be the reflection of P in the line of symmetry of triangle ABC. Join Q to A, B, C and P. Since PQ and CB are both perpendicular to the line of symmetry they are parallel and $\angle PQC = \angle QCB = \phi$. Thus triangle PQC is isosceles with $PQ = PC$.

Reflect triangle ACP in AC. Join P to its image R. Then $PC = RC$ and $\angle PCR = 60°$, so triangle PCR is equilateral and $PR = PC$.

Since $AR = AP = AQ$ and $PR = PQ$, triangles APR and APQ are congruent. Noting that $\angle RAC = \angle CAP = \angle BAQ = \theta$, we find that $\angle PAQ = 2\theta$ and $\angle BAC = 4\theta$.

The interior angle sum of triangle $\angle ABC = 4\theta + 2(30° + 2\phi) = 180°$ and it follows that $\theta + \phi = 30°$. In our special case $\phi = 20°$, so $\theta = 10°$.

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107.42 On the Lemoine line for the triangle

For obtuse triangles, the circumcentre and orthocentre are external points so the centroid G is the unique point on the Euler line which lies within every triangle ABC.

Now the isogonal conjugate of G is the Lemoine or symmedian point K and we will use areal coordinates to find the largest segment on the Lemoine line GK produced which lies within every triangle.

The coordinates

With the area of the triangle of reference denoted by ΔABC, a point $P[x, y, z]$ has actual areal coordinates $x = \frac{\Delta PBC}{\Delta ABC}$, $y = \frac{\Delta PCA}{\Delta ABC}$, $z = \frac{\Delta PAB}{\Delta ABC}$ $(x + y + z = 1)$.

We use [ ] for actual and () for relative coordinates (e.g. vertex $A[1, 0, 0]$ and centroid $G(1, 1, 1)$).

Isogonal conjugacy

If the cevians from A, B, C are concurrent at a point P, then the reflections in the internal angle bisectors at A, B, C are concurrent at P', the isogonal conjugate of P ([1]).

It follows that interior points $P[X, Y, Z]$ and $P'[X', Y', Z']$ are isogonal if $\frac{XX'}{a^2} = \frac{YY'}{b^2} = \frac{ZZ'}{c^2}$.

Now the centroid $G(1, 1, 1)$ and symmedian point $K(a^2, b^2, c^2)$ are clearly internal conjugates so GK is an internal segment of the Lemoine line for all triangles.

The GK extension

FIGURE 1

Points to the right of K in Figure 1 may be written $t[K] + (1 - t)[G]$ $(t > 1)$, so these are external points for sufficiently obtuse triangles. However, the valid internal segment extends beyond K' (the symmedian point of the medial triangle $A'B'C'$).

We next show that GK extends to Q where G is the KQ midpoint.

Since $[Q] = 2[G] - [K]$, relative coordinates of the point Q are given by

$$\left(2b^2 + 2c^2 - a^2,\ 2c^2 + 2a^2 - b^2,\ 2a^2 + 2b^2 - c^2\right).$$

Then (without loss of generality and using $a < b + c$) for the x-coordinate we have

$$2b^2 + 2c^2 - a^2 > 2b^2 + 2c^2 - (b + c)^2 = (b - c)^2 \geqslant 0,$$

so the segment KQ lies within all triangles,

The reader may wish to check the trigonometrical alternative

$$4 \cot A + \cot B + \cot C > 0.$$

Concluding remarks

That the skew segment KQ lies within every triangle ABC, notwithstanding the isogonal symmetry of G and K, may surprise.

Let U and V represent the respective centroid/circumcentre and centroid/Steiner point midpoints and M the Euler/Fermat line meet. Then M, U, Q, V are concyclic by reflecting a known circle across the centroid G ([2]).

Note finally that the symmdian point/Kiepert centre join on the Fermat line also lies within every triangle.

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107.43 Generalised averages of polygons

As is well known, the midpoints of the sides of a quadrilateral form a parallelogram. In [1], David Wells extends the previous result to any hexagon and, in [2], Nick Lord extends it further proving the following:

> Take any $2n$-sided polygon, and find the centres of gravity of each set of n consecutive vertices. These form a $2n$-sided polygon whose opposite sides are equal and parallel in pairs.

Retaining the notation provided by the above-cited authors, in particular, denoting by (A_k) an n-sided polygon with vertices $A_1A_2\dots A_n$, we can generalise the previous statement to any n-sided polygons as follows:

> Take any n-sided polygon (A_k), and find the centres of gravity $(B_{h,k})$ of each set of h consecutive vertices (with $1 \leqslant h \leqslant n-1$) and the centres of gravity $(B_{n-h,k})$ of each set of $n-h$ consecutive vertices. Then, each side $B_{h.k}B_{h.k+1}$ of $(B_{h,k})$ is $\frac{n-h}{h}$-times in length and parallel to the side $B_{n-h.k+h}B_{n-h.k+h+1}$ of $(B_{n-h,k})$. Therefore, $(B_{h,k})$ and $(B_{n-h,k})$ are similar with ratio $\frac{n-h}{h}$.

Indeed, let $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ be the position vectors of consecutive vertices of (A_k) and continue the labelling cyclically so that $\mathbf{a}_{n+i} = \mathbf{a}_i$ for $1 \leqslant i \leqslant n-1$. The position vectors of the vertices $(B_{h,k})$ of the derived n-sided polygon are then given by $\mathbf{b}_{h,k} = \frac{1}{h}\left(\mathbf{a}_k + \mathbf{a}_{k+1} + \dots + \mathbf{a}_{k+h-1}\right)$ for $1 \leqslant k \leqslant n$. It follows that $\mathbf{b}_{h,k+1} - \mathbf{b}_{h,k} = \frac{1}{h}\left(\mathbf{a}_{k+h} - \mathbf{a}_k\right)$, and, replacing k by $k+h$ and h by $n-h$, that $\mathbf{b}_{n-h,k+h+1} - \mathbf{b}_{n-h,k+h} = \frac{1}{n-h}\left(\mathbf{a}_{k+n} - \mathbf{a}_{k+h}\right)$. Thus, the side $B_{h,k}B_{h,k+1}$ of $(B_{h,k})$ is $\frac{n-h}{h}$–times and parallel to the side $B_{n-h,k+h}B_{n-h,k+h+1}$ of $(B_{n-h,k})$. Consequently, the angles of $(B_{h,k})$ and $(B_{n-h,k})$ are equal in pairs and all the ratios of the pairs of the corresponding sides are equal to $\frac{n-h}{h}$. Therefore, $(B_{h,k})$ and $(B_{n-h,k})$ are similar with ratio $\frac{n-h}{h}$.