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CORRECTION TO ON THE BRAUER GROUP OF ALGEBRAS HAVING A GRADING AND AN ACTION

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Let R be a commutative ring, G a finite abelian group. Let A be an R-algebra which is graded by G (i.e. $A = \sum \oplus_{\sigma \in G} A_\sigma$, where $A_\sigma A_\tau \subset A_{\sigma\tau}$ for σ, τ in G) and for which A_1 is an R-module of finite type. In Remark 4.1 (a) of [**1**] we asserted that under these hypotheses if u is in A and $u + pA$ is homogeneous in A/pA for each maximal ideal p of R then u is homogeneous in A. We used this assertion for u a unit in A such that $a \rightarrow uau^{-1}$ is a grading-preserving homomorphism. K. Ulbrich has kindly pointed out a counterexample to the assertion: $R = \mathbf{Z}/4\mathbf{Z}$, $G = \{1, \sigma\}$, $u = 2\sigma + 1$, $p = 2R$. Proposition 4.2 of [**1**] uses the erroneous result and is in turn invoked later in the paper. The theorem stated and proved below gives a statement that can be invoked instead of Proposition 4.2 in the proof of Theorem 4.4. However, our proof is not valid for all R and our restriction must be imposed in Sections 4 and 5 of [**1**].

THEOREM. *Let R be a connected commutative ring for which* Spec (R) *has finitely many irreducible components. Let A be an Azumaya R-algebra graded by a finite abelian group G. Let u be a unit in A such that $a \rightarrow uau^{-1}$ preserves the grading on A. Then u is homogeneous.*

Proof. If $R \rightarrow S$ is any homomorphism of commutative rings then the hypotheses on A, u hold for $S \otimes_R A$, $1 \otimes u$. We shall use this fact repeatedly without explicitly mentioning it.

If R is a field the proof of Proposition 4.2 of [**1**] applies to show our theorem is valid in this case. It follows that the theorem holds for R any integral domain.

We show next that the theorem holds if R is any local ring. Let m be the maximal ideal of R. Write $u = \sum_{\sigma \in G} u_\sigma$ with u_σ in A_σ. In A/mA, $u + mA = u_\gamma + mA$ for some fixed γ in G and $u_\gamma \notin mA$. The proof of Proposition 4.2 of [**1**] applies to this case up to where the equation

(1) $\quad wu = \sum w(\sigma) u_\sigma = \sum r u_\sigma$

is obtained for w an arbitrary element of GR. Because u_σ is in A_σ and $A = \sum \oplus A_\sigma$, $w(\sigma) = r u_\sigma$ for each σ in G. In particular

(2) $\quad (w(\gamma) - r) u_\gamma = 0.$

Because R is local and A is an Azumaya R-algebra, A_σ is R-free for each σ. Let $x_1, \ldots, x_s$ be an R-basis for A_γ and write $u_\gamma = \sum_{i=1}^{s} a_i x_i$. Equation

(2) yields $(w(\gamma) - r)a_i = 0$ for each $i = 1, \ldots, s$. Because $u_\gamma \notin mA$, at least one a_i is a unit in R. Hence $w(\gamma) = r$. Equation (1) then gives $wu = w(\gamma)u$. But this implies u is in A_γ, since in the proof of Proposition 4.2 of [**1**] it is established that

$$A_\sigma = \{x \in A | wx = w(\sigma)x \text{ for } w \text{ in } GR\}$$

for each σ in G. Thus the theorem holds for R a local ring.

Now let R be a ring satisfying the hypotheses of the theorem. For p a prime ideal of R, R/p is an integral domain. Then $u + pA$ is homogeneous in A/pA. Let $\sigma(p)$ be the unique element of G satisfying $u_{\sigma(p)} + pA = u + pA$. Then $\sigma(\ )$: Spec $(R) \to G$ gives a map which clearly satisfies the condition

(3) $p \subset q$ implies $\sigma(p) = \sigma(q)$ for p, q in Spec (R).

This condition implies σ is constant on irreducible components of Spec (R). These components are closed and finite in number. Since Spec (R) is connected it follows that no union of irreducible components is disjoint from the union of the rest. Hence $\sigma(\ )$ is constant on Spec (R). Let $\sigma(p) = \gamma$. For each p, $u/1$ is homogeneous in the localization A_p. Because R_p/pR_p is the quotient field of R/p, it follows that $u/1$ is homogeneous of grade γ in A_p for each prime ideal p of R. Thus, for $\sigma \neq \gamma$, $u_\sigma/1$ is 0 in A_p for each prime ideal p of R. Hence $u_\sigma = 0$ for $\sigma \neq \gamma$, so u is homogeneous.

Remark. Condition (3) by itself, without some assumption on R such as finitely many irreducible components, is not sufficient to conclude that $\sigma(\ )$ is constant on Spec (R). As C. A. Weibel has pointed out to us, the ring R of continuous real-valued functions on the unit interval I admits the following structure on its prime ideals: Each prime ideal is contained in a unique maximal ideal. The maximal ideals of R are in bijective correspondence with the points of I, so the functions $\sigma(\ )$: Spec $(R) \to G$ satisfying (3) are in bijective correspondence with the set maps of I to G.

Reference

1. M. Orzech, *On the Brauer group of algebras having a grading and an action*, Can. J. Math. *28* (1976), 533–552.

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