A NOTE ON COMMUTATIVE BAER RINGS III

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(Received 29 July 1970)

Communicated by G. B. Preston

Introduction

If R is a commutative semiprime ring with identity Kist [4], [5] has shown that R can be embedded into a commutative Baer ring $B(R)$, and has given some properties of this embedding. More recently Mewborn [7] has given a construction which embeds R into a commutative Baer ring with the stronger property that every annihilator is generated by an idempotent. Both of these constructions involve a representation of R as a ring of global sections of a sheaf over a Boolean space.

In this note we do two things — firstly we give a unification of the abovementioned results by constructing a family of extensions of R, the smallest of which is Kist's and the largest Mewborn's; secondly we give entirely algebraic constructions which relate to ones used in the theory of l-groups [2], [3]. Our extensions reduce to the familiar m-completions of R [8] in the case R a Boolean algebra, and we thus generalise the result of Brainerd and Lambek, see [6].

The author would like to thank Dr. A. C. Mewborn for supplying the preprint [7].

1. Preliminaries

For our notation we follow the previous notes: in particular if $a \in R$ where R is a commutative ring, we write $(a)_R$ $[(a)^*_R]$ for the principal ideal generated by [annihilator of] a; subscripts will be dropped when no confusion is likely to result. $|A|$ denotes the cardinality of A and m denotes a cardinal greater than 1. Also we write E_R for the Boolean algebra of idempotents of the commutative ring R.

DEFINITION 1.1 *A commutative ring B is called a commutative Baer m-ring [commutative complete Baer ring] if for any $S \subseteq B$ with $|S| \leqq m$ [if for any $S \subseteq B$] there is an idempotent $e \in E_B$ with $(S)^*_B = (e)_B$.*

In the case m finite we call B a commutative Baer ring, see [10], [11].

https://doi.org/10.1017/S1446788700012702 Published online by Cambridge University Press

DEFINITION 1.2 *A ring morphism* $\phi: R \to R'$ *is said to be* ρ_m*-compatible if for any* $S, T \subseteq R$ *with* $|S| \leqq m$, $|T| \leqq m$ *and* $(S)^*_R = (T)^*_R$, *we have* $(S\phi)^*_{R'} = (T\phi)^*_{R'}$.

LEMMA 1.3 *Let J be an ideal of the commutative Baer m-ring B. Then the following are equivalent*:

(i) *For any* $S \subseteq B$ *with* $|S| \leqq m$, *we have* $S \subseteq J$ *iff* $(S)^{**} \subseteq J$.

(ii) *For* $S, T \subseteq B$ *with* $|S| \leqq m$, $|T| \leqq m$ *and* $(S)^* = (T)^*$ *we have* $S \subseteq J$ *iff* $T \subseteq J$.

(iii) *J is a Baer ideal (see* [10]) *and* $J \cap E_B$ *is a Boolean m-ideal.*

The simple proof is omitted.

We call an ideal J satisfying the conditions of 1.3 a *Baer m-ideal*; if J is a Baer m-ideal for all cardinals m, we call J a *complete Baer ideal.* The following lemma also has its easy proof omitted.

LEMMA 1.4 *Let* $\phi: B \to B'$ *be a surjective ring morphism between two commutative Baer m-rings. Then the following are equivalent*:

(i) ϕ *is* ρ_m*-compatible.*

(ii) *ker* ϕ *is a Baer m-ideal.*

(iii) ϕ *is a Baer morphism (see* [10]) *and* $\phi | E_B$ *is a Boolean m-morphism.*

A ring morphism satisfying the conditions of 1.4 is called a Baer *m-morphism*; if ϕ is a Baer m-morphism for all cardinals m, we call ϕ a *complete Baer morphism.*

2. Baer *m*-Extensions

The construction which follows was suggested by Conrad's direct limit construction of the orthocompletion of a representable l-group [3] which goes back, via Bernau [2], to Amemiya [1]. We also recall that Kist's construction of the Baer extension of a commutative semiprime ring derived from [1].

Let R be a commutative semiprime ring and denote by $A(R)$ the complete Boolean lattice of all annihilator ideals of R, Lambek [6] p. 43. If we write $\mu_R = \{(a)^{**} : a \in R\}$ then it is easily seen that μ_R is a dense sub-semi-lattice of $A(R)$ and that $A(R)$ is the normal completion of $\bar{\mu}_R$, the Boolean sublattice of $A(R)$ generated by μ_R. By $A_m(R)$ (m an infinite cardinal) we mean the Boolean m-sublattice of $A(R)$ generated by μ_R; clearly $A_m(R)$ is the Boolean m-completion of $\bar{\mu}_R$.

A finite partition of $A_m(R)$ is a family $\mathscr{D}$ of elements of $A_m(R)$ such that for distinct $D, E \in \mathscr{D}$ we have $D \cap E = (0)$, and $(\Sigma_{D \in \mathscr{D}} D)^{**} = R$. Let $\pi_m(R)$ be the (directed) set of all finite partitions of $A_m(R)$; we will define a family $\{R_{\mathscr{D}} : \mathscr{D} \in \pi_m(R)\}$ of commutative rings and a family of ring morphisms $\pi_{\mathscr{C}\mathscr{D}} : R_{\mathscr{C}} \to R_{\mathscr{D}}$ whenever $\mathscr{C} \leqq \mathscr{D}$.

https://doi.org/10.1017/S1446788700012702 Published online by Cambridge University Press

(i) For $\mathscr{D} \in \pi_m(R)$ put $R_{\mathscr{D}} = \times_{D \in \mathscr{D}} R/D^*$.

(ii) For $\mathscr{C} \leqq \mathscr{D}$ in $\pi_m(R)$ we proceed as follows: write $C = (\sum_\delta D_\delta)^{**}$ for any $C \in \mathscr{C}$; then $C^* = \cap_\delta D_\delta^*$ and so we obtain a canonical isomorphism of R/C^* into $\times_\delta R/D_\delta^*$. Doing this for all C we obtain a canonical isomorphism

$$\pi_{\mathscr{C}\mathscr{D}}: \mathop{\times}_{C \in \mathscr{C}} R/C^* \to \mathop{\times}_{D \in \mathscr{D}} R/D^*.$$

Now the family $\{R_{\mathscr{D}}, \pi_{\mathscr{C}\mathscr{D}}: \mathscr{C}, \mathscr{D} \in \pi_m(R), \mathscr{C} \leqq \mathscr{D}\}$ forms a direct system of commutative rings and we write $B_m(R) = \varinjlim_{\mathscr{D}} R_{\mathscr{D}}$ for the direct limit taken as $\mathscr{D} \in \pi_m(R)$. Let $\beta: R \to B_m(P)$ be the injection embedding R into $B_m(R)$ as a subring.

Lemma 2.1 *Let $x \in B_m(R)$. Then there is a family $\{e_i: 1 \leqq i \leqq n\}$ of orthogonal idempotents such that $\sum_i e_i = 1$ and a family $\{a_i: 1 \leqq i \leqq n) \subseteq R$ such that*

$$x = \sum_i (a_i\beta)e_i$$

Further the idempotents $\{e_i\}$ can be represented by elements $\{\langle 1 + D_i^*, 0 + D_i\rangle\}$ where $\{D_i: 1 \leqq i \leqq n\} \in \pi_m(R)$.

Proof. By our construction x can be represented by an ordered n-tuple

$$\langle x_i + D_i^*: 1 \leqq i \leqq n\rangle$$

where $\{D_i: 1 \leqq i \leqq n\} \in \pi_m(R)$ and $\{x_i\} \subseteq R$. Put

$$e_i = \langle \delta_{ij} + D_j^*: 1 \leqq j \leqq n\rangle$$

where δ_{ij} is the Kronecker delta, $a_i = x_i$ and the Lemma follows.

Call the representation given in 2.1 the *standard form* for $x \in B_m(R)$.

Lemma 2.2 *$B_m(R)$ is a commutative Baer ring.*

Proof. For $a \in R$ we define $(a\beta)^* \in B_m(R)$ to be the element represented by

$$\langle 0 + (a)^*, \ 1 + (a)^{**}\rangle,$$

and we will prove that $(a\beta)^*_{B_m(B)} = ((a\beta)^*)_{B_m(R)}$, i.e. that the annihilator of $a\beta$ is the principal ideal of $B_m(R)$ generated by the idempotent element $(a\beta)^*$ just defined.

Take $x \in B_m(R)$ in standard form, $x = \sum_i (a_i\beta)e_i$. Then $(a\beta)x =$ if and only if

$$(a\beta)(a_i\beta)e_i = 0 \ \ (1 \leqq i \leqq n),$$

and we will now show that this is the case if and only if

$$(a\beta)^* \ (a_i\beta)e_i = (a_i\beta)e_i \ (1 \leqq i \leqq n).$$

Suppose $e_i = \langle 1 + D_i^*, 0 + D_i\rangle$ where $D_i \in A_m(R)$; then

$$(a_i\beta)e_i = \langle a_i + D_i^*, \ 0 + D_i\rangle.$$

We now refine $\{(a)^*, (a)^{**}\}$ and $\{D_i^*, D_i\}$ to

$$\{(a)^* \cap D_i^*,\ (a)^{**} \cap D_i^*,\ (a)^* \cap D_i,\ (a)^{**} \cap D_i\}$$

and then relative to this partition we have:

$$(a_i\beta)e_i = \langle a_i + (D_i \cap (a)^*)^*,\ a_i + (D_i \cap (a)^{**})^*,\ 0 + (D_i^* \cap (a)^*)^*,\ 0 + (D_i^* \cap (a)^{**})^* \rangle$$

$$(a_i\beta)^* = \langle 1 + (D_i \cap (a)^*)^*,\ 0 + (D_i \cap (a)^{**})^*,\ 1 + (D_i^* \cap (a)^*)^*,\ 0 + (D^* \cap (a)^{**})^* \rangle$$

From these expressions we see that we have $(a\beta)^*(a_i\beta)e_i = (a_i\beta)e_i$ if and only if $a_i \in (D_i \cap (a)^{**})^*$, while $(a\beta)(a_i\beta)e_i = 0$ iff $aa_i \in D_i^*$. Thus we will be nearly through when we have proved the following:

SUBLEMMA. $a_i \in (D_i \cap (a)^{**})^*$ iff $a\,a_i \in D_i^*$.

PROOF. Suppose $aa_i \in D_i^*$ and let $t \in D_i \cap (a)^{**}$. Then $taa_i \in (a)^{**}$ and also $t\,a\,a_i = 0$ whence $t\,a_i = 0$ proving that $D_i \cap (a)^{**} \subseteq (a_i)^*$ and so

$$a_i \in (a_i)^{**} \subseteq (D_i \cap (a)^{**})^*.$$

For the converse assume that $a_i \in (D_i \cap (a)^{**})^*$ and take $t \in D_i$. Then $t\,a \in D_i \cap (a)^{**}$ whence $t\,a\,a_i = 0$ proving that $a\,a_i \in D_i^*$.

We have thus shown that for any $a \in R$ the annihilator $(a\beta)_{B_m(R)}$ is a direct summand of $B_m(R)$. Now for an arbitrary element y (in standard form) $y = \Sigma_j (b_j\beta)f_j$

$$(y)^*_{B_m(R)} = \bigcap_j ((b_j\beta)f_j)^*_{B_m(R)} = \bigcap_j ((b_j\beta)^{**}f_j)^*_{B_m(R)}$$

which is certainly idempotent generated. Thus $B_m(R)$ is a commutative Baer ring.

LEMMA 2.3 *$B_m(R)$ is a commutative Baer m-ring.*

PROOF. In the previous lemma we saw that for any $x \in B_m(R)$ there was an idempotent x^* such that $(x)_{B_m(R)} = (x^*)_{B_m(R)}$. An examination of the construction shows that each such x^* is of the form $\langle 0 + D^*, 1 + D\rangle$ for some $D \in A_m(R)$. Take a subset $S \subseteq B_m(R)$ with $|S| \leqq m$; then

$$\begin{aligned} S^* &= \cap\{(s)^* : s \in S\} \\ &= \cap\{(s^*) : s \in S\} \\ &= \cap\{(\langle 0 + D(s)^*,\ 1 + D(s)\rangle) : s \in S\} \text{ where } D(s) \in A_m(R) \text{ for } s \in S, \\ &= (e)_{B_m(R)}, \end{aligned}$$

where $e = \langle 0 + \cap_s D(s)^*, 1 + (\cap_s D(s)^*)^* \rangle$. Thus $B_m(R)$ is a commutative Baer m-ring; in fact we have also proved the following:

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COROLLARY 2.4 *The map* $D \to \langle 1 + D^*, 0 + D \rangle$ *defines an isomorphism* $A_m(R) \cong E_{B_m(R)}$.

We now collect the preceding results and prove a characterisation of the extension $B_m(R)$ of R.

THEOREM 2.5 *Let R be a commutative semiprime ring. Then there is a commutative Baer m-ring* $B_m(R)$ *and a* ρ_m*-compatible ring monomorphism* $\beta: R \to B_m(R)$ *with the following property: for any* ρ_m*-compatible ring morphism* $\phi: R \to B$ of R *into a commutative Baer m-ring B there is a unique Baer m-morphism* $\bar{\phi}: B_m(R) \to B$ *such that* $\beta \circ \bar{\phi} = \phi$. *Further, the pair* $(\beta, B_m(R))$ *is unique.*

PROOF. We refer to the preceding lemmas for the construction of $B_m(R)$ with the embedding β. To prove that β is ρ_m-compatible take, S, $T \subseteq R$ with $|S| \leqq m$, $|T| \leqq m$ and $S^* = T^*$ in R. Then $(S\beta)^*_{B_m(R)} = (e)_{B_m(R)}$ and we readily see that $e = \langle 0 + S^*, 1 + S^{**} \rangle$ whence $e = \langle 0 + T^*, 1 + T^{**} \rangle$ and so $(S\beta)^*_{B_m(R)} = (T\beta)^*_{B_m(R)}$.

Let $\phi: R \to B$ be a ρ_m-compatible ring morphism into a commutative Baer m-ring. We extend ϕ to $B_m(R)$ as follows: for $a \in R$ put $(\alpha\beta)\bar{\phi} = a\phi$; for $e = \langle 0 + S^*, 1 + S^{**} \rangle$ put $e\bar{\phi} = (S\phi)^*$ where $(S\phi)^*$ is the idempotent generator of $(S\phi)_{B^*}$ in B. Finally if $x = \Sigma_i (a_i\beta)e_i$ put $x\bar{\phi} = \Sigma(a_i\phi)(e_i\bar{\phi})$. It is easy to check that $\bar{\phi}$ is well defined and it also follows from the fact that ϕ is ρ_m-compatible that $\bar{\phi}$ is a Baer m-morphism. Clearly $\beta \circ \bar{\phi} = \phi$.

Finally standard category arguments establish that the pair $(\beta, B_m(R))$ is unique; we have in fact constructed a left adjoint for the forgetful functor from commutative Baer m-rings (with Baer m-morphisms) to commutative semiprime rings (with the usual ring morphisms).

The following theorem (whose proof we omit) gives another characterisation of the pair $(\beta, B_m(R))$.

THEOREM 2.6 *The pair* $(\beta, B_m(R))$ *satisfy the following conditions*:

(i) $\beta: R \to B_m(R)$ *is a* ρ_m*-compatible ring monomorphism of R into a commutative Baer m-ring*;

(ii) *The induced map* $\beta_*: \mu_R \to E_{B_m(R)}$ *given by* $(a)^{**}\ \beta_* = (a\beta)^{**}$ *embeds* μ_R *as a dense subsemi-lattice of* $E_{B_m(R)}$ *and* β_* *lifts to an isomorphism* $A_m(R) \cong E_{B_m(R)}$;

(iii) *For any* $x \in B_m(R)$ *there are elements* $\{a_i\} \subseteq R$ *and orthogonal idempotents* $\{e_i\}$ *such that* $\Sigma_i\, e_i = 1$ *and* $x = \Sigma_i (a_i\beta)e_i$.

Conversely, if $(k, K_m(R))$ *is an extension of R satisfying* (i), (ii), (iii) *above, then* $K_m(R)$ *and* $B_m(R)$ *are Baer m-isomorphic over R.*

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3. Relation to $Q(R)$

We close this note by indicating how the extensions $B_m(R)$ fit inside $Q(R)$, the complete ring of quotients of R, as R-subalgebras.

PROPOSITION 3.1. *$B_m(R)$ is isomorphic to an R-subalgebra of $Q(R)$.*

PROOF. We recall that $Q(R) = \lim_{\rightarrow} \mathrm{Hom}_R(\Delta, R)$ where the direct limit is taken over all dense ideals Δ of R. Now for $\mathscr{D} \in \pi_m(R)$, $\Sigma_{D \in \mathscr{D}} D$ is a dense ideal of R, and we will see that there is a canonical isomorphism:

$$\delta_{\mathscr{D}}: R_{\mathscr{D}} \to \mathrm{Hom}_R\left(\sum_{D \in \mathscr{D}} D, R\right).$$

For, if $\langle x(D) + D^* \rangle_{D \in \mathscr{D}}$ is an element of $R_{\mathscr{D}}$, then the map $\langle x(D) + D^* \rangle \delta_{\mathscr{D}}$ which sends $\Sigma_{D \in \mathscr{D}} a_D$ to $\Sigma_{D \in \mathscr{D}} x(D)\, a_D$ is easily seen to be an R-homomorphism from $\Sigma_{D \in \mathscr{D}} D$ to R. Further the map $\delta_{\mathscr{D}}$ is a monomorphism.

Now if $\mathscr{C} \leqq \mathscr{D}$, it is clear that $\Sigma_{C \in \mathscr{C}} C \supseteq \Sigma_{D \in \mathscr{D}} D$, and the map

$$pr_{\mathscr{C}\mathscr{D}}: \mathrm{Hom}_R\left(\sum_{C \in \mathscr{C}} C, R\right) \to \mathrm{Hom}_R\left(\sum_{D \in \mathscr{D}} D, R\right)$$

is simply given by restriction. A calculation which we omit shows that the following diagram:

$$\begin{array}{ccc} R_{\mathscr{C}} & \xrightarrow{\pi_{\mathscr{C}\mathscr{D}}} & R_{\mathscr{D}} \\ \delta_{\mathscr{C}} \downarrow & & \downarrow \delta_{\mathscr{D}} \\ \mathrm{Hom}_R\left(\sum_{C \in \mathscr{C}} C, R\right) & \xrightarrow{pr_{\mathscr{C}\mathscr{D}}} & \mathrm{Hom}_R\left(\sum_{D \in \mathscr{D}} D, R\right) \end{array}$$

is commutative. Thus the monomorphisms $\{\delta_{\mathscr{D}}\}$ lift to define a monomorphism

$$\delta: \lim_{\substack{\rightarrow \\ \mathscr{D}}} R \to \lim_{\substack{\rightarrow \\ \Delta}} \mathrm{Hom}_R(\Delta, R),$$

and the proposition is proved.

COROLLARY 3.2 *$B_m(R)$ is a ring of quotients of R.*

PROOF. This is immediate from 3.1 and Proposition 6 page 40 of [6].

From now on we identify $B_m(R)$ with its isomorphic copy in $Q(R)$ and turn to giving a simple description of it. For any $S \in A_m(R)$ consider the idempotent $f_S \in Q(R)$ given by

$$f_S: S + S^* \to R: a + b \mapsto a.$$

The Boolean algebra of all such idempotents f_S is a subring of $Q(R)$ isomorphic to $A_m(R)$ which we denote by $\bar{A}_m(R)$.

THEOREM 3.3 *$B_m(R)$ is the R-subalgebra of $Q(R)$ generated by $\bar{A}_m(R)$.*

PROOF. This is immediate from the standard form for elements of $B_m(R)$ and the description of $\bar{A}_m(R)$ just given.

COROLLARY 3.4 *The Baer hull of R (Mewborn [7]) is identical with the complete Baer extension of R.*

PROOF. This follows from Proposition 2.5 of [7] and 3.3 above.

COROLLARY 3.5 *The Baer extension of R (Kist [4]) is a ring of quotients of R.*

PROOF. This follows from 3.3.

REMARKS. It can also be shown that the classical ring of quotients of $B(R)$ is the epimorphic hull [7] of R. This will appear in a forthcoming paper of M. W. Evans.

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https://doi.org/10.1017/S1446788700012702 Published online by Cambridge University Press