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# RATIONAL GROUP ACTIONS ON AFFINE PI-ALGEBRAS

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*To Kenny Brown and Toby Stafford on the occasion of their 60th birthdays*

**Abstract.** Let $R$ be an affine PI-algebra over an algebraically closed field $\Bbbk$ and let $G$ be an affine algebraic $\Bbbk$-group that acts rationally by algebra automorphisms on $R$. For $R$ prime and $G$ a torus, we show that $R$ has only finitely many $G$-prime ideals if and only if the action of $G$ on the centre of $R$ is multiplicity free. This extends a standard result on affine algebraic $G$-varieties. Under suitable hypotheses on $R$ and $G$, we also prove a PI-version of a well-known result on spherical varieties and a version of Schelter's catenarity theorem for $G$-primes.

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## 1. Introduction.

**1.1.** This paper addresses the following general question:

> Suppose a group $G$ acts by automorphisms on a ring $R$. When does $R$ have only finitely many $G$-prime ideals?

Recall that a proper $G$-stable (two-sided) ideal $I$ of $R$ is called *$G$-prime* if $AB \subseteq I$ for $G$-stable ideals $A$ and $B$ of $R$ implies that $A \subseteq I$ or $B \subseteq I$. The set of all $G$-prime ideals of $R$ will be denoted by

$$G\text{-Spec}\, R.$$

Continuing our investigations in [**21**] and [**22**], our main focus will be on the case where $R$ is an algebra over an algebraically closed base field $\Bbbk$ and $G$ is an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$; see 2.2 below for a brief reminder on rational actions. This setting will be assumed for the remainder of the Introduction. For noetherian $R$ and an algebraic torus $G$, the above question was stated as Problem II.10.6 in [**7**].

**1.2** The question in Section 1 is motivated in part by the stratification of the prime spectrum Spec $R$ that is induced by the action of $G$. Namely, there is a surjection

$$\text{Spec}\, R \twoheadrightarrow G\text{-Spec}\, R$$

sending a given prime ideal $P$ to the largest $G$-stable ideal of $R$ that is contained in $P$. A precise description of the fibres of this map in terms of *commutative* algebras is given in [**22**, Theorem 9]. Hence, from a noncommutative perspective, the focus shifts to the description of $G$-Spec $R$, with finiteness being the optimal scenario. It turns out that, as long as the deformation parameters are sufficiently generic, $G$-Spec $R$ is indeed finite for all quantized coordinate algebras $R = \mathcal{O}_{\mathbf{q}}(X)$ that have been analysed in detail thus far, the acting group $G$ typically being a suitably chosen algebraic torus. Notable examples include the (generic) quantized coordinate rings of all semisimple algebraic groups (Joseph [**14**], Hodges, Levasseur and Toro [**12**]), quantum matrices and quantum Grassmannians (Cauchon, Lenagan and others; e.g, [**8**], [**9**] and [**19**]). Finiteness of $G$-Spec $R$ has also been observed for Leavitt path algebras $R$, again for the action of a suitable torus $G$ [**1**]. These finiteness results depend either on long calculations in $R$ or else on finding a presentation of $R$ as an iterated skew polynomial algebra, a class of algebras for which a finiteness result due to Goodearl and Letzter [**10**, **11**] is available. A general finiteness criterion for $G$-Spec $R$ is currently lacking.

**1.3.** In this paper, we concentrate on the case of an affine $\Bbbk$-algebra $R$ satisfying a polynomial identity (PI). The class of PI-algebras, whose structural theory was pioneered by Kaplansky [**15**], Amitsur [**2**] and Procesi [**27**], combines aspects of noncommutativity with geometric properties that are familiar from commutative algebras. In order to give the finiteness problem in Section 1 a geometric perspective, we mention the following connection with $G$-orbits of rational ideals. Here, a prime ideal $P$ of $R$ is called *rational* if $\mathcal{C}(R/P) = \Bbbk$, where $\mathcal{C}(\cdot)$ denotes the centre of the classical ring of quotients of the ring in question. Rational primes are exactly the closed points of Spec $R$; see Section 2.3.4 below for several equivalent characterizations of rationality. An ideal $P \in G$-Spec $R$ is said to be $G$-*rational* if the algebra of $G$-invariants $\mathcal{C}(R/P)^G$ coincides with $\Bbbk$. The subset of Spec $R$ consisting of all rational primes of $R$ will be denoted by Rat $R$, and $G$-Rat $R \subseteq G$-Spec $R$ will denote the set of all $G$-rational ideals. Since $R$ satisfies the ascending chain condition for semiprime ideals (2.3.1), the Nullstellensatz (2.3.3) and the Dixmier–Mœglin equivalence (2.3.4), the following result is a special case of [**22**, Proposition 14].

PROPOSITION 1. *Let $R$ be an affine PI-algebra over the algebraically closed field $\Bbbk$ and let $G$ be an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$. Then the following are equivalent:*
(i) $G$-Spec $R$ *is finite;*
(ii) $G$-Rat $R$ *is finite;*
(iii) *$G$ has finitely many orbits in* Rat $R$*;*
(iv) $G$-Rat $R = G$-Spec $R$.

Thus, the finiteness problem at hand amounts to determining when all $G$-primes of $R$ are $G$-rational.

**1.4.** In studying the finiteness question in Section 1 we may assume without loss that $G$ is connected. In this case, all $G$-primes of $R$ are actually prime, and hence $G$-Spec $R$ is the set of all $G$-stable prime ideals of $R$; see Lemma 4 below. The main result of this note concerns the special case where $R$ is an affine PI-algebra and $G$ is a torus; it generalizes a standard result on affine algebraic $G$-varieties [**17**, II.3.3 Satz 5].

THEOREM 2. *Let $R$ be a prime affine PI-algebra over the algebraically closed field $\Bbbk$ and let $G$ be an algebraic $\Bbbk$-torus that acts rationally by $\Bbbk$-algebra automorphisms on $R$. Then* $G$-Spec $R$ *is finite if and only if the action of $G$ on the centre $\mathcal{Z}(R)$ is multiplicity*

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*free: for each rational character* $\lambda\colon G \to \Bbbk^\times$*, the weight space* $\mathcal{Z}(R)_\lambda = \{r \in \mathcal{Z}(R) \mid g.r = \lambda(g)r$ *for all* $g \in G\}$ *has dimension at most* 1.

The proof of Theorem 2 will be given in Section 3 after deploying some auxiliary results and a generous amount of background material in Section 2. We remark that if $R$ is also assumed noetherian, then Theorem 2 is quite a bit easier, being an immediate consequence of Proposition 7 and Lemma 8(b) below. We conclude, in Section 4, with two results for noetherian $R$, namely a PI-version of a standard result on spherical varieties (Proposition 10) and a version of Schelter's catenarity theorem for $G$-primes (Proposition 11).

NOTATIONS AND CONVENTIONS. All rings have a 1 which is inherited by subrings and preserved under homorphisms. The action of the group $G$ on the ring $R$ will be written as $G \times R \to R$, $(g, r) \mapsto g.r$. For any ideal $I$ of $R$, we will write $I{:}G = \bigcap_{g\in G} g.I$; this is the largest $G$-stable ideal of $R$ that is contained in $I$. The symbol $\subset$ denotes a proper inclusion.

## 2. Preliminaries.

**2.1. Finite centralizing ring extensions.** A ring extension $R \subseteq S$ is called *centralizing* if $S = R\mathbf{C}_S(R)$, where $\mathbf{C}_S(R) = \{s \in S \mid sr = rs$ for all $r \in R\}$. In this case, for any prime ideal $P$ of $S$, the contraction $P \cap R$ is easily seen to be a prime ideal of $R$. A centralizing extension $R \subseteq S$ is called *finite*, if $S$ is finitely generated as left or, equivalently, right $R$-module. By results of Bergman [**3**, **4**] (see also [**29**]), the classical relations of lying over and incomparability for prime ideals hold in any finite centralizing extension $R \subseteq S$:

- given $Q \in \operatorname{Spec} R$, there exists $P \in \operatorname{Spec} S$ such that $Q = P \cap R$ (Lying Over);
- if $P, P' \in \operatorname{Spec} S$ are such that $P \subset P'$ then $P \cap R \subset P' \cap R$ (Incomparability).

LEMMA 3. *Let* $R \subseteq S$ *be a finite centralizing extension of rings and let* $G$ *be a group acting by automorphisms on* $S$ *that stabilize* $R$*. Assume that every ideal* $A$ *of* $S$ *contains a finite product of primes each of which contains* $A$*. Then contraction yields a surjective map*

$$G\text{-}\operatorname{Spec} S \twoheadrightarrow G\text{-}\operatorname{Spec} R\,, \qquad I \mapsto I \cap R$$

*with finite fibres. In particular, if one of* $G$-Spec $S$ *or* $G$-Spec $R$ *is finite then so is the other.*

*Proof.* First, we note that the $G$-primes of $S$ are exactly the ideals of the form $P{:}G$ with $P \in \operatorname{Spec} S$. Indeed, it is straightforward to check that $P{:}G$ is $G$-prime. Conversely, for any given $I \in G\text{-}\operatorname{Spec} S$, there are finitely many $P_i \in \operatorname{Spec} S$ (not necessarily distinct) with $I \subseteq P_i$ and $\prod_i P_i \subseteq I$. But then $I \subseteq P_i{:}G$ for each $i$ and $\prod_i P_i{:}G \subseteq I$, whence $I = P_i{:}G$ for some $i$. In particular, each $I \in G\text{-}\operatorname{Spec} S$ is semiprime. The group $G$ permutes the finitely many primes of $S$ that are minimal over $I$ and $G$-primeness forces these primes to form a single $G$-orbit. Therefore, we may write $I = P{:}G$ with $P \in \operatorname{Spec} S$ having a finite $G$-orbit. Similar remarks apply to the ring $R$, because every ideal $B$ of $R$ also contains a finite product of primes each of which contains $B$; this follows from the fact that $B$ contains some finite power of $BS \cap R$ by [**20**, Corollary 1.4].

Now let $I \in G\text{-}\operatorname{Spec} S$ be given and let $A, B$ be $G$-stable ideals of $R$ such that $AB \subseteq I \cap R$. Then, $AS = SA$ is a $G$-stable ideal of $S$ and similarly for $B$. Since $(AS)(BS) =$

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$ABS \subseteq I$, we must have $AS \subseteq I$ or $BS \subseteq I$ and hence $A \subseteq I \cap R$ or $B \subseteq I \cap R$. Thus, contraction yields a well-defined map $G\text{-Spec}\, S \to G\text{-Spec}\, R$.

For surjectivity of the contraction map, let $J \in G\text{-Spec}\, R$ be given and write $J = Q{:}G$ with $Q \in \operatorname{Spec} R$. By Lying Over we may choose $P \in \operatorname{Spec} S$ with $Q = P \cap R$. Putting $I = P{:}G$ we obtain a $G$-prime of $S$ such that $J = I \cap R$.

Finally, assume that $I \in G\text{-Spec}\, S$ contracts to a given $J \in G\text{-Spec}\, R$. Write $I = P{:}G$ with $P \in \operatorname{Spec} S$ having a finite $G$-orbit. We claim that $P$ must be minimal over the ideal $JS$. Indeed, suppose that $JS \subseteq P' \subset P$ for some $P' \in \operatorname{Spec} S$. Then Incomparability gives $P \cap R \supset P' \cap R \supseteq J = \bigcap_{g \in G} g.(P \cap R)$. Since the last intersection is finite and $P' \cap R$ is prime, we conclude that $g.(P \cap R) \subseteq P' \cap R$ for some $g \in G$. Consequently, $g.(P \cap R) \subset P \cap R$, which is impossible. This proves minimality of $P$ over $JS$. It follows that there are finitely many possibilities for $P$, and hence there are finitely many possibilities for $I$. This completes the proof of the lemma. □

The hypothesis that every ideal of $S$ contains a finite product of prime divisors is of course satisfied, by Noether's classical argument, if $S$ satisfies the ascending chain condition for ideals. More importantly, for our purposes, the hypothesis also holds for any affine PI-algebra $S$ over some commutative noetherian ring by Braun's theorem [**30**, 6.3.39].

**2.2. Rational group actions.** Let $G$ be an affine algebraic $\Bbbk$-group, where $\Bbbk$ is an algebraically closed field, and let $\Bbbk[G]$ denote the Hopf algebra of regular functions on $G$. A $\Bbbk$-vector space $M$ is called a *$G$-module* if $M$ is a $\Bbbk[G]$-comodule; see Jantzen [**13**, 2.7–2.8] or Waterhouse [**32**, 3.1–3.2]. Writing the comodule structure map $\Delta_M \colon M \to M \otimes \Bbbk[G]$ as $\Delta_M(m) = \sum m_0 \otimes m_1$, the group $G$ acts by $\Bbbk$-linear transformations on $M$ via

$$g.m = \sum m_0 m_1(g) \qquad (g \in G, m \in M).$$

Such $G$-actions, called *rational $G$-actions*, are in particular locally finite: the $G$-orbit of any $m \in M$ is contained in the finite-dimensional $\Bbbk$-subspace of $M$ that is generated by $\{m_0\}$. If $G$ acts rationally on $M$ then it does so on all $G$-subquotients of $M$. Moreover, every irreducible $G$-submodule of $M$ is finite dimensional, and the sum of all irreducible $G$-submodules is an essential $G$-submodule of $M$; it is called the *socle* of $M$ and denoted by $\operatorname{soc}_G M$. In the following, we will denote the set of isomorphism classes of irreducible $G$-modules by $\operatorname{irr} G$ and, for each $E \in \operatorname{irr} G$, we let

$$[M : E] \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$$

denote the *multiplicity* of $E$ as a *composition factor* of $M$; see [**13**, I.2.14].

We will be primarily concerned with the situation where $G$ acts rationally by algebra automorphisms on a $\Bbbk$-algebra $R$. This is equivalent to $R$ being a right $\Bbbk[G]$-comodule algebra in the sense of [**26**, 4.1.2]. In the special case where $G \cong (\Bbbk^\times)^d$ is an algebraic torus, we have $\operatorname{irr} G = X(G) \cong \mathbb{Z}^d$, the lattice of rational characters $\lambda \colon G \to \Bbbk^\times$. We will usually write $\Lambda = X(G)$. A rational $G$-action on $R$ is equivalent to a $\mathbb{Z}^d$-grading $R = \bigoplus_{\lambda \in \mathbb{Z}^d} R_\lambda$ of the algebra $R$. This follows from the fact that $\Bbbk[G]$ is the group algebra $\Bbbk\Lambda$ of the lattice $\Lambda \cong \mathbb{Z}^d$, and $\Bbbk\Lambda$-comodule algebras are the same as $\Lambda$-graded algebras; see [**26**, 4.1.7]. The homogeneous component of $R$ of degree $\lambda$ is the

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weight space

$$R_\lambda = \{r \in R \mid g.r = \lambda(g)r \text{ for all } g \in G\},$$

and $[R : \lambda] = \dim_{\Bbbk} R_\lambda$.

The following lemma was referred to in the Introduction.

LEMMA 4. *Let $R$ be a $\Bbbk$-algebra, where $\Bbbk$ is an algebraically closed field, and let $G$ be an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$. If $G^0 \subseteq G$ denotes the connected component of the identity, then $G^0$-Spec $R$ consists of all ordinary prime ideals of $R$ that are $G^0$-stable. Moreover, $G$-Spec $R$ is finite if and only if $G^0$-Spec is finite.*

*Proof.* For the assertion that all $G^0$-primes are prime, see [**21**, Proposition 19(a)].

The second assertion, that $G$-Spec $R$ is finite if and only if $G^0$-Spec is so, actually holds for any (normal) subgroup $N \trianglelefteq G$ having finite index in $G$ in place of $G^0$. Putting $\mathcal{G} = G/N$, we first note that the $G$-primes of $R$ are exactly the ideals of the form $P = \bigcap_{x\in\mathcal{G}} x.Q$ with $Q \in N$-Spec $R$. Indeed, $\bigcap_{x\in\mathcal{G}} x.Q$ is easily seen to be $G$-prime. Conversely, any $P \in G$-Spec $R$ has the form $P = P'{:}G$ with $P' \in$ Spec $R$ by [**21**, Proposition 8], and hence we may take $Q = P'{:}N$. Moreover, the intersection $\bigcap_{x\in\mathcal{G}} x.Q$ determines the $N$-prime $Q$ to within $\mathcal{G}$-conjugacy, because all $x.Q$ are $N$-prime ideals of $R$ and $\mathcal{G}$ is finite. Therefore, finiteness of $N$-Spec $R$ is equivalent to finiteness of $G$-Spec $R$. □

**2.3. Some ring theoretic background on affine PI-algebras.** Let $R$ be an affine PI-algebra over a commutative noetherian ring $\Bbbk$. The following facts are well known.

**2.3.1. Semiprime ideals.** The ring $R$ satisfies the ascending chain condition for semiprime ideals and, for each ideal $I$ of $R$, there are only finitely many primes of $R$ that are minimal over $I$. If $I$ is semiprime then $R/I$ is a right and left Goldie ring and the extended centroid of $R/I$, in the sense of Martindale [**24**], is given by $\mathcal{C}(R/I) = \mathcal{Z}(Q(R/I))$, the centre of the classical ring of quotients of $R/I$. If $I$ is prime, then $\mathcal{C}(R/I)$ is identical to the field of fractions of $\mathcal{Z}(R/I)$ by Posner's theorem. See [**30**, 6.1.30, 6.3.36′], [**25**, 13.6.9], [**21**, 1.4.2] for all this.

**2.3.2. *G*-prime ideals.** By Braun's theorem [**30**, 6.3.39], every ideal $I$ of $R$ contains a finite product of primes that contain $I$. As in the proof of Lemma 3, it follows that for any group $G$ acting by ring automorphisms on $R$, the $G$-primes of $R$ are exactly the ideals of the form $P{:}G$ with $P \in$ Spec $R$. Moreover, $P$ can be chosen to have a finite $G$-orbit. In particular, every $I \in G$-Spec $R$ is semiprime. The ring of $G$-invariants $\mathcal{C}(R/I)^G$ is a field for every $I \in G$-Spec $R$; see [**21**, Proposition 9].

**2.3.3. Nullstellensatz.** If $\Bbbk$ is a Jacobson ring then so is $R$: every prime ideal of $R$ is an intersection of primitive ideals. Moreover, if $P$ is a primitive ideal of $R$ then $P$ is maximal; in fact, $\Bbbk/P \cap \Bbbk$ is a field and $R/P$ is a finite-dimensional algebra over this field; see [**30**, 6.3.3].

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**2.3.4. Rational ideals and the Dixmier–Mœglin equivalence.** Now assume that $\Bbbk$ is an algebraically closed field. Recall that a prime ideal $P$ of $R$ is said to be *rational* if $\mathcal{C}(R/P) = \Bbbk$ or, equivalently, $\mathcal{Z}(R/P) = \Bbbk$. By Posner's theorem [**30**, 6.1.30], this forces $P$ to have finite $\Bbbk$-codimension in $R$. In fact, for any prime ideal of $R$, the following properties coincide (*Dixmier–Mœglin equivalence*), the implications $\Rightarrow$ being either trivial or immediate from the Nullstellensatz:

$$\text{finite codimensional} \equiv \text{maximal} \equiv \text{locally closed in Spec}\, R \equiv \text{primitive} \equiv \text{rational}.$$

**2.4. The trace ring of a prime PI-ring.** Let $R$ be a prime PI-ring with centre $C = \mathcal{Z}(R)$. By Posner's theorem [**30**, 6.1.30], the central localization $Q(R) = R_{C\setminus\{0\}}$ is a central simple algebra over the field of fractions $F = Q(C) = \mathcal{C}(R)$. For each $q \in Q(R)$ we can consider the *reduced characteristic polynomial* $c_q(X) \in F[X]$. In detail, letting $F^{\text{alg}}$ denote an algebraic closure of $F$, we have an isomorphism of $F^{\text{alg}}$-algebras

$$\varphi\colon Q(R) \otimes_F F^{\text{alg}} \cong \mathrm{M}_n(F^{\text{alg}}) \tag{1}$$

for some $n$. This isomorphism allows us to define $c_q(X)$ as the characteristic polynomial of the matrix $\varphi(q \otimes 1) \in \mathrm{M}_n(F^{\text{alg}})$. One can show that $c_q(X)$ has coefficients in $F$ and is independent of the choice of the isomorphism $\varphi$; see [**28**, Section 9a] or [**6**, Section 12.3].

The *commutative trace ring* of $R$, by definition, is the $C$-subalgebra of $F$ that is generated by the coefficients of all polynomials $c_r(X)$ with $r \in R$; this algebra will be denoted by $T$. The *trace ring* of $R$, denoted by $TR$, is the $C$-subalgebra of $Q(R)$ that is generated by $R$ and $T$. The following result is standard; see [**25**, 13.9.11] or [**31**, 3.2].

LEMMA 5. *Let $R$ be a prime PI-ring that is an affine algebra over some commutative noetherian ring $\Bbbk$. Then $T$ is an affine commutative $\Bbbk$-algebra and $TR$ is a finitely generated $T$-module. Furthermore, $TR$ is finitely generated as $R$-module if and only if $R$ is noetherian.*

Now suppose that a group $G$ acts by ring automorphisms on $R$. The action of $G$ extends uniquely to an action on the trace ring $TR$, and this action stabilizes $T$. To see this, note that the $G$-action on $R$ extends uniquely to an action on the ring of fractions $Q(R)$. Each $g \in G$ stabilizes $F = \mathcal{Z}(Q(R))$, and hence $g$ yields an automorphism of $F[X]$ via its action on the coefficients of polynomials. The reduced characteristic polynomials of $q \in Q(R)$ and of $g.q$ are related by

$$c_{g.q}(X) = g.c_q(X). \tag{2}$$

Indeed, extending $g$ to a field automorphism of $F^{\text{alg}}$, we obtain automorphisms $\mathrm{M}_n(g) \in \operatorname{Aut} \mathrm{M}_n(F^{\text{alg}})$ and $\alpha_g \in \operatorname{Aut} Q(R) \otimes_F F^{\text{alg}}$, the latter being given by $\alpha_g(q \otimes f) = g.q \otimes g.f$. Fixing $\varphi$ as in (1) we obtain an isomorphism of $F^{\text{alg}}$-algebras $\mathrm{M}_n(g)^{-1} \circ \varphi \circ \alpha_g\colon Q(R) \otimes_F F^{\text{alg}} \cong \mathrm{M}_n(F^{\text{alg}})$. Using this isomorphism to compute reduced characteristic polynomials, we see that $c_q(X) = g^{-1}.c_{g.q}(X)$, proving (2). Since $g.r \in R$ for $r \in R$, equation (2) shows that the commutative trace ring $T$ is stable under the action of $G$ on $Q(R)$, and hence so is the trace ring $TR$. For rational actions, we have the following result of Vonessen [**31**, Proposition 3.4].

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LEMMA 6 (Vonessen [31]). *Let $R$ a prime PI-algebra over an algebraically closed field $\Bbbk$ and let $G$ be an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$. Then the induced $G$-actions on $TR$ and on $T$ are rational as well.*

In general, the finiteness problem stated in the Introduction transfers nicely to trace rings.

PROPOSITION 7. *Let $R$ be a prime PI-ring that is an affine algebra over some commutative noetherian ring. Let $G$ be a group acting by ring automorphism on $R$ and consider the induced $G$-actions on $T$ and on $TR$. Then $G$-Spec $T$ is finite if and only if $G$-Spec $TR$ is finite. If $R$ is noetherian, then this is also equivalent to $G$-Spec $R$ being finite.*

*Proof.* Lemma 3, applied to the finite centralizing extension $T \subseteq TR$ (Lemma 5), tells us that finiteness of $G$-Spec $TR$ is equivalent to finiteness of $G$-Spec $T$. If $R$ is noetherian, then we may argue in the same way for the finite centralizing extension $R \subseteq TR$. □

**3. Main result.** *Throughout this section, $R$ denotes an affine PI-algebra over an algebraically closed field $\Bbbk$ and $G$ will be an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$.*

**3.1. Sufficient criteria for $G$-rationality.** By Proposition 1 we know that $G$-Spec $R$ is finite if and only if all $G$-primes of $R$ are $G$-rational. Therefore, $G$-rationality criteria are essential. As usual, the algebra $R$ will be called $G$-prime if the zero ideal of $R$ is $G$-prime; similarly for $G$-rationality.

LEMMA 8. *Assume that $R$ is $G$-prime.*

(a) *If there is an $N \in \mathbb{Z}$ such that $[\mathrm{soc}_G\, \mathcal{Z}(R) : E] \le N$ for all $E \in \mathsf{irr}\, G$ then $R$ is $G$-rational.*

(b) *If $G$ is connected solvable then $R$ is $G$-rational if and only if $[\mathrm{soc}_G\, \mathcal{Z}(R) : E] \le 1$ for all $E \in \mathsf{irr}\, G$.*

*Proof.* (a) For a given $q \in \mathcal{C}(R)^G$ put $I = \{r \in R \mid qr \in R\}$; this is a nonzero $G$-stable ideal of $R$. Therefore, $J = I^N \cap \mathcal{Z}(R)$ is a nonzero $G$-stable ideal of $\mathcal{Z}(R)$; see [**30**, 6.1.28]. Note that $q^i J \subseteq \mathcal{Z}(R)$ for $0 \le i \le N$. We have $E \hookrightarrow J$ for some $E \in \mathsf{irr}\, G$ and multiplication with $q^i$ yields a $G$-equivariant map $E \hookrightarrow J \to \mathcal{Z}(R)$. Since $\dim_{\Bbbk} \mathrm{Hom}_G(E, \mathcal{Z}(R)) = [\mathrm{soc}_G\, \mathcal{Z}(R) : E] \le N$, there are $k_i \in \Bbbk$, not all 0, such that $c = \sum_{i=0}^N k_i q^i$ annihilates $E$. But nonzero elements of $\mathcal{C}(R)^G$ are invertible; so we must have $c = 0$. Thus, $q$ is algebraic over $\Bbbk$ and so $q \in \Bbbk$.

(b) The condition is sufficient by part (a). For the converse, assume that $E_1 \oplus E_2 \subseteq \mathcal{Z}(R)$ for isomorphic $E_i \in \mathsf{irr}\, G$. By the Lie–Kolchin Theorem [**5**, III.10.5], $E_i = \Bbbk x_i$ for suitable $x_i$. Since $x_i$ generates a $G$-stable two-sided ideal, $x_i$ is regular in $R$. The quotient $x_1 x_2^{-1} \in \mathcal{C}(R)$ is a nonscalar $G$-invariant; so $R$ is not $G$-rational. □

REMARK. A simplified version of the argument in the proof of (a), without recourse to [**30**, 6.1.28], establishes the following general fact: Let $A$ be an arbitrary (associative) $\Bbbk$-algebra and let $G$ be a group that acts on $A$ by locally finite $\Bbbk$-algebra automorphisms. If there is an $N \in \mathbb{Z}$ such that $[A : E] \le N$ holds for all finite-dimensional irreducible

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$\Bbbk G$-modules $E$, where $[A : E]$ denotes the multiplicity of $E$ as composition factor of $A$ as in [**13**, I.2.14], then $G$-Spec $A = G$-Rat $A$.

**3.2. Regular primes.** Recall from (1) that if $R$ is prime, then the classical ring of quotients $Q(R)$ is a central simple algebra over the field of fractions $F = Q(\mathcal{Z}(R))$. The *PI degree* of $R$, by definition, is the degree of this central simple algebra: $\operatorname{PI deg} R = \sqrt{\dim_F Q(R)}$. For any $P \in \operatorname{Spec} R$, one has $\operatorname{PI deg} R/P \leq \operatorname{PI deg} R$. The prime $P$ is called *regular* if equality holds here. The regular primes form an open subset of Spec $R$. See [**30**, p. 104] or [**25**, 13.7.2] for all this.

Now let $G$ be an algebraic $\Bbbk$-torus. In particular, $G$ is connected and so $G$-Spec $R$ consists of the $G$-stable prime ideals of $R$ by Lemma 4.

LEMMA 9. *Let $G$ be an algebraic $\Bbbk$-torus and assume that $R$ is prime. Then, for every regular $P \in G$-Spec $R$, we have $\operatorname{tr deg}_{\Bbbk} \mathcal{C}(R/P)^G \leq \operatorname{tr deg}_{\Bbbk} \mathcal{C}(R)^G$. Consequently, if $R$ is $G$-rational then all regular primes in $G$-Spec $R$ belong to $G$-Rat $R$.*

*Proof.* Let $P \in G$-Spec $R$ be regular. Put $n = \operatorname{PI deg} R$ and let $g_n(R)^+$ denote the Formanek centre of $R$; this is a $G$-stable ideal of $\mathcal{Z}(R)$ such that $g_n(R)^+ \nsubseteq P$ (cf. [**30**, 6.1.37] or [**25**, 13.7.2(i)]). Therefore, we may choose a semi-invariant $c \in g_n(R)^+_\lambda$ with $c \notin P$. The group $G$ acts rationally on the localization $R_c = R[1/c]$ and $R_c$ is Azumaya by the Artin-Procesi theorem [**25**, 13.7.14]. Therefore, $\mathcal{Z}(R_c)$ maps onto $\mathcal{Z}(R_c/PR_c)$ and $\mathcal{Z}(R_c)_\lambda$ maps onto $\mathcal{Z}(R_c/PR_c)_\lambda$ for all $\lambda \in X(G)$. The map $\mathcal{Z}(R_c) \twoheadrightarrow \mathcal{Z}(R_c/PR_c)$ extends to a $G$-equivariant epimorphism $\mathcal{Z}(R_\mathfrak{p}) \twoheadrightarrow \mathcal{C}(R/P) = Q(\mathcal{Z}(R_c/PR_c))$, where $\mathfrak{p} = P \cap \mathcal{Z}(R)$. Since $\mathcal{Z}(R_\mathfrak{p})^G \subseteq \mathcal{C}(R)^G$, it suffices to show that $\mathcal{Z}(R_\mathfrak{p})^G$ maps onto $\mathcal{C}(R/P)^G$. But, given $q \in \mathcal{C}(R/P)^G$, we can find a semi-invariant $0 \neq x \in \mathcal{Z}(R_c/PR_c)_\lambda$ such that $qx \in \mathcal{Z}(R_c/PR_c)$, and we can further find $y, z \in \mathcal{Z}(R_c)_\lambda$ with $y \mapsto x$ and $z \mapsto qx$. Then $zy^{-1} \in \mathcal{Z}(R_\mathfrak{p})^G$ maps to $q$. This proves the lemma. □

**3.3. Proof of Theorem 2.** Let $G$ be an algebraic $\Bbbk$-torus and assume that $R$ is prime. We need to show that $G$-Spec $R$ is finite if and only if the action of $G$ on $\mathcal{Z}(R)$ is multiplicity free. By Lemma 8(b), the latter property is equivalent to $G$-rationality of $R$, and this is certainly necessary for $G$-Spec $R$ to be finite by Proposition 1.

Now assume that $R$ is $G$-rational. By Proposition 1 we must show that all $G$-primes of $R$ are $G$-rational. Lemma 9 ensures this for the regular $G$-primes. In particular, we may assume that $n := \operatorname{PI deg} R > 1$. Now consider $P \in G$-Spec $R$ with $\operatorname{PI deg} R/P < n$. Then $P$ contains the ideal $\mathfrak{a} = g_n(R)R \subseteq R$; this is a nonzero $G$-stable common ideal of $R$ and of the trace ring $R' := TR$ of $R$ (cf. [**30**, 6.1.37 and 6.3.28]). All primes of $R$ that are minimal over $\mathfrak{a}$ are $G$-stable. Let $Q$ be one of these primes such that $Q \subseteq P$. It suffices to show that $Q$ is $G$-rational. For, then we may replace $R$ by $R/Q$, and since $\operatorname{PI deg} R/Q < n$, we may argue by induction that $P$ is $G$-rational.

First, we claim that there exists $Q' \in G$-Spec $R'$ with $Q' \cap R = Q$. Indeed, choosing $Q'$ to be a $G$-stable ideal of $R'$ that is maximal subject to the condition $Q' \cap R \subseteq Q$, it is straightforward to see that $Q'$ is $G$-prime. If $Q' \cap R \neq Q$ then $Q' \nsupseteq \mathfrak{a}$ by minimality of $Q$ over $\mathfrak{a}$. Thus, $Q' + \mathfrak{a}$ is a $G$-stable ideal of $R'$ which properly contains $Q'$ and yet also satisfies $(Q' + \mathfrak{a}) \cap R = (Q' \cap R) + \mathfrak{a} \subseteq Q$. Since this contradicts our maximal choice of $Q'$, we must have $Q' \cap R = Q$ as claimed.

Next, we show that $Q'$ is $G$-rational. To see this, recall from Lemma 6 that $G$ acts rationally on the trace rings $T$ and $R'$. Moreover, $T$ is an affine commutative $\Bbbk$-algebra that is $G$-rational, because $Q(T)^G = \mathcal{C}(R)^G = \Bbbk$. Therefore, by the case $n = 1$, we know

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that $G$-Spec $T$ is finite. By Proposition 7, $G$-Spec $R'$ is finite as well, and in view of Proposition 1, this forces $Q'$ to be $G$-rational.

Finally, we show that $Q$ is $G$-rational; this will finish the proof. But $\mathcal{C}(R/Q) \subseteq \mathcal{C}(R'/Q')$ and $\mathcal{C}(R'/Q')^G = \Bbbk$ by the foregoing. Therefore, $\mathcal{C}(R/Q)^G = \Bbbk$ as desired.

**4. Related results.** *In this section, $R$ and $G$ are as in the previous section and $R$ is also assumed noetherian.*

**4.1. Actions of reductive groups.** Recall from Lemma 6 that the induced $G$-action on the commutative trace ring $T$ is rational. This fact allows us to invoke results from algebraic geometry.

PROPOSITION 10. *Let $R$ be an affine noetherian PI-algebra over an algebraically closed field $\Bbbk$ and let $G$ be an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$. Assume that $R$ is prime and that $G$ is connected reductive. Let $F = Q(\mathcal{Z}(R))$ denote the field of fractions of the centre of $R$, and let $F^B \subseteq F$ denote the invariant subfield of a Borel subgroup $B \leq G$. If $F^B = \Bbbk$ then $B$-Spec $R$ is finite (and hence $G$-Spec $R$ is finite as well).*

*Proof.* By Proposition 7, $B$-Spec $R$ is finite if and only if $B$-Spec $T$ is finite. Now, $T$ is an affine commutative domain over $\Bbbk$ and the field of fractions of $T$ is $F$. By a standard result on spherical varieties [**16**, Corollary 2.6], the condition $F^B = \Bbbk$ implies that there are only finitely many $B$-orbits in Rat $T$. The latter fact is equivalent to finiteness of $B$-Spec $T$ by Proposition 1, which proves the proposition. □

**4.2. Catenarity.** A partially ordered set $(P, \leq)$ is said to be *catenary* if, given any two $x < x'$ in $P$, all saturated chains $x = x_0 < x_1 < \cdots < x_r = x'$ have the same finite length $r = r(x, x')$.

In the commutative case, the following observation goes back to conversations that I had with R. Rentschler a long time ago; cf. [**23**, Section 3]. As usual, GK dim denotes Gelfand–Kirillov dimension.

PROPOSITION 11. *Let $R$ be an affine noetherian PI-algebra over an algebraically closed field $\Bbbk$ and let $G$ be an affine algebraic $\Bbbk$-group that acts rationally by $\Bbbk$-algebra automorphisms on $R$. If the connected component of the identity of $G$ is solvable then the poset $(G\text{-Spec}\, R, \subseteq)$ is catenary. In fact, every saturated chain $P = P_0 \subset P_1 \subset \cdots \subset P_r = P'$ in $G$-Spec $R$ has length $r = \text{GK dim}\, R/P - \text{GK dim}\, R/P'$.*

*Proof.* First assume that $G$ is connected; so $G$-Spec $R$ consists of the $G$-stable primes of $R$. In view of Schelter's catenarity theorem for Spec $R$ [**30**, 6.3.43], we need to show that any two neighbours $Q \subset P$ in $G$-Spec $R$ are also neighbours when viewed in Spec $R$. Passing to $R/Q$ we may assume that the algebra $R$ is prime and $P$ is a minimal nonzero member of $G$-Spec $R$, and we need to show that $P$ has height 1 in Spec $R$. But $P \cap \mathcal{Z}(R)$ is a nonzero $G$-stable ideal of $\mathcal{Z}(R)$ and hence the Lie–Kolchin theorem provides us with a $G$-eigenvector $0 \neq z \in P \cap \mathcal{Z}(R)$. The ideal $P$ is a minimal prime over $(z)$. For, if $(z) \subseteq P' \subset P$ for some $P' \in$ Spec $R$ then $(z) \subseteq P'{:}G \subset P$ and $P'{:}G \in G$-Spec $R$, contradicting the fact that $P$ is a minimal nonzero member of $G$-Spec $R$. Thus, $P$ is minimal over $(z)$ as claimed, and the principal ideal theorem [**25**, 4.1.11] gives that $P$ has height 1 as desired.

In general, let $G^0$ denote the connected component of the identity of $G$ and put $\mathcal{G} = G/G^0$. Write $Q = \bigcap_{x\in\mathcal{G}} x.\widetilde{Q}$ and $P = \bigcap_{x\in\mathcal{G}} x.\widetilde{P}$ for suitable $\widetilde{Q}, \widetilde{P} \in G^0$-Spec $R$ as in the proof of Lemma 4. Since these intersections are finite intersections of $G^0$-primes of $R$, we can arrange that $\widetilde{Q} \subset \widetilde{P}$. The ideals $\widetilde{Q}$ and $\widetilde{P}$ are neighbours in $G^0$-Spec $R$. For, if $\widetilde{Q} \subset \widetilde{T} \subset \widetilde{P}$ for some $\widetilde{T} \in G^0$-Spec $R$ then $Q \subset \bigcap_{x\in\mathcal{G}} x.\widetilde{T} \subset P$ since $\mathcal{G}$ is finite, which contradicts the fact that $Q$ and $P$ are neighbours in $G$-Spec $R$. By the first paragraph of the proof, $\widetilde{Q}$ and $\widetilde{P}$ are also neighbours in Spec $R$, and hence GK dim $R/\widetilde{Q} =$ GK dim $R/\widetilde{P} + 1$ by Schelter's theorem. Since GK dim $R/Q =$ GK dim $R/\widetilde{Q}$ and GK dim $R/P =$ GK dim $R/\widetilde{P}$ by [**18**, Corollary 3.3], the proof is complete. □

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