Can fish count? by Brian Butterworth, pp. 373, £20 (hard), ISBN 978-1-52941-125-6, Quercus Books (2022)

This fascinating book gives the general reader access to a wide variety of academic research on the mathematical abilities of members of the animal world, including humans and, of course, fish. It starts with reflections on the fundamental place of mathematics in our world, including the assertion by Galileo that the universe is written in mathematics and we cannot read it unless we become familiar with the characters in which it is written.

In the sections of the book relating to humans, the author explores number vocabulary (some of which, in many languages, including English, is linked to the names of various parts of the body), number systems, and counting and recording devices. An intriguing series of experiments, initiated by the author and an Australian-based colleague, demonstrated that Aboriginal children whose language included no number or counting vocabulary could perform as well as English-speaking children on simple counting and calculating tasks.

Evidence of non-human animals' ability to make use of the mathematics of the universe in their efforts to satisfy hunger, to mate and to avoid danger is given in the descriptions and interpretations of many ingenious studies, some of which the author was and is personally involved in conducting. There are chapters discussing experiments with and observations of apes and monkeys, other mammals, birds, amphibians and reptiles, fish and even invertebrates. Bees and ants, for example, demonstrate the ability to pass mathematical information about the location and size of sources of food to other members of their groups.

Particularly interesting for me was to find out why turtles don't always end up in exactly the right place, why Clever Hans the horse was clever but not in the way that was originally thought, and the extent of the capabilities of Alex the parrot, who learnt to use number words with apparent understanding and to calculate with digits.

An added bonus is that the author of this book is an expert in dyscalculia and in one of the introductory chapters there is a brief but welcome discussion of the disadvantages, to individuals and to the economy, of poor numeracy and, worse, dyscalculia. Humans and animals alike benefit from being appropriately mathematically competent.

10.1017/mag.2024.46

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Irrationality and transcendence in number theory by David Angell, pp. 242, £59.99 (hard), ISBN 978-0-367-62837-6, Chapman and Hall/CRC (2022)

Although it is easy to prove that e is irrational, the proof that π is also irrational is rather challenging. Historically, the result was first established in 1761 by J. H. Lambert using the continued fraction expansion for the tangent function. In 1873 C. Hermite introduced radical ideas which could be used to establish the irrationality of various numbers related to the exponential and logarithm functions, leading in more recent times to some slick proofs that π is irrational. Indeed the ideas were developed further to deal with the much deeper problem of the transcendence of similar numbers, and the method naturally becomes correspondingly more difficult. One has to know the underlying reason for the seemingly mystical construction of complicated auxiliary functions in order to appreciate the method for the proof. The challenge of an expositor on transcendental number theory is to provide the motivation and the background to these and other ideas, and to explain the subtleties and intricacy involved in the proofs.

The book being reviewed is suitable for third year undergraduate and postgraduate mathematics students. The titles of the seven chapters are: 1. Introduction; 2. Hermite's method; 3. Algebraic and transcendental numbers; 4. Continued fractions; 5. Hermite's method for transcendence; 6. Automata and transcendence; 7. Lambert's irrationality proofs. The thoughtfully organised book is very well written—meticulous care has been taken to make the material accessible to students new to the topics. The delightfully written text is sprinkled with relevant examples and useful comments, and the required background materials in number theory, mathematical analysis or algebra are set out in appendices at the end of each chapter. There are also plenty of well constructed exercises, with generous hints to most of them at the end of the book. I was astonished to find, on page 13 of the first chapter, the irrationality of $e^{\sqrt{2}}$ being set out as an exercise! The corresponding hint has been bafflingly fumbled—nevertheless it is enough for me to appreciate D. W. Masser's brilliant idea for the relatively simple solution.

Besides dealing with irrational surds, the criterion for the rationality of decimals is established in Chapter 1, so that objects such as Liouville numbers and the Champernowne constant are shown to be irrational. Hermite's method is introduced in Chapter 2, and it is applied to establish the irrationality of e^r when r is a non-zero rational, and also π and various other values of trigonometric functions. Irreducibility lemmas of Gauss and Eisenstein are proved, and the closure property of algebraic numbers is sketched in Chapter 3. Dirichlet's approximation theorem and Liouville's theorem are proved; Roth's theorem is stated together with remarks on this celebrated result, and there is also a sketch of Apéry's argument for the irrationality of $\zeta(3)$. Chapter 4 on continued fractions contains the usual material relevant to approximations; there is a derivation of the continued fraction of e, which is rarely given in textbooks. The two chosen applications are the explanations of the Gregorian calendar, and the chromatic scale in musical theory, although such 'explanations' are considered only with hindsight because, as the author points out, Pope Gregory's scholars were unversed in continued fractions, and our present tonal system was developed in accordance with the needs of composers and performers, who knew little, and cared less, about such fractions.

The learning curve turns steep for the next chapter, in which the transcendence of e and π are established, together with Lindemann's theorem on e^{α}, where α is a non-zero algebraic number. Instead of just delivering the proofs, much background preparation on the choice of the auxiliary functions is given, including the reason for, and a proof of, the symmetric polynomial theorem, which is needed for the case of π. The reader is thus ready to make good sense of the Gelfond-Schneider theorem on the transcendence of α^{β}, when $\alpha \neq 0, 1$ is algebraic, and β is algebraic and irrational, and perhaps also Baker's theorem on linear forms in logarithms of algebraic numbers.

In Chapter 6 the notion of a deterministic finite automaton is explained—roughly speaking, it is a machine for sorting strings of letters into classes, and the notion can be applied to transcendence theory. For example, certain decimals in which the digits form a simple 'pattern' can be generated as the output of such an automaton, and it is explained how to arrive at the generating function $f(z) = \sum_{m \geqslant 0} z^{2^m}$; there then follows Mahler's method on transcendence and his proof

https://doi.org/10.1017/mag.2024.47 Published online by Cambridge University Press

that $f(1/10)$ is transcendental. The text rounds up nicely with a chapter on Lambert's irrationality proofs, which involve a generalisation of simple continued fractions.

There is no shortage of number theory books, and many of them include some of the topics being mentioned here. However, as a book devoted entirely to the subject of irrationality and transcendence at this level, I only know of [1], from which I first learned about such matters, and the more recent [2]. All three books are excellent but, perhaps surprisingly, neither [1] nor [2] is mentioned in the otherwise adequate bibliography.

References

1. Ivan Niven, *Irrational Numbers*, Carus Mathematical Monographs, Series Number 11, The Mathematical Association of America (1956).
2. Edward B. Burger and Robert Tubbs, *Making Transcendence Transparent: An intuitive approach to classical transcendental number theory*, Springer (2004).

10.1017/mag.2024.47 © The Authors, 2024
Published by Cambridge University Press
on behalf of The Mathematical Association

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Theory of infinite sequences and series by Ludmila Bourchtein and Andrei Bourchtein, pp 377, £54.99 (paper), ISBN 978-3-03079-430-9, Springer Verlag (2022)

The notion of a limit of sequence is fundamental in analysis and with it the related and perhaps more important notion of a sum of an infinite series. This textbook is divided in five chapters: the first two are devoted to sequences and series of numbers, followed by two chapters on sequences and series of functions and the last, fifth chapter is on power series.

The first chapter on sequences of numbers contains the standard material found in any analysis textbook. Much attention is paid to calculation of limits and to this end the Stolz-Cesaro theorems, which are discrete versions of L'Hôpital's rules, are used for evaluation of indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$. There is no mention of the important notion of accumulation points of sequences nor of recurrent sequences like the famous Fibonacci sequence. We encounter only explicitly defined sequences throughout. It is strange that a sequence is denoted simply by a_n, which is the same as the general term, not using the common notation (a_n).

After some introductory examples of convergent and divergent series of numbers, Chapter 2 deals mainly with various tests for convergence of series. First the familiar D'Alembert's test (ratio test) and Cauchy's test (root test) are treated, followed by the integral, comparison and Cauchy condensation tests. As a nice application of the last-named, the authors give the convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$, which in most textbooks is treated with the integral test. When the first two tests fail, there are more delicate ones to be tried, like the Raabe and the Jamet test. These are shown to be by-products of a more general Kummer and Cauchy chain of tests. Other topics discussed are absolute and conditional convergence, the Cauchy product and the astonishing Riemann theorem for the existence of rearrangement of conditionally convergent series to any previously chosen value.

In Chapter 3, after a brief discussion of pointwise convergence, the fundamental concept of uniform convergence of sequences of functions is thoroughly discussed. Under uniform convergence the limit function f inherits the nice properties of the functions f_n such as boundedness, continuity, differentiability and integrability. For pointwise convergence this is not the case and the authors demonstrate this with plenty of examples and counterexamples. This chapter is preparatory for the next,