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ON SINGULAR NORMAL LINEAR INTEGRAL EQUATIONS

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In this work we consider the equation

(1) $Q(x) = f(x)+\lambda \int_0^1 K(x, t)Q(t)\,dt$ where $K(x, y)$ is singular in the sense that it does not properly belong to L_2 and $f(x)$ is an arbitrary L_2 function.

A Lebesgue measurable function $K(x, y)$ of two variables, having real values on $[0.1]\times[0.1]$ is called a singular normal kernel of

(i) $\int_0^1 |K(x, y)|^2\,dy < \infty, \qquad \int_0^1 |K(x, y)|^2\,dx < \infty$

There exists approximating kernels $K_m(x, y)$ satisfying

(ii) $K_m(x, y)$ is L_2 in (x, y) $\quad |K_m(x, y)| < |K(x, y)|$

(iii) $\int_0^1 K_m(x, t)K_m(y, t)\,dt = \int_0^1 K_m(t, x)K_m(t, y)\,dt$

(iv) $\lim_{m\to\infty} K_m(x, y) = K(x, y)$

It is seen in [1] that the approximating kernel $K_m(x, y)$ admits a representation $K_m(x, y)=K'_m(x, y)+iK''_m(x, y)$ where K' and K'' are symmetric, having one and same system of characteristic functions. The real and imaginary parts of the characteristic numbers of the kernel $K_m(x, y)$ are characteristic numbers of K'_m and K''_m respectively. The approximating kernels behave like symmetric kernels with the single exception that the characteristic values may be complex valued.

We associate with equation (1) the equation

$$Q(x) = f(x)+\lambda \int_0^1 K_m(x, t)Q(t)\,dt. \tag{2}$$

The Fredholm theory is applicable to (2), thus there exists a unique solution Q_m in L_2 of (2) for every non-characteristic value λ_{mv} of K_m. Consequently the Schmidt representation extended to the solution of (2) yields,

$$Q_m(x) = f(x)+\lambda \sum_v \frac{f_{mv}}{\lambda_{mv}-\lambda}\,\phi_{mv}, \qquad f_{mv} = \int_0^1 f(s)\phi_{mv}(s)\,ds \tag{3}$$

and

$$\int_0^1 Q_m(x)\,\phi_{mj}(x)\,dx = \frac{\lambda_{mj}}{\lambda_{mj}-\lambda}\int_0^1 f(x)\phi_{mj}(x)\,dx \tag{4}$$

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or equivalently we may write

$$
(5)\qquad \begin{aligned} Q_m(x) &= f(x)+\lambda\int_0^1 K_m(x,t)f(t)\,dt \\ &\quad +\lambda^2\sum_v \frac{1}{\lambda_{mv}-\lambda}\int_0^1\int_0^1 K_m(x,\tau)\phi_{mv}(\tau)f(t)\,d\tau\,dt. \end{aligned}
$$

As is well known from classical theory, the series in (5) converges uniformly. Multiplying by $\overline{Q_m(x)}$ and integrating one gets,

$$
(6)\qquad \begin{aligned} \int_0^1 |Q_m^2(x)|\,dx &= \int_0^1 f(x)\bar{Q}_m(x)\,dx+\lambda\sum_v\frac{1}{\lambda_{mv}-\lambda} \\ &\quad \times\int_0^1 Q_m(x)\phi_{mv}(x)\,dx\int_0^1 \phi_{mv}(t)f(t)\,dt \\ &\le \left|\int_0^1 f(x)\bar{Q}_m(x)\,dx\right|+\frac{|\lambda|}{\delta}\left\{\sum_v\left|\int_0^1 \bar{Q}_m(x)\phi_{mv}(x)\,dx\right|^2\right. \\ &\quad \left.\times\sum_v\left|\int_0^1 f(x)\phi_{mv}(x)\,dx\right|^2\right\}^{1/2} \end{aligned}
$$

where λ is in the compliment $C(\bar{E})$ of the closure $\bar{E}$ of the set of characteristic values $\lambda_{mv}(m, v=1,2,\ldots)$ of K_m and δ the distance (necessarily positive) from λ to $\bar{E}$.

From Bessel's inequality

$$
(7)\qquad \int_0^1 |Q_m^2(x)|\,dx \le \left\{\int_0^1 |f|^2\,dx\int_0^1 |Q_m|^2\,dx\right\}^{1/2}+\frac{|\lambda|}{\delta}\left\{\int_0^1 |Q_m|^2\,dx\int_0^1 |f|^2\,dx\right\}^{1/2}
$$

and

$$
(8)\qquad \int_0^1 |Q_m^2(x)|\,dx \le \left\{1+\frac{|\lambda|}{|\delta|}\right\}^2\int_0^1 |f^2(x)|\,dx.
$$

We now cite the following well known lemmas of F. Riesz [3].

Lemma 1. *If* $f_v(x)\in L_2[0,1]$, $v=1,2,\ldots$, *and if* $\int_0^1 |f_v(x)|^2\,dx<M$, $v=1,2,\ldots$, *then there exists a subsequence* $\{f_{v_j}(x)\}$ *such that as* $j\to\infty$, $f_{v_j}(x)\to f(x)$ *weakly, i.e.* $\lim_j \int_0^x f_{v_j}(x)dx=\int_0^x f(x)\,dx$, $0<x<1$. *Furthermore* $\int_0^1 f^2(x)\,dx<M$. (*M is a constant independent of j.*)

Lemma 2 [2, p. 132]. *If* $\int_0^1 f_v^2(x)\,dx<c$, $f_v\to f(x)$ *weakly while* $g_n(x)\to g(x)$ *and* $|g_n(x)|<\gamma(x)\subset L_2$, $n=1,2,\ldots$, *then*

$$
\lim_n \int_0^1 f_n(x)g_n(x)\,dx = \int_0^1 f(x)g(x)\,dx.
$$

Lemma 3. *If a sequence* $\{f_v(x)\}$ *converges weakly to* $f(x)$ *and converges in the ordinary sense to* $F(x)$ *then*

$$
f(x) = F(x) \quad \text{a.e.}
$$

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From (8) and Lemma 1 there exists a subsequence $\{Q_{mn}\}$ converging weakly to a function $Q(x)$ in L_2.

From Lemma 2 it follows that for almost all x in [0, 1]

$$\lim_{v\to\infty}\int_0^1 K_{nv}(x,y)Q_{nv}(y)\,dy = \int_0^1 K(x,y)Q(y)\,dy. \tag{9}$$

With the aid of (2), it follows from Lemma 3 that $\lim_v Q_{nv}=Q(x)$ is a solution of (1).

We have thus proved the following:

THEOREM. *For λ in the compliment $C(\bar{E})$ of the closure $\bar{E}$ of the set of characteristic values λ_{mv} $(m, v=1, 2, \ldots)$ of the approximating kernels $K_m(x, y)$, the equation $Q(x)=f(x)+\lambda\int_0^1 K(x,y)Q(y)\,dy$ with singular normal kernel K, and f in L_2, possesses a solution $Q(x)$ in L_2.*

We now examine the solution $Q(x)$ obtained above. We suppose in what follows that

$$\int_0^1 |K_m(x,t)-K_m(x',t)|^2\,dt < \sigma(|x-x'|) \tag{10}$$

where σ is independent of m and approaches zero as $x\to x'$.

In view of (8).

$$\begin{aligned}|Q_m(x)-f(x)| &\le |\lambda|\left\{\int_0^1 |K_m^2(x,y)|\,dy\int_0^1 |Q^2(y)|\,dy\right\}^{1/2}\\ &\le C\,|\lambda|\left\{1+\frac{|\lambda|}{8}\right\}^2 K(x)\end{aligned} \tag{11}$$

where

$$C=\int_0^1 f^2(y)\,dy \quad\text{and}\quad K(x)=\left\{\int_0^1 K(x,y)^2\,dy\right\}^{1/2}.$$

Likewise

$$\begin{aligned}&|Q_m(x)-f(x)-(Q_m(x')-f(x'))|\\ &\qquad\le |\lambda|\left\{\int_0^1 Q_m^2(y)\,dy\right\}^{1/2}\left\{\int_0^1 |K_m(x,y)-K_m(x',y)|^2\,dy\right\}^{1/2}\\ &\qquad\le C\,|\lambda|\left\{1+\frac{|\lambda|}{\delta}\right\}^2\left\{\int_0^1 |K_m(x,y)-K_m(x',y)|^2\,dy\right\}.\end{aligned} \tag{12}$$

In view of (10)

$$|(Q_m(x)-f(x))-(Q_m(x')-f(x'))| \le C\,|\lambda|\left\{1+\frac{|\lambda|}{\delta}\right\}^2\sigma(|x-x'|). \tag{13}$$

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It follows as in [2, pp. 54–55] with the aid of Vitali's theorem that $\lim_{m\to\infty} Q_m(x) = Q(x)$ is analytic in λ for every λ for which Theorem 1 holds.

COROLLARY. *Suppose Ω is a region in $C(\bar{E})$ and the homogeneous equation $Q(x)=\lambda\int_0^1 K(x,y)Q(y)\,dy$ has no nonzero solutions in L_2 for $\lambda=\lambda_1,\lambda_2,\ldots,$ the λ_n, $n=1,2,\ldots,$ being distinct points in Ω with a limiting point in Ω, then there exists no solution $Q=Q(x,\lambda)$, distinct from zero and analytic in λ.*

Proof. Results directly from the identity theorem of analytic functions.

REMARK. For any multiple connected region in $C(\bar{E})$, the function Q may be multiple valued. As λ describes some closed path in $C(\bar{E})$, Q may change to another function ψ. However any such circuit will leave equation (1) unchanged, so ψ will also be a solution of (1).

It follows [2, p. 56] that there exists a linear operator $T_{x\lambda}$ (dependent on K) and a subsequence $\{m_j\}$ (independent of f) such that with Q_m denoting a solution of (2) for λ in $C(\bar{E})$ we have

$$\lim_j Q_{m_j} = Q = T_{x\lambda}(f) \text{ is a solution of (1).} \tag{14}$$

THEOREM 2. *The operator $T_{x\lambda}(f)$ of (14) satisfies*

$$\int_0^1 f_1(x)T_{x\lambda}(f_2)\,dx = \int_0^1 f_2(x)T_{x\lambda}(f_1)\,dx \tag{15}$$

for all f_i in L_2, $i=1,2,\ldots,$ and λ in $C(\bar{E})$.

Proof. Let $Q_{i,m}(x)$ be a solution of

$$Q(x) = \lambda\int_0^1 K_m(x,y)Q(y)\,dy + f_i, \quad i = 1,2,\ldots \tag{16}$$

Multiply the equation (16) for $Q_{1,m}$ by $Q_{1,m}$ and integrate. Similarly multiply the equation for $Q_{2,m}$ by $Q_{1,m}$ and integrate. Subtracting the second expression from the first as in [2] we get

$$\int_0^1 f_1(x)\cdot Q_{2,m}(x)\,dx = \int_0^1 f_2(x)\cdot Q_{1,m}(x)\,dx. \tag{17}$$

Setting $m=m_j$, from (14) and noting

$$\int_0^1 |Q_{i,m_j}(x)|^2\,dx \le M, \quad i = 1,2,\ldots$$

we may pass to the limit in (17). Our result now follows from (14). This result is similar to that in [2, p. 56].

However, since our kernel is now normal and not necessarily symmetric, the development as in [2, Ch. II and III] relating to kernels of Class 1 (those generating selfadjoint operators) cannot be made without additional assumptions.

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REMARK. A spectral theory paralleling that in [2] or [4]–[5] can be developed for the kernels considered here on the basis of which, properties 1–4 of Theorem 10.4 [4, pp. 406–407] can be demonstrated.

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