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Errata and a New Result on Signs

WILLIAM M. McGOVERN*
Department of Mathematics, Box 354350, University of Washington, Seattle, WA 98195, U.S.A.; e-mail: mcgovern@math.washington.edu

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Abstract. We correct the errors in [*Compositio Math.* **101** (1996), 77–98] and [*Compositio Math.* **101** (1996), 99–108] and sharpen Theorem 4.3 in [*Compositio Math.* **101** (1996), 77–98].

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1. Introduction

Let W be a finite Weyl group of type B_n, C_n, or D_n. In this paper we correct Theorems 4.2 of [Mc1] and 2.1 and 2.2 of [Mc2], which are false as stated. We also give corrected proofs of the results of [Mc1] and [Mc2] depending on these theorems. In so doing we will also prove a more precise version of Theorem 4.3 of [Mc1]: we will show how to write down explicit bases of the irreducible consituents of a left or right cell $\mathcal{C}$ of W in terms of the Kazhdan–Lusztig basis of $\mathcal{C}$ itself. More precisely, given $\mathcal{C}$, a right cell $\mathcal{R}$ meeting $\mathcal{C}$, and a representation σ appearing in both $\mathbb{C}\mathcal{C}$ and $\mathbb{C}\mathcal{R}$, we will construct a weighted sum R_σ of Kazhdan–Lusztig basis vectors C_x for $x \in \mathcal{C} \cap \mathcal{R}$ such that the right or left W-submodule generated by R_σ is irreducible and W acts on it by σ. All coefficients in the weighted sums R_σ will be ± 1 and we will give a simple rule for deciding which are positive. This rule is the 'new result on signs' promised in the title; it sharpens Theorem 4.3 of [Mc1]. If we then hold $\mathcal{C}$ and σ fixed and let $\mathcal{R}$ run through the right cells of W such that σ appears in $\mathbb{C}\mathcal{R}$, we obtain a basis for the σ-isotypic component of $\mathbb{C}\mathcal{C}$, which is irreducible. Using this basis one can compute Langlands parameters in the Barbasch–Vogan character formulas for special unipotent representations of complex groups and extend these formulas to certain representations of real classical groups. As in [Mc1, Mc2], the main tool is the ordered pair of standard domino tableaux of the same shape attached by Garfinkle to every $w \in W$ [G1], which determines w uniquely.

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2. Isotypic Basis Vectors

Retain the above notation and let $\mathcal{C}$ resp. $\mathcal{R}$ be a left resp. right cell of W. Assume that $\mathcal{C}$ and $\mathcal{R}$ are contained in a common double cell $\mathcal{D}$, so that the intersection $\mathcal{C} \cap \mathcal{R}$ is nonempty. It is well known that the complex spans $\mathbb{C}\mathcal{C}$, $\mathbb{C}\mathcal{R}$ of $\mathcal{C}$, $\mathcal{R}$ in $\mathbb{C}W$ are multiplicity-free as W-modules (for W classical) and the number of their common constituents equals the cardinality of the intersection $\mathcal{C} \cap \mathcal{R}$. More precisely and generally, for any pair $\mathcal{C}_1, \mathcal{C}_2$ of left cells in any finite Weyl group, the cardinality of the intersection $\mathcal{C}_1 \cap \mathcal{C}_2^{-1}$ equals the dimension of the space of W-homomorphisms from $\mathbb{C}\mathcal{C}_1$ to $\mathbb{C}\mathcal{C}_2$. In the classical case, there is a bit more structure: the sets $\{\mathcal{C}\}$, $\{\mathcal{R}\}$ of irreducible representations of W appearing in $\mathbb{C}\mathcal{C}$, $\mathbb{C}\mathcal{R}$ themselves have the structure of elementary abelian 2-groups, as does their intersection $\{\mathcal{C} \cap \mathcal{R}\} := \{\mathcal{C}\} \cap \{\mathcal{R}\}$ [Lu1, Lu2]. Given an element w of $\mathcal{C} \cap \mathcal{R}$, we may parametrize and compute $\{\mathcal{C} \cap \mathcal{R}\}$ as follows. Start with the pair $(T_L(w), T_R(w))$ of standard domino tableaux attached to w. Recall that $T_L(w)$ does not depend on $\mathcal{C}$ alone, but there is a unique tableau of special shape in the sense of [G1] attached to $T_L(w)$ which is a complete invariant of $\mathcal{C}$ [G1, G3, G4]. Following [G1], we group the dominos of $T_L(w)$ into cycles, some open and the others closed, and we do the same for $T_R(w)$. In types B and C our convention here differs from that of [G1, G2]: in these cases we call a cycle open only if does not involve the square in the upper left corner of the tableau. Given any set Σ of open cycles in $T_L(w)$, we may move $T_L(w)$ through Σ; this involves changing the positions of the dominos in the cycles in Σ but no others [G1, G2]. Our convention on open cycles guarantees that moving through any set of them does not change the type of the tableau, if this is B or C. Similarly, we may move $T_R(w)$ through any set of its open cycles, subject to the convention above. We now recall from [G2] the notion of extended open cycles of $T_L(w)$ relative to $T_R(w)$. These are just the minimal nonempty sets Σ of open cycles of $T_L(w)$ such that moving $T_L(w)$ through Σ yields a tableau T_Σ of the same shape as moving $T_R(w)$ through some set of *its* open cycles. Now, following [Lu1], we have shown in [Mc1, Thm 3.2] how to map tableau shapes or partitions to elements of $\widehat{W}$ in such a way that the shapes of T_Σ parametrize the elements of $\{\mathcal{C} \cap \mathcal{R}\}$ as Σ runs through the extended open cycles of $T_L(w)$ relative to $T_R(w)$. This parametrization is one-to-one in types B and C; in type D it is two-to-one unless the double cell $\mathcal{D}$ containing $\mathcal{C}$ and $\mathcal{R}$ corresponds to a very even partition, in which case it is one-to-one.

We can now state our rule for constructing the R_σ. Denote the Kazhdan–Lusztig basis of $\mathbb{C}W$ by $\{C_w\}$. Fix some $x \in \mathcal{C} \cap \mathcal{R}$ arbitrarily. Let the shape of σ (or either of the corresponding shapes in type D) be obtained from the special shape corresponding to $\mathcal{C}$ (or either special shape in type D) by moving through the extended open cycles $e_1, \ldots, e_t$. Given $w \in \mathcal{C} \cap \mathcal{R}$, let $T_L(w)$ be obtained from $T_L(x)$ by moving through the extended open cycles $f_1, \ldots, f_s$. Put $\sigma_w = \pm 1$ according as an even or odd number of f_i appear among the e_j. Set $R_\sigma := \sum_{w \in \mathcal{C} \cap \mathcal{R}} \sigma_w C_w$. Then R_σ has the desired property; note that its definition depends on the choice of

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x only up to sign. If W is replaced by the set of double cosets in W of a suitable parabolic subgroup, then our R_σ coincides up to scalar with the R_σ defined in [BV] for certain left cells and generalized to certain others in [Jo]. Before we can prove our main result, we need a technical fact about W-equivariant maps between left cells and the elements R_σ.

LEMMA 1. *Let $\mathcal{C}$, $\mathcal{C}'$ be two left cells lying in the same double cell $\mathcal{D}$, and let σ be a representation lying in both $\{\mathcal{C}\}$ and $\{\mathcal{C}'\}$. Then there is a W-equivariant map π_σ from $\mathbb{C}\mathcal{C}$ to $\mathbb{C}\mathcal{C}'$ sending basis vectors to sums of basis vectors which is injective on the copy E_σ of σ in $\mathbb{C}\mathcal{C}$. The map π_σ sends $\mathbb{C}\mathcal{C} \cap \mathbb{C}\mathcal{R}$ to $\mathbb{C}\mathcal{C}' \cap \mathbb{C}\mathcal{R}$ for any right cell $\mathcal{R}$ lying in $\mathcal{D}$, and whenever σ is a representation lying in all three of $\{\mathcal{C}\}, \{\mathcal{C}'\}, \{\mathcal{R}\}$, then π_σ sends the R_σ defined above for $\mathcal{C} \cap \mathcal{R}$ to that for $\mathcal{C}' \cap \mathcal{R}$ up to a sign.*

Proof. Brian Hopkins has shown that whenever two elements x, y have the same right tableau T_R, then each can be obtained from the other by a composition of certain maps T_i [Ho]. The maps T_i are not defined on all of W, but whenever T_i is defined on an element w, it is defined on any other element w' with the same left tableau as w and moreover the left tableaux of $T_i(w)$ and $T_i(w')$ are the same, as are the right tableaux of $w, T_i(w)$, and $T_i(w')$. Thus the maps T_i preserve right cells and map left cells to left cells. Whenever the map T_i sends the left cell $\mathcal{C}_1$ to $\mathcal{C}_2$, there is a corresponding left W-equivariant map $\widetilde{T}_i$ defined on the complex span $\mathbb{C}\,\mathcal{C}_1$ of $\mathcal{C}_1$ and mapping it to $\mathbb{C}\,\mathcal{C}_2$. It sends basis vectors in $\mathbb{C}\mathcal{C}_1$ either to basis vectors or to sums of two basis vectors. It is left W-equivariant because it acts by right multiplication by a suitable simple reflection (in the coherent continuation representation) followed by projection to $\mathbb{C}\,\mathcal{C}_2$. The maps $\widetilde{T}_i$ are just the wall-crossing operators $T_{\alpha\beta}, T'_{\alpha\beta}, T_D$ of [Mc1,Mc2], together with four additional operators defined in [Ho]. Note that the left and right tableaux of a typical $\widetilde{T}_i(w)$, unlike those of $T_i(w)$, depend on the type B, C, or D in which w lives and not just on the left and right tableaux of w. In particular the maps $\widetilde{T}_i$ differ in types C and D even though the map sending w to $(T_L(w), T_R(w))$ is the same for both of these types. Now the maps $\widetilde{T}_i$ come as close to being isomorphisms as possible. More precisely, the wall-crossing operator $T_{\alpha\beta}$ of [Mc1, Mc2] restricts to an isomorphism on the complex span of any left cell for which it is defined. The other $\widetilde{T}_i$ send spans $\mathbb{C}\,\mathcal{C}_1$ of left cells to spans $\mathbb{C}\,\mathcal{C}_2$ of cells such that the cells $\mathcal{C}_1$ and $\mathcal{C}_2$ are adjacent in the sense of [Mc2], so that exactly half of the representations appearing in $\mathbb{C}\mathcal{C}_1$ also appear in $\mathbb{C}\mathcal{C}_2$. Then the restriction of $\widetilde{T}_i$ to any simple constituent of $\mathbb{C}\mathcal{C}_1$ is an isomorphism if this constituent has an isomorphic copy in $\mathbb{C}\mathcal{C}_2$ and is identically 0 otherwise.

Under the hypotheses of the lemma, we know from Theorem 3.2 of [Mc1] that there are elements x, y lying respectively in $\mathcal{C} \cap \mathcal{R}$, $\mathcal{C}' \cap \mathcal{R}$ whose left or right tableau shapes correspond to the representation σ in the sense of [Mc1]. Then there is a composition of maps T_i sending x to y. Taking the composition of the corresponding maps $\widetilde{T}_i$, we find that it sends E_σ to the unique copy E'_σ of σ lying in $\mathbb{C}\,\mathcal{C}'$ isomorphically. It only remains to verify the last assertion. This follows from the description of the maps $\widetilde{T}_i$ on the level of tableaux in [G2] and [Ho]: when restricted to the nonempty intersection $\mathbb{C}(\mathcal{C} \cap \mathcal{R})$ of (the spans of) any left and right cell on which they are defined, they act by moving through fixed sets of extended open cycles depending only on $\mathcal{C}$ and $\mathcal{R}$. □

THEOREM 1. *With notation as above, the right or left W-submodule generated by R_σ is irreducible and W acts on it by σ.*

Proof. Let T_σ be the unique element of $\mathbb{C}\mathcal{C} \cap \mathbb{C}\mathcal{R}$ such that T_σ satisfies the conclusion of Theorem 1 and the basis vector C_x appears with coefficient one in T_σ. Then we must show that $R_\sigma = T_\sigma$. Note first that T_σ also generates an irreducible right W-submodule acted on by σ (as claimed in the Introduction); this is an elementary fact about group algebras, or more generally semisimple Artinian rings.

Next we observe that the subgroup $G_\mathcal{C} := J_{\mathcal{C}^{-1}\cap\mathcal{C}}$ of the asymptotic Hecke algebra J of W is an elementary abelian 2-group acting transitively on $\mathcal{C} \cap \mathcal{R}$ by right multiplication [Lu2, Lu3]. It also acts W-equivariantly on $\mathbb{C}\,\mathcal{C}$ and in fact spans the full endomorphism algebra of this left W-module. It follows that the T_σ span the common eigenlines of this action, so that one of them is the sum of the C_w for $w \in \mathcal{C} \cap \mathcal{R}$ while the others are alternating sums of C_w. We also see that the T_σ form an orthogonal basis for $\mathbb{C}\mathcal{C} \cap \mathbb{C}\mathcal{R}$ if we regard the latter as an inner product space by declaring that the basis vectors C_w form an orthonormal basis.

We now show that $R_\sigma = T_\sigma$ by induction on the complexity of σ, as defined in the proof of [Mc1, Thm 4.3]. The only σ with complexity zero is the special representation s, whose corresponding (tableau) shape is also special. In this case, the theorem says that the coefficient of every basis vector in T_s is one, which is indeed the case (as remarked at the end of [Mc2]). We next claim that, if $R_\sigma = T_\sigma$ for some cell intersection $\mathcal{C}' \cap \mathcal{R}'$ with $\sigma \in \{\mathcal{C}' \cap \mathcal{R}'\}$, then this equality continues to hold for every cell intersection $\mathcal{C}'' \cap \mathcal{R}''$ with $\sigma \in \{\mathcal{C}'' \cap \mathcal{R}''\}$. This follows at once from Lemma 1 and its analogue for right cells (which uses the right analogues of the maps $\widetilde{T}_i$ mentioned in the proof of Lemma 1), applied first to $\mathcal{C}'$, $\mathcal{C}''$ and then to $\mathcal{R}'$, $\mathcal{R}''$.

Now we can complete the induction: assume that $R_\mu = T_\mu$ whenever μ has complexity less than m, and let σ have complexity m. Write $m = t + u$ and let σ correspond to a sum $\ell + r$ as in the proof of [Mc1, Thm 4.3]. Then it is easy to construct a left cell $\mathcal{C}'$ and a right cell $\mathcal{R}'$ such that σ is the unique element of $\{\mathcal{C}' \cap \mathcal{R}'\}$ of largest complexity, which is m. Since each T_σ agrees with some R_μ for $\{\mathcal{C}' \cap \mathcal{R}'\}$, and $R_\gamma = T_\gamma$ for all $\gamma \in \{\mathcal{C}' \cap \mathcal{R}'\}$ except possibly σ, we see that in fact $R_\sigma = T_\sigma$ for $\mathcal{C}' \cap \mathcal{R}'$ as well, using the orthogonality of the T_σ noted above. By the preceding paragraph, we deduce that this equality remains true for the intersection $\mathcal{C} \cap \mathcal{R}$. This completes the proof. □

Theorem 1 is a sharper version of Theorem 4.3 of [Mc1]. Theorems 4.2 of [Mc1] and 2.1, 2.2 of [Mc2] are false as stated but become correct if ‘sequence

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of maps …' is replaced throughout by 'W-equivariant map'. For Theorem 4.2 of [Mc1], let u be the union of extended open cycles of $\mathcal{C}_1$ relative to $\mathcal{R}$ such that moving the left tableau $T_L(w_1)$ through u gives a tableau of the same shape as $T_L(w_2)$. Then define the map from $\mathbb{C}(\mathcal{C}_1 \cap \mathcal{R})$ to $\mathbb{C}(\mathcal{C}_2 \cap \mathcal{R})$ on the level of tableaux of basis vectors by first moving the left tableau through the cycles in u, then moving the right tableau through the corresponding set of its open cycles, and then finally sending the resulting pair (T, T') of tableaux of the same shape to the unique pair $(T_L(y), T_R(y))$ of left and right tableaux with this shape of an element y in $\mathcal{C}_2 \cap \mathcal{R}$. A similar argument proves Theorems 2.1 and 2.2 of [Mc2] as amended above. More precisely, one can show that the W-equivariant maps of the aforementioned theorems of [Mc1, Mc2] can be taken to be linear combinations of compositions of the maps $\widetilde{T}_i$ occurring in the proof of Lemma 1. The amended versions of these theorems now suffice to make the proofs of Theorem 5.1 of [Mc1] and 3.2 of [Mc2] go through. The other results of [Mc1] and [Mc2] are correct as stated.

We conclude by remarking that one can use Theorem 1 to derive a somewhat more explicit version of Theorem III of [BV] in the classical case and to extend it under certain additional hypotheses to real classical groups.

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