} \leqq q \leqq N^{\frac{1}{2}k}, \qquad |\alpha q - a| < N^{-\frac{1}{2}k}, \qquad (a, q) = 1.$$

Then

$$\left| \sum_{p \leqq N} (\log p) e(\alpha p^k) \right| \ll N^{1+\varepsilon-\gamma}, \tag{1}$$

where

$$\gamma = (k2^k)^{-1}.$$

THEOREM 2. *For $\varepsilon > 0$, β an arbitrary real number and α irrational, there are infinitely many solutions of the inequality*

$$\|\alpha p^k + \beta\| < p^{-\xi+\varepsilon}. \tag{2}$$

Here

$$\xi = (2^{k+1} + (2^{k+1} - 1 - 2k)/k)^{-1} \quad \textit{and} \quad k \geqq 3.$$

THEOREM 3. *Let k be an integer $\geqq 4$ and $f(x)$ a real polynomial in x with irrational leading coefficient. Then, for a given $\varepsilon > 0$, there are infinitely many solutions of the inequality*

$$\|f(p)\| < p^{-\tau+\varepsilon}. \tag{3}$$

Here, for $k \leqq 11$,

$$\tau = (2T + (2^{k+1} - 1 - 2k)/k)^{-1},$$

where T is defined by the following table.

k	4	5	6	7	8	9	10	11
T	46	110	240	414	672	1080	1770	3000

† Written while the author held a London University Postgraduate Studentship.

For $k \geqq 2$, we have

$$\tau = (12[k^2(\log k + \tfrac{1}{2}\log\log k + 1\cdot 3)])^{-1}.$$

THEOREM 4. *Suppose f is a real polynomial of degree $k \geqq 2$ with an irrational leading coefficient. Then, for $\varepsilon > 0$, there are infinitely many solutions of*

$$\|f(P_2)\| < P_2^{-\sigma+\varepsilon}, \tag{4}$$

where $\sigma = (2^k + 2)^{-1}$.

For a brief history of results of the types (1), (2) and (3), we refer the reader to [**6**]. Theorem 1 improves Theorem 1 of [**6**] in certain cases. Theorem 2 represents a substantial improvement on Theorem 2 of [**6**] for the specific case of monomials (e.g., for $k = 3$, 1/19 replaces 1/32, for $k = 8$, $(573\frac{3}{8})^{-1}$ replaces $(32{,}768)^{-1}$). Theorem 3 improves upon Theorem 2 of [**6**] for $k \geqq 4$ (e.g., for $k = 8$, we get $(1407\frac{7}{8})^{-1}$), and improves upon the present Theorem 2 for large k. No results of the type (4) seem to have appeared in the literature before, but S. W. Graham has shown [**5**] that there are infinitely many solutions of

$$\|\alpha P_2\| < P_2^{-\frac{1}{3}}(\log P_2)^{18}. \tag{5}$$

See also [**3**] and [**4**] for related work. His method is an application of the small sieve. In Section 3 we show that the exponent of $\log P_2$ in (5) may be reduced to 4/3 using the *large* sieve. For large k, it follows from Chapter 5 of [**12**], with slight modifications, that we can get the same answer for $f(P_2)$ as the best currently known results for $f(n)$. In particular, it follows from [**1**], with some alterations, that, for monomials, we can take

$$\sigma = (2(\log k + 1)(3\cdot 25 + (k+1)\log(k(\log k + 1)))/\log k)^{-1},$$

which is better that the present result for $k \geqq 7$.

The new ideas in this paper compared with [**6**] involve Vinogradov's method [**11**] of relating a trigonometrical sum to an integral with a simple arithmetical interpretation, and the combination of this method with the ideas of Section 2 of [**6**]. I would like to thank the referee for his comments, also I would like to express my thanks to R. C. Baker for his helpful criticisms of my manuscripts.

2. Some preparatory lemmas. By $\sum'$ we indicate that the variable summed over takes values coprime to the number q which will appear in the statements of lemmas.

LEMMA 1. *For $\log q \ll \log N$, f a real valued function, we have*

$$\sum_{n=1}^{N} \Lambda(n)e(f(n)) = O(N^{\frac{1}{3}}) + S_1 - S_2 - S_3, \tag{6}$$

where

$$S_1 = \sum_{d \leqq N^{\frac{1}{3}}}' \mu(d) \sum_{l \leqq Nd^{-1}}' (\log l) e(f(dl)),$$

$$S_2 = \sum_{r \leqq N^{\frac{2}{3}}}' \phi_1(r) \sum_{m \leqq Nr^{-1}}' e(f(rm)),$$

$$S_3 = \sum_{N^{\frac{1}{3}} < m \leqq N^{\frac{2}{3}}}' \phi_2(m) \sum_{N^{\frac{1}{3}} < n < Nm^{-1}}' \Lambda(n) e(f(mn))$$

and for any $\delta > 0$,

$$|\phi_1(m)| = o(m^\delta), \qquad |\phi_2(m)| = o(m^\delta).$$

Proof. This is essentially given by Vaughan in **[9]** and **[10]**.

We remark that it follows from this lemma (removing the $\log l$ factor in S_1 by partial summation as in **[6]**) that we need only estimate two types of sum:

(I)
$$\sum_{Y < y \leqq 2Y}' \psi(y) \sum_{x \leqq Ny^{-1}}' \phi(x) e(f(xy)), \tag{7}$$

where $N^{\frac{1}{3}} \leqq Y \leqq N^{\frac{1}{2}}$;

(II)
$$\sum_{Y < y \leqq 2Y}' \psi(y) \sum_{x \leqq Ny^{-1}}' e(f(xy)). \tag{8}$$

Here $Y < N^{\frac{1}{3}}$. Both ϕ and ψ in (7) and (8) can be assumed to satisfy

$$\phi(u) \ll u^\delta, \qquad \psi(v) \ll v^\delta$$

for every $\delta > 0$.

LEMMA 2. *For any positive integers W, q and real number ρ, we have, for $\varepsilon > 0$,*

$$\left| \sum_{u=1}^{W,} e(\rho u^k) \right|^R \ll \max_{\substack{d \mid q \\ \mu(d) \neq 0 \\ d \leqq W}} (Wq)^\varepsilon \left(\frac{W}{d}\right)^{R-k} \sum_{z=1}^{J} \min\left(\frac{W}{d}, \frac{1}{\|\rho z\|}\right). \tag{9}$$

Here $R = 2^{k-1}$ and $J = (k!)dW^{k-1}$.

We remark that by the conventional method of estimating sums of the type which occurs on the right of (9), the estimate is a decreasing function of d. Thus d can essentially be thought of as 1 in (9). This gives the usual Weyl inequality result, but we have removed all numbers from the sum on the left of (9) not coprime to q.

Proof. It is easily shown (see Lemma 2, Chapter 9 of **[12]**) that

$$\sum_{u=1}^{W,} e(\rho u^k) = \sum_{d \mid q} \mu(d) S(d), \tag{10}$$

where

$$S(d)=\begin{cases} 0 & \text{if} \quad d>W, \\ \sum\limits_{u=1}^{[Wd^{-1}]} e(\rho u^k d^k) & \text{for} \quad d \leqq W. \end{cases} \tag{11}$$

By Lemma 2 of [**4**],

$$\begin{aligned} |S(d)|^R &\ll W^{\frac{1}{2}\varepsilon}\left(\frac{W}{d}\right)^{R-k} \sum_{z=1}^{(k!)(W/d)^{k-1}} \min\left(\frac{W}{d}, \frac{1}{\|\rho z d^k\|}\right) \\ &\ll W^{\varepsilon}\left(\frac{W}{d}\right)^{R-k} \sum_{z=1}^{J} \min\left(\frac{W}{d}, \frac{1}{\|\rho z\|}\right). \end{aligned} \tag{12}$$

Combining (10), (11) and (12) gives (9) since the number divisors of q is $\ll q^{\varepsilon}$. Similarly we may prove

$$\sum_{l=1}^{L} \left| \sum_{u=1}^{W,}{}' e(lf(u)) \right|^R \ll \max_{\substack{d \mid q \\ \mu(d)\neq 0 \\ d \leqq W}} (LWq)^{\varepsilon}\left(\frac{W}{d}\right)^{R-k} \sum_{z=1}^{X} \min\left(\frac{W}{d}, \frac{1}{\|\alpha z\|}\right), \tag{13}$$

where $X=(k!)dLW^{k-1}$, and $f(u)=\alpha u^k+\ldots+\omega$.

LEMMA 3. *Suppose* $Y \leqq N^{\frac{1}{2}}, 1 \leqq m \leqq k, |q\alpha - a| < q^{-1}, (a, q)=1$. *Then, for* $\varepsilon>0$,

$$\begin{aligned} &\sum_{Y<y\leqq 2Y}{}' \psi(y) \sum_{x \leqq Ny^{-1}}{}' \phi(x) e(\alpha y^k x^k) \\ &\qquad \ll FN^{1+\varepsilon}\left(\left(\frac{Y^{k-m+1}}{N}\right)^{2^{2-m-k}} + \left(Y^{k-m}\left(\frac{1}{q}+\frac{q}{N^k}\right)\right)^{2^{2-m-k}} + \delta_m Y^{-2^{1-m}}\right), \end{aligned} \tag{14}$$

where $\phi(x)$, $\psi(y)$ *are real valued functions;* $\phi(x)\equiv 1$ *is an additional necessary condition if* m *is taken as* 1. *Here*

$$\delta_m=\begin{cases} 0 & \textit{if } m=1, \\ 1 & \textit{otherwise} \end{cases}$$

and

$$F=\max_u |\phi(u)| \max_v |\psi(v)|. \tag{15}$$

Proof. For $m=1$, this is Lemma 4 of [**6**] (replacing Lemma 2 there by the present Lemma 2). For $m=k$, it is the Corollary to Lemma 3 (making the same replacement). For $1<m<k$, the result follows by only applying the Weyl differencing technique, for the variable y, $m-1$ times in Lemma 3 of [**6**] (stopping the induction at $s=m-1$, not $s=k-1$ as there).

Henceforth in this paper whenever the letter F appears it is defined by (15).

LEMMA 4. *Suppose* $Y \leqq N^{\frac{1}{2}}, |\alpha - a/q| \leqq (N^k L)^{-1}, (a, q) = 1, N > L \geqq 1$. *Put*

$$S_L = \sum_{l=1}^{L} \left| \sum_{Y<y<2Y}{}' \psi(y) \sum_{x \leqq Ny^{-1}} \phi(x) e(\alpha l (xy)^k) \right|.$$

Then, if $\phi(x) = 1$ *for all* x,

$$S_L \ll F(NL)^{1+\varepsilon} \left(\frac{Y}{N} + \frac{1}{q} + \frac{q}{L} \left(\frac{Y}{N} \right)^k \right)^{2^{1-k}}. \tag{16}$$

Otherwise

$$S_L \ll F(NL)^{1+\varepsilon} \left(\frac{1}{Y^{\frac{1}{2}}} + \left(\frac{qY^k}{N^k L} + \frac{Y^k}{q} + \frac{Y}{N} \right)^{2^{-k}} \right). \tag{17}$$

Proof. We write

$$S_L = \sum_{l=1}^{L} A_l$$

and prove (16) first. By partial summation

$$A_l = \sum_{Y<y\leqq 2Y}{}' \left(\sum_{x \leqq Ny^{-1}} \delta_y(x) S_{x,y}(l) + \psi(y) e(\alpha' l y^k ([Ny^{-1}] + 1)^k) S_{[Ny^{-1}],y}(l) \right).$$

Here

$$\delta_y(x) = \psi(y) e(\alpha' l (yx)^k) - \psi(y) e(\alpha' l (x+1)^k y^k), \qquad \alpha' = \alpha - aq^{-1}$$

and

$$S_{x,y}(l) = \sum_{n \leqq x}{}' e\left(\frac{al(ny)^k}{q} \right).$$

Clearly $\delta_y(x) \ll Fly^k x^{k-1} (LN^k)^{-1}$. Thus

$$\sum_{l=1}^{L} |A_l| \ll \sum_{Y<y\leqq 2Y}{}' \left(\sum_{x \leqq N_y^{-1}} \frac{Fy^k x^{k-1}}{N^k} \sum_{l=1}^{L} |S_{x,y}(l)| + F \sum_{l=1}^{L} |S_{[Ny^{-1}],y}(l)| \right). \tag{18}$$

By (13), the fact that $(y, q) = 1$ and Hölder's inequality we find that

$$\sum_{l=1}^{L} |S_{x,y}(l)| \ll (Lx)^{1+\frac{1}{2}\varepsilon} \left(\frac{1}{x} + \frac{1}{q} + \frac{q}{x^k L} \right)^{2^{1-k}}. \tag{19}$$

It is now easy to deduce (16) from (18) and (19).

To prove (17), we use Cauchy's inequality to obtain

$$|A_l|^2 \leqq \left(\sum_{x \leqq NY^{-1}} \phi(x)^2 \right) \left(\sum_{x \leqq NY^{-1}} \sum_{Y<v_1<H_x} \psi(v_1) \sum_{Y<v_2<H_x} \psi(v_2) e(\alpha x^k l (v_1^k - v_2^k)) \right),$$

where

$$H_x = \min(2Y, Nx^{-1}) \leqq \frac{N}{Y} \max_u |\phi(u)|^2 S_l, \tag{20}$$

say. We now remove all the terms with $v_1 = v_2$ from S_l to leave a sum A_l', say. The terms with $v_1 = v_2$ contribute

$$\ll \max_v |\psi(v)|^2 N$$

to S_l and hence

$$\ll \frac{FNL}{Y^{\frac{1}{2}}} \tag{21}$$

to S_L. We proceed to estimate A_l' as we treated A_l above. We now get sums $S_{x,y_1,y_2}(l)$ to estimate given by

$$S_{x,y_1,y_2}(l) = \sum_{n \leqq x}' e\left(\frac{a l n^k (y_1^k - y_2^k)}{q}\right).$$

The complication arises that $y_1^k - y_2^k$ may not be coprime to q. It turns out that quite a crude argument will suffice for the applications (the $Y^k q^{-1}$ term in (17) can be improved but not the $qY^kN^{-k}L^{-1}$ term). We have

$$(q, (y_1^k - y_2^k)) \leqq |y_1^k - y_2^k| \ll Y^k; \tag{22}$$

thus

$$\sum_{l=1}^{L} |S_{x,y_1,y_2}(l)| \ll (Lx)^{1+\varepsilon}\left(\frac{1}{x} + \frac{Y^k}{q} + \frac{q}{x^k L}\right)^{2^{1-k}}. \tag{23}$$

Hence

$$\sum_{l=1}^{L} A_l' \ll \max_v |\psi(v)|^2 \, Y(LN)^{1+\varepsilon}\left(\frac{Y}{N} + \frac{Y^k}{q} + \frac{qY^k}{N^k L}\right)^{2^{1-k}}. \tag{24}$$

A combination of (20), (21) and (24) together with Cauchy's inequality then yields (17) as desired. We note that there are no technical difficulties involved in replacing αn^k by a polynomial of degree k with leading coefficient α.

LEMMA 5. *Let $\phi(x)$ be an arbitrary function. Let B and A be positive integers. Then, for $\delta > 0$, we have*

$$I = \int_0^1 \left| \sum_{A < u \leqq A+B} \phi(u) e(yu^k) \right|^{2^k} dy \ll B^{2^k - k + \delta} \max_{A < u \leqq A+B} |\phi(u)|^{2^k}. \tag{25}$$

Proof. The lemma follows easily from Theorem 7 of [**7**].

We remark now that the drawback of the results of Lemmas 3 and 4 is that their estimates become trivial for Y near $N^{\frac{1}{2}}$ (in fact the situation is even worse in Lemma 3 for small m). The following lemma deals with estimation of sums where both ranges are quite large.

LEMMA 6. *Suppose* $Y \leqq N^{\frac{1}{2}}$, $q \leqq N^{\frac{1}{2}k}L^{\frac{1}{2}}$, $l \leqq L \leqq N$, $|\alpha q - a| < (N^k L)^{-\frac{1}{2}}$, $(a, q) = 1$, $\varepsilon > 0$. *Then*

$$\sum_{Y<y\leqq 2Y}' \psi(y) \sum_{x\leqq Ny^{-1}}' \phi(x)e(\alpha l y^k x^k) \ll FN^{1+\varepsilon}\left(\frac{1}{Y}+\frac{\lambda}{q}\right)^{2^{-k}}\left(1+\frac{L^{\frac{1}{2}}Y^k}{N^{\frac{1}{2}k}}\right)^{2^{-k}}. \tag{26}$$

Here $\lambda = (l, q)$.

Proof. Without loss of generality $Y = 2^t$, where t is an integer. Some notation is required in order to split the trigonometric sum in (26) into subsums. We define sets of integers C_m as follows for $0 \leqq m \leqq t$:

$$C_0 = \{Y\},\ C_m = \{y_m : y_m = Y + rY2^{1-m}, 0 \leqq r \leqq 2^{m-1}\}.$$

We put $Y_m = Y2^{-m}$, and write $\theta(y_m)$ for the set of integers x with $N(y_m + 2Y_m)^{-1} < x \leqq N(y_m + Y_m)^{-1}$ for $m > 0$. We define $\theta(y_0)$ as the set of integers x with $0 < x \leqq N(2Y)^{-1}$. Clearly

$$\sum_{Y<y\leqq 2Y}' \psi(y) \sum_{x\leqq N/y}' \phi(x)e(\alpha l y^k x^k) = \sum_{m=0}^{t} \sum_{y_m \in C_m} S(y_m) + O(N/y), \tag{27}$$

where

$$S(y_m) = \sum_{y=y_m}^{y_m+Y_m}{}' \psi(y) \sum_{x\in\theta(y_m)}' \phi(x)e(\alpha l x^k y^k). \tag{28}$$

We write $S_l(y)$ for the inner sum in (28). We shall consider m fixed at the moment and concentrate on one subsum $S(y_m)$. In the following the summation over x will be for $x \in \theta(y_m)$. We note that there are $\ll NY_m/Y^2$ numbers in $\theta(y_m)$, and $\ll Y/Y_m$ numbers in C_m. Write $X = Ny^{-1}$.

We now relate $S_l(y)$ to integrals in accordance with one of Vinogradov's methods. We make one important change in that we will use an infinite series of integrals rather than one integral plus an error. The saving this apparent innovation produces is only significant for small k; it makes no real difference to the result of Theorem I of Chapter 6 of [**12**], for instance. We have

$$|S_l(y)|^{2^k} \ll \sum_{r=0}^{\infty} \frac{X^{-rk2^k} I_r(y) X^k}{r!}, \tag{29}$$

where

$$I_r(y) = \int_{\mathscr{I}(y)} \left|\sum_x{}' x^{rk} e(ux^k)\phi(x)\right|^{2k} du. \tag{30}$$

Here

$$\mathscr{I}(y)=[ay^k l-\tfrac{1}{2}X^{-k},\ ay^k l+\tfrac{1}{2}X^{-k}].$$

(The reader familiar with Vinogradov's work should note that we have been able to make $\mathscr{I}(y)$ somewhat larger than usual; this requires us to use an infinite series but it will become apparent that this is no real problem.) To obtain (29), note that for any u,

$$\begin{aligned} S_l(y)&=\sum_x{}' \phi(x)e(ux^k)e(x^k(\alpha y^k l-u))\\ &=\sum_x{}' e(ux^k)\sum_{r=0}^{\infty}\frac{x^{kr}(\alpha y^k l-u)^r(2\pi i)^r}{r!}\phi(x). \end{aligned}$$

The interchanges of orders of summation and integration in the following working are easily justified. We have

$$\begin{aligned} S_l(y)&=X^k\int_{\mathscr{I}(y)}\sum_{r=0}^{\infty}\sum_x{}'\frac{e(x^k u)x^{kr}(\alpha y^k l-u)^r(2\pi i)^r\phi(x)}{r!}\,du\\ &=\sum_{r=0}^{\infty}\frac{(2\pi i)^r X^k}{r!}\int_{\mathscr{I}(y)}(ay^k l-u)^r\sum_x{}' e(x^k u)x^{kr}\phi(x)\,du\\ &\ll\sum_{r=0}^{\infty}\frac{(2\pi)^r X^k}{r!}\int_{\mathscr{I}(y)}|u-\alpha y^k l|^r\left|\sum_x{}' x^{kr}\phi(x)e(x^k u)\right|du\\ &\ll X^k\sum_{r=0}^{\infty}\frac{(2\pi)^r X^{-kr}}{r!}\int_{\mathscr{I}(y)}\left|\sum_x{}'\phi(x)x^{kr}e(x^k u)\right|du. \end{aligned}$$

By Holder's inequality,

$$\begin{aligned} |S_l(y)|^{2^k}&\ll X^{k2^k}\left(\sum_{r=0}^{\infty}\frac{(2\pi)^{2r}}{r!}\right)^{2^k-1}\sum_{r=0}^{\infty}\frac{X^{-kr2^k}}{r!}\left(\int_{\mathscr{I}(y)}\left|\sum_x{}'\phi(x)x^{kr}e(x^k u)\right|\right)^{2^k}du\\ &\ll X^k\sum_{r=0}^{\infty}\frac{X^{-rk2^k}}{r!}I_r \end{aligned}$$

by another application of Hölder's inequality. This establishes (29).

Our next task is to relate $S(y_m)$ to integrals over $[0,1)$ in a manner similar to Vinogradov (see [**11**]), and use Lemma 5 to obtain a good estimate for the integrals. We say two intervals $\mathscr{I}(y_1)$, $\mathscr{I}(y_2)$ *overlap mod* 1 if there is a real number x and an integer n such that $x\in\mathscr{I}(y_1)$ and $n+x\in\mathscr{I}(y_2)$. We will show that not many of the $\mathscr{I}(y)$ overlap mod 1. Using the periodicity of the integrand in I_r we may then get our required integrals.

Suppose $\mathscr{I}(y_1)$, $\mathscr{I}(y_2)$ overlap mod 1; then

$$\alpha l(y_1^k-y_2^k)=h+O(X^{-k}),$$

where h is an integer. Thus

$$\begin{aligned} al(y_1^k-y_2^k)&=hq+O(qX^{-k})+O(Y^k l(N^{\frac{1}{2}k}L^{\frac{1}{2}})^{-1})\\ &=hq+O(L^{\frac{1}{2}}Y^kN^{-\frac{1}{2}k}). \end{aligned}$$

https://doi.org/10.1017/S0017089500005024 Published online by Cambridge University Press

Since $(y_1, q) = (y_2, q) = 1$, there are

$$\ll \left(\frac{\lambda Y_m}{q}+1\right)q^{\varepsilon}$$

solutions of $y_1^k al \equiv b(\bmod q)$ in y_1, with $y_m \leqq y_1 \leqq y_m + Y_m$. Thus only

$$\left(1+\frac{L^{\frac{1}{2}}y^k}{N^{\frac{1}{2}k}}\right)\left(\frac{\lambda Y_m}{q}+1\right)q^{\varepsilon}$$

intervals $\mathscr{I}(y_1)$ can overlap mod 1 with a given $\mathscr{I}(y_2)$.

Write $V = \max_v |\psi(v)|^{2^k}$, $U = \max_u |\phi(u)|^{2^k}$. Then we deduce with one further application of Hölder's inequality that

$$|S(y_m)|^{2^k} \ll Y_m^{2^k} q^{\varepsilon}\left(1+\frac{L^{\frac{1}{2}}y^k}{N^{\frac{1}{2}k}}\right)\left(\frac{\lambda}{q}+\frac{1}{Y_m}\right) V \sum_{r=0}^{\infty} \frac{X^{k-rk2^k}}{r!} I'_r,$$

where

$$I'_r = \int_0^1 \left|\sum_x{}' x^{kr} e(ux^k)\phi(x)\right|^{2^k} du \ll X^{kr2^k}\left(\frac{NY_m}{Y^2}\right)^{2^k-k+\varepsilon} U$$

by Lemma 5. Hence

$$|S(y_m)|^{2^k} \ll F^{2^k} Y_m^{2^k} q^{\varepsilon}\left(1+\frac{L^{\frac{1}{2}}Y^k}{N^{\frac{1}{2}k}}\right)\left(\frac{\lambda}{q}+\frac{1}{Y_m}\right)\left(\frac{Y_m}{Y}\right)^{2^k-k} X^{2^k+\varepsilon}.$$

Thus

$$\begin{aligned}
&\left|\sum_{y_m \in C_m} S(y_m)\right|^{2^k} \\
&\quad \ll F^{2^k} Y_m^{2^k} q^{\varepsilon}\left(1+\frac{L^{\frac{1}{2}}Y^k}{N^{\frac{1}{2}k}}\right)\left(\frac{\lambda}{q}+\frac{1}{Y_m}\right)\left(\frac{Y_m}{Y}\right)^{2^k-k} X^{2^k+\varepsilon}\left(\frac{Y}{Y_m}\right)^{2^k} \\
&\quad < (F(XY)^{\frac{1}{2}\varepsilon+1})^{2k}\left(1+\frac{L^{\frac{1}{2}}Y^k}{N^{\frac{1}{2}k}}\right)\left(\frac{\lambda}{q}+\frac{1}{Y}\right) \qquad (31)
\end{aligned}$$

since

$$\left(\frac{Y_m}{Y}\right)^{2^k-k}\left(\frac{1}{Y_m}\right) \leqq \frac{1}{Y}.$$

The result of Lemma 6 follows easily from (31) since there are $O(\log N)$ subsums as given in (27) to consider. Slight modifications are necessary in the working for the sum with y range of length Y since the inner sum over x has the form $0 < x \leqq NY^{-1}$, but there are no added difficulties.

LEMMA 7. *Suppose we have the hypotheses of Lemma* 6, *with the added condition that*

$$N^{\frac{1}{2}k}L^{\frac{1}{2}} \ll q \ll N^{\frac{1}{2}k}L^{\frac{1}{2}}.$$

Then

$$\sum_{l=1}^{L} |\sigma_l| \ll (LN)^{1+\varepsilon} F\left(\frac{1}{LY} + \frac{1}{N^{\frac{1}{2}}}\right)^{2-k}, \tag{32}$$

where σ_l is the sum on the left hand side of (26).

Proof. We shall only outline the necessary modifications to the proof of Lemma 6. By the modulus inequality it suffices to estimate sums of the form (in the notation of (27), (28))

$$\sum_{Y_m \in C_m} \sum_{y}{}' |\psi(y)| \sum_{l=1}^{L} |S_l(y)|.$$

We proceed as before, relating $S_l(y)$ to the same series of integrals. This time however, we are interested in how the intervals are distributed as both y and l vary. We thus require the number of solutions of

$$y^k al \equiv b (\text{mod } q)$$

for y in a given range of Y_m numbers, $1 \leqq l \leqq L$. We have

$$ly^k = C + mq,$$

with

$$C \equiv ba^{-1} (\text{mod } q),$$

$$0 \leqq C < q, \qquad 0 \leqq m \leqq LY^k q^{-1} \ll L^{\frac{1}{2}} Y^k N^{-\frac{1}{2}k}.$$

The number of intervals $\mathscr{I}(y)$ which overlap with a given interval is thus

$$\ll \left(1 + \frac{L^{\frac{1}{2}}Y^k}{N^{\frac{1}{2}k}}\right)^2 q^{\varepsilon}$$

$$\ll (1 + LY^{2k}N^{-k})q^{\varepsilon}.$$

This is a saving of a factor $(1/LY_m + Y^{2k}/N^k Y_m)q^{\varepsilon}$ over the trivial estimate. The remainder of the proof follows without difficulty.

3. Proof of Theorems 1, 2 and 4.

Proof of Theorem 1. The sum on the left of (1) differs from the sum on the left of (6) by $O(N^{\frac{1}{2}})$; so we need only estimate sums of the type I and II ((7) and (8)). For $Y \geqq N^{1/k}$, we use Lemma 6 (with $l = L = 1$). This gives an upper bound

$$\ll N^{1+\frac{1}{2}\varepsilon}(N^{-1/k} + q^{-1})^{2-k}$$

$$\ll N^{1-\gamma+\frac{1}{2}\varepsilon}. \tag{33}$$

For $N^{1/k} > Y \geqq N^{(k2^{k-1})^{-1}}$, we use Lemma 3 with $m = 2$. This also gives an estimate (33).

Finally, for $Y < N^{(k2^{k-1})^{-1}}$, we apply Lemma 3 with $m = 1$, which also gives the bound (33). As there are only $O(\log N)$ sums of the type I and II, (1) follows from (33). We note that if we put a more restrictive condition on the size of q the result could be somewhat improved. In general, however, the application of results like (1) require as large a range of q as possible (see **[2]** where we use this result).

Proof of Theorem 2. As α is irrational there are infinitely many convergents to its continued fraction. Let a/q be one such convergent. Pick N so that

$$q = [N^{\frac{1}{2}k + \frac{1}{2}\xi - \frac{1}{2}\varepsilon}]$$

and put

$$L = N^{\xi - \varepsilon}.$$

It follows from Lemma 5 of **[6]** that we need only show that

$$\sum_{l=1}^{L} \left| \sum_{p \leqq N} (\log p) e(\alpha p^k l) \right| = o(N) \tag{34}$$

in order to establish a solution of (2) with $N^{\frac{1}{2}} < p \leqq N$. Since α is irrational and we picked a sequence of convergents with $q \to \infty$ the result of Theorem 2 follows. As in the case of Theorem 1, we need only consider sums of the type I and II, but we here add on extra summation over l.

Put

$$\rho = k(2k + 2 + (2^k - 1)^{-1})^{-1}.$$

Then

$$\xi + \rho = 2^k \xi, \tag{35}$$

and

$$k\rho - \tfrac{1}{2}\xi - \tfrac{1}{2}k = -2^k \xi, \qquad \rho - 1 < -2^k \xi. \tag{36}$$

We estimate sums of type (I) by Lemma 7 if $N^{\rho} < Y \leqq N^{\frac{1}{2}}$. There are $\ll \log N$ such sums, and by (35), (32) we get an upper bound for the total of these sums of

$$\begin{aligned}
&\ll (LN)^{\frac{1}{4}\varepsilon + 1} N^{-(\xi - \varepsilon + \rho)2^{-k}} (\log N) \\
&= N^{\xi + 1 - \xi - \varepsilon + \frac{1}{4}\varepsilon - \frac{1}{4}\varepsilon^2 + \varepsilon 2^{-k} + \frac{1}{4}\xi\varepsilon} (\log N) \\
&= o(N).
\end{aligned}$$

Assuming, as we may, that ε is sufficiently small. We have used the fact that $F = O(N^{\varepsilon/8})$ to obtain this result.

For $N^{\frac{1}{3}} \leqq Y \leqq N^{\rho}$, we estimate sums of type (I) by the case of Lemma 4 with $\phi(x) \not\equiv 1$. It follows from (36) that we get a bound which is $o(N)$ for these sums as well.

We estimate sums of type (II) by the case of Lemma 4 with $\phi(x) \equiv 1$. Here the estimate is

$$N^{1+\varepsilon}L(N^{-\frac{1}{6}k}+N^{-\frac{2}{3}})^{2^{1-k}}$$

which is certainly $o(N)$. This establishes (34) and thus completes the proof of Theorem 2.

Proof of Theorem 4. Let α be the leading coefficient of f. Choose N and L as in the proof of Theorem 2, but replacing ξ by σ. Write $Y=N^{2\sigma}$. Let N be the collection of all numbers of the form p_1p_2 where p_1, p_2 are primes and

$$Y<p_1\leqq 2Y, \qquad 4Y^2<p_1p_2\leqq N.$$

We note there are $\gg N(\log N)^{-2}$ such numbers. It thus suffices, by Lemma 5 of [**6**], to prove that

$$\sum_{l=1}^{L}\left|\sum_{n\in N} e(lf(n))\right| = o(N(\log \mathrm{N})^{-2}). \tag{37}$$

For $k=2$, (37) follows from a suitable variant of Lemma 3 (by adding an extra range of summation), taking $m=2$ and making obvious choices for ϕ and ψ. For $k\geqq 3$, (37) is established directly from Lemma 4 (17). This completes the proof of Theorem 4.

We now include a brief demonstration of an improvement upon Graham's result. *For irrational α and arbitrary real β there are infinitely many solutions of*

$$\|\alpha P_2+\beta\|<cP_2^{-\frac{1}{3}}(\log P_2)^{\frac{4}{3}}.$$

Here c is a numerical constant which can be evaluated.

To prove this, let a/q be a convergent to the continued fraction of α, $q>10^6$. Choose X as the largest integer with

$$q>X^2(\log Xq)^{-2}.$$

Put

$$N=Xq, \qquad L=[c_1N^{\frac{1}{3}}(\log N)^{-\frac{4}{3}}],$$

where c_1 is a constant <1. We note that $LX<q$. From Lemma 5 of [**6**] it suffices to show that

$$S_L=\sum_{l=1}^{L}\left|\sum_{X^{\frac{1}{2}}<p_1<X}\ \sum_{X<p_2\leqq q} e(\alpha l p_1 p_2)\right|<\frac{M}{6},$$

where M is the number of P_2 numbers of the form p_1p_2 occurring in the above sum. Clearly $M\gg N(\log N)^{-2}$. We have, by the modulus inequality,

$$S_L\leqq \sum_{m=1}^{LX} h(m)\left|\sum_{x<p\leqq q} e(\alpha m p)\right|.$$

Here $h(m)$ is the number of representations of m as lp_1 with $X^{\frac{1}{2}}<p_1\leqq X$. We observe that

$h(m) \leqq 5$ since $m < X^{5/2}$. As $h(m)$ is non-zero for $\ll LX(\log X)^{-1}$ numbers m, by Cauchy's inequality we have

$$S_L^2 \ll \frac{LX}{(\log X)} \sum_{m=1}^{LX} \left| \sum_{X<p\leqq q} e(\alpha mp) \right|^2.$$

We may now use the well known large sieve inequality (see [**8**]) to estimate the above sum. This is more economical than a similar method in [**3**] which was first given by Vinogradov. We get

$$\begin{aligned} S_L^2 &\ll \frac{LX}{\log X} q \sum_{X<p\leqq q} 1 \\ &\ll \frac{c_1 q^3}{(\log N)^2} \\ &\ll \frac{c_1 N^2}{(\log N)^4}. \end{aligned}$$

By choosing c_1 sufficiently small the result follows.

We remark that the above method can be adapted to prove a result like Theorem 4 but with the weaker exponent $k((2^k-1)(2k-1)+2^k)^{-1}$.

4. Proof of Theorem 3. We first require some more lemmas.

LEMMA 8. *Let $\phi(x)$ be an arbitrary function, A and B integers. Then, for $\varepsilon > 0$, we have*

$$I = \int_0^1 \dots \int_0^1 \left| \sum_{A<u\leqq A+B} \phi(u) e(\alpha_k u^k + \alpha_{k-1} u^{k-1} + \dots + \alpha_1 u) \right|^T d\alpha_1 \dots d\alpha_k$$

$$\ll B^{T-\frac{1}{2}k(k+1)+\varepsilon} \max_{A<u\leqq A+B} |\phi(u)|^T, \quad (38)$$

where T is given by the table in the statement of Theorem 3 for $k \leqq 11$. For $k \geqq 12$, we take

$$T = 4[k^2(\log k + \tfrac{1}{2}\log\log k + 1.3)].$$

Proof. The lemma follows easily from Theorem 7 of [**7**] and Theorem 4 of [**13**].

LEMMA 9. *Under all the hypotheses of Lemma 7 with α as the leading coefficient of f, we have*

$$\sum_{l=1}^{L} |\sigma_l| \ll (LN)^{1+\varepsilon} F\left(\frac{1}{LY} + \frac{1}{N^{1/2}}\right)^{1/T}, \quad (41)$$

where T is as given in Lemma 8, and σ_l is the sum of Lemma 7 with $f(n)$ replacing αn^k.

Proof. The proof follows as for Lemmas 6 and 7 with Lemma 8 replacing Lemma 5.

The only real difference is that (29) becomes

$$|S_l(y)|^T \ll \sum_{r_1=0}^{\infty} \dots \sum_{r_k=0}^{\infty} \frac{X^{-RT} I_R(y) X^{\frac{1}{2}k(k+1)}}{r_1! r_2! \dots r_k!},$$

where

$$R = r_1 + 2r_2 + \dots + kr_k.$$

$$I_R = \int_{\mathcal{I}_1(y)} \dots \int_{\mathcal{I}_k(y)} \left| \sum_x{}' \phi(x) X^R e(\alpha_k x^k + \dots + \alpha_1 x) \right|^T d\alpha_1 \dots d\alpha_k$$

and

$$\mathcal{I}_s(y) = [a_s y^s l - \tfrac{1}{2} X^{-s}, a_s y^s l + \tfrac{1}{2} X^{-s}] \quad (s = 1, \dots, k),$$

where a_s is the coefficient of x^s in $f(x)$ (so $a_k = \alpha$).

Lemma 10. *Suppose* $k \geqq 12$, $N^{\frac{1}{2}k} \leqq q \leqq N^{13k/24}$, $L \leqq N$, $(a, q) = 1$, $|\alpha - a/q| \leqq (N^k L)^{-1}$, $Y \leqq N^{\frac{1}{3}}$. *Write, for* $l \leqq L$,

$$S_l = \sum_{Y \leqq y \leqq 2Y}{}' \psi(y) \sum_{x \leqq Ny^{-1}}{}' e(lf(xy)).$$

Then

$$S_l \ll FN^{1-(3T)^{-1}}.$$

Proof. Working analogously to Lemmas 2 and 4, we need only estimate

$$S_x(d) = \sum_{n=1}^{x/d} e\left(\frac{aly^k d^k n^k}{q} + lg(ydn) \right).$$

Here $x \leqq NY^{-1}$, g is a polynomial of degree $k-1$ and $(y, q) = 1$. Sums with $d > N^{\frac{1}{4}}$ contribute $\ll N^{\frac{3}{4}+\varepsilon}$ to S_l by a trivial estimate; so we may assume $d \leqq N^{\frac{1}{4}}$. Similarly we can presume $x/d \geqq N^{\frac{5}{8}}$. Let

$$\frac{aly^k d^k}{q} = \frac{b}{q'} \quad \text{with} \quad (b, q') = 1.$$

We have

$$\left(\frac{x}{d}\right)^2 \leqq N^2 \leqq \frac{N^{\frac{1}{4}k}}{L} \leqq q' \leqq N^{\frac{5}{8}k - \frac{1}{12}k} \leqq \left(\frac{x}{d}\right)^{k-1}.$$

We are thus able to apply Theorem I of Chapter 6 of **[12]** to $S_x(d)$ to get the estimate

$$\left(\frac{x}{d}\right) N^{-\delta},$$

where $\delta = 5(24\, k^2 \log(12k(k+1)))^{-1}$. This is more than good enough to prove this lemma. We remark that although we have thrown a lot away in this proof, there is no point in being more precise, since the sticking point in the proof of Theorem 3 is the estimation of sums of type I (i.e. (7)).

Proof of Theorem 3. For $k \leqq 11$, the proof follows as for Theorem 2, only using Lemma 9 in place of Lemma 7. As we remarked at the end of Lemma 4's proof, there is no problem in changing αn^k to $f(n)$. The value corresponding to ρ in the proof of Theorem 2 is

$$\rho' = (T-1)(2T + (2^{k+1} - 1 - 2k)/k)^{-1},$$

which satisfies

$$\rho' + \tau = T\tau,\ k\rho' - \tfrac{1}{2}\tau - \tfrac{1}{2}k = -2^k\tau;\ \rho' - 1 < -2^k\tau.$$

For $k \geqq 12$, the proof follows from Lemmas 1, 9 and 10.

REFERENCES

1. R. C. Baker, Small fractional parts of the sequence αn^k, *Michigan Math. J.* **28** (1981), 223–228.

2. R. C. Baker and G. Harman, Diophantine approximation by prime numbers, *J. London Math. Soc.* (2) **25** (1982), 201–215.

3. T. Estermann, A new result in the additive prime number theory, *Quart. J. Math. Oxford Ser.* (2) **8** (1937), 32–38.

4. O. M. Fomenko, The distribution of numbers with a fixed number of different prime divisors, *Studies in number theory 5, Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov.* (*LOMI*) **82** (1979), 158–164, 168.

5. S. W. Graham, Diophantine approximation by almost primes, unpublished.

6. G. Harman, Trigonometric sums over primes I, *Mathematika*, **28** (1981), 249–254.

7. L. K. Hua, *Additive theory of prime numbers*, *Amer. Math. Soc. Transl.* **13** (1965).

8. H. L. Montgomery, The analytic principle of the large sieve, *Bull. Amer. Math. Soc.* **84** (1978), 547–567.

9. R. C. Vaughan, Sommes trigonometriques sur les nombres premiers, *C.R. Acad. Sci. Paris Ser. A*, **285** (1977), 981–983.

10. R. C. Vaughan, An elementary method in prime number theory, *Acta Arith.* **37** (1980), 111–115.

11. I. M. Vinogradov, On the estimates of a trigonometric sum over primes, *Izv. Akad. Nauk SSSR Ser. Mat.* **12** (1948), 225–248.

12. I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers* (English translation by A. Davenport and K. F. Roth of 1947 Russian monograph, Wiley-Interscience, 1954).

13. I. M. Vinogradov, *The method of trigonometrical sums in the theory of numbers* (Russian revised edition, Moscow, 1971).

Royal Holloway College
Egham
Surrey, TW20 OEX