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THE SECOND-ORDER ANALYSIS OF STATIONARY POINT PROCESSES

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Abstract

This paper provides a rigorous foundation for the second-order analysis of stationary point processes on general spaces. It illuminates the results of Bartlett on spatial point processes, and covers the point processes of stochastic geometry, including the line and hyperplane processes of Davidson and Krickeberg. The main tool is the decomposition of moment measures pioneered by Krickeberg and Vere-Jones. Finally some practical aspects of the analysis of point processes are discussed.

MOMENT MEASURES; STATIONARY POINT PROCESSES; SPATIAL POINT PROCESSES; LINE PROCESSES; HYPERPLANE PROCESSES; STOCHASTIC GEOMETRY

1. Introduction

We assume throughout this paper that X is a topological space and G is a topological group acting continuously on X (i.e. there is a continuous map $(g,x)\to gx$ from $G\times X$ to X satisfying $g(hx)=(gh)x$ and $ex=x$). We suppose both G and X are LCD spaces, that is locally compact Hausdorff spaces with countable bases (which thus are σ-compact). A typical example is the group G of rigid motions acting on the plane X. Let $\mathscr{A}$ denote the Borel (equivalently, Baire) σ-field of X and $\mathscr{B}$ the class of relatively compact sets (in this example the usual bounded sets). The realizations of a point process on X will be *locally finite multi-sets*, i.e. collections of points from X, possibly repeated, but with only a finite number of occurrences of points from any member of $\mathscr{B}$. (For our point process theory we follow Ripley (1976b) which contains proofs of our assertions.) We can identify this class with N, the class of σ-additive functions $n:\mathscr{A}\cap\mathscr{B}\to\mathbf{Z}_+$, n corresponding to the multi-set of $n(\{x\})$ x's for each x. Let $\mathscr{N}$ be the smallest σ-field on N making all the evaluation maps measurable; one may count the number of points in any member of $\mathscr{C}=\mathscr{A}\cap\mathscr{B}$, the class of bounded measurable sets. (This is the natural σ-field on N.) We define a *point process* on X to be a measurable map Z from a probability space to $(N,\mathscr{N})$, and its

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distribution to be the probability Z induces on $\mathcal{N}$. We say the point process is *simple* if its range is in the measurable subset $N_0 = \{n \mid n(\{x\}) \leqq 1 \forall x \in X\}$. This subset corresponds to the class of locally finite *sets* (with no repeated points). We define G to act on N by $gn(A) = n(g^{-1}A)$ for $A \in \mathscr{C}$ (G maps $\mathscr{C}$ onto $\mathscr{C}$), corresponding to transforming the multi-set by g. Then, for the Borel σ-field on G, $(g, n) \to gn$ is measurable (Mecke (1967), Anhang B, Ripley (1976c), Proposition 6). We say the point process is *stationary* if its distribution is invariant under G. Theorem 4 of Ripley (1976b) shows we need only check invariance for certain special sets in $\mathcal{N}$. For our example these could be the finite-dimensional distributions (or, for a simple point process, the hitting probabilities) for sets from a class $\mathscr{T}$, for instance the class of open discs or the classes of open or half-open rectangles.

We may wish to consider marked point processes, an observation or type being associated with each point. Let $(K, \mathscr{K})$ be the measurable space of marks. We replace $(X, \mathscr{A}, \mathscr{B})$ by $(X \times K, \mathscr{A} \otimes \mathscr{K}, \mathscr{B} \times \{K\})$. Precisely how we treat marks will depend on their nature; they occur in the examples.

2. Moment measures

Moment measures (Moyal (1962), Krickeberg (1970), (1973), Vere-Jones (1968), (1970)) are the modern versions of product densities (Bhabha (1950), Ramakrishnan (1950)). We shall be concerned with the first two moments; higher moments may be treated analogously.

For a point process Z and $m = 1, 2, \cdots$ define

$$\mu^m\left(\prod_1^m A_i\right) = \mathbf{E}\left(\prod_1^m Z(A_i)\right), \text{ for } A_i \in \mathscr{C}.$$

We say the process is *second-order* if μ^2 is finite on $\mathscr{C} \times \mathscr{C}$. Each μ^m is the expectation of the measure Z^m on the semi-ring $\mathscr{C}^m$, so μ^m is a measure which may be extended to the σ-ring generated by $\mathscr{C}^m$, which is the Borel σ-field of X^m. For a second-order process and $m \in \{1, 2\}$ this extension is unique and a Radon measure (i.e. finite on compact sets), called the *mth moment measure.* If Z is stationary then both μ^1 and μ^2 are invariant under G (acting on X^2 by $g(x, y) = (gx, gy)$). From now on we assume all point processes have invariant Radon first- and second-moment measures.

We now assume G acts transitively on X, so any point in X may be transformed to any other. Then X is a topological homogeneous space (of the group G) (Bourbaki (1963) Appendix 1, Lemma 2), so μ^1 is proportional to the invariant Radon measure ν on X, unique up to a scalar factor but possibly null (Nachbin (1965), p. 138). Thus we can have non-trivial processes only on spaces for which ν is non-null.

G defines an equivalence relation on $Y = X^2$ by $(x_1, x_2) \sim (y_1, y_2)$ if there is

$g \in G$ with $y_i = gx_i$, $i = 1, 2$. Let T be the quotient space and p the quotient map $Y \to T$. We will represent μ^2 by a measure on T. To do so it transpires we need a σ-field $\mathcal{T}$ on T and a normalizing function $h: Y \to \mathbf{R}$ satisfying:

(a) $p: Y \to (T, \mathcal{T})$ is measurable;

(b) $(T, \mathcal{T})$ is countably separated, i.e. there is a countable subclass of $\mathcal{T}$ separating points of T;

(c) h is non-negative, measurable and bounded;

(d) $\{h > 0\} \cap Y_t$ has a non-void relative interior in Y_t for each t, where $Y_t = p^{-1}(\{t\})$;

(e) there is a countable measurable partition (T_n) of T such that $\{h > 0\} \cap p^{-1}(T_n)$ is relatively compact for each n.

Then we have the following results (Ripley (1976a)). For each $t \in T$ there is at most one invariant Radon measure ν_t on Y, concentrated on Y_t, with $\nu_t(h) = 1$. If there is no such measure we set $\nu_t \equiv 0$. If Y_t is a G_δ in Y (in particular, if Y_t is closed) it may be identified with a homogeneous space, and ν_t is proportional to the invariant Radon measure on that space. There is a σ-finite measure κ on T, finite on $\{T_n\}$, such that

$$\mu^2(A) = \int_T \nu_t(A)\, d\kappa(t)$$

for all Borel sets A of Y. If no ν_t is null then κ is unique. Notice the diagonal D of Y is always a closed equivalence class Y_0 which may be identified with X as a topological homogeneous space. Thus ν_0 is proportional to the measure m defined by $m(A) = \nu(\pi_1(D \cap A))$, π_1 being the first co-ordinate projection.

Thus the first two moment measures may be summarized by the intensity λ, where $\mu^1 = \lambda\nu$, and κ, the reduced second-moment measure. These results should be clarified by the following examples. Ripley (1976a) contains further material on whether a pair $(\mathcal{T}, h)$ can be chosen. This is always so if every equivalence class is closed.

3. Points in space

Let X be $\mathbf{R}^2$ or $\mathbf{R}^3$ and G be the appropriate group of rigid motions with the usual topology (Nachbin (1965)). The equivalence classes in Y are specified by the distance between a pair of points, so we may identify T with $[0, \infty)$, Y_0 being D. The distance function is jointly continuous, so the usual topology on $[0, \infty)$ makes p continuous. We take $\mathcal{T}$ to be the Borel σ-field of this topology and h to be the indicator of $B \times X$, B being the open unit cube. If we take $T_n = [n, n + 1)$, (a)–(e) are satisfied, so κ will be finite on each T_n and so be a Radon measure. For $t \neq 0$ ν_t may be found by the usual methods of integral geometry. The distribution of the inter-point distances considered by Bartlett (1964) is closely related to κ, as we shall see in Section 6.

A variation of this example is to take $X = \mathbf{S}^2$, the sphere, and $G = SO(3)$, the appropriate set of rigid motions (the group of all rotations). Again the equivalence classes are specified by the (great-circle) distance between any pair of points, so we identify T with $[0, \pi]$, and take $h \equiv 1$. We can identify Y_π with X, since Y_π consists of pairs of anti-polar points, and so obtain ν_π. We can identify Y_t, $t \not\in \{0, \pi\}$ with G, and so find ν_t from the left Haar measure on G.

4. Groups

Suppose X is itself the group. The interesting special cases will be $\mathbf{R}^n$ (especially $\mathbf{R}^1$), the circle $\mathbf{S}$, and the group of rigid motions of the plane, (which will occur in Section 5). For $\mathbf{R}^1$ the moment measures and their decompositions have been studied by Vere-Jones (1968), (1970), (1971). Note that, in the terminology of Cox and Lewis (1966), we are considering only second-order properties of *counts*; the concept of the *interval* between occurrences does not generalize from $\mathbf{R}^1$.

Let $S = \{e\} \times X$. Then S is a section of Y and so may be identified with T, as well as with $X = G$. Let $\mathcal{T}$ be the largest σ-field on T making p measurable, so $\psi = p \mid S$ is measurable. Let $\mathcal{U}$ denote the Borel σ-field of the second countable $T1$ quotient topology. Then $\mathcal{U} \subset \mathcal{T}$ and $(T, \mathcal{U})$ is countably separated, so ψ is an isomorphism $S \to (T, \mathcal{U})$ (Christensen (1974), Theorem 2.4); thus we may identify $(T, \mathcal{T})$, $(T, \mathcal{U})$ and $(X, \mathcal{A})$.

We choose K, a compact neighbourhood of the identity in G, and normalize the left Haar measure ν so that $\nu(K) = 1$. Let h be the indicator of $K \times X$. Let (T_n) be the partition formed by a countable family of compact sets (K_n) with $K_n \subset K^0_{n+1}$ and $K_n \uparrow X$. Then (a)–(e) are satisfied.

Let $\phi_t: G \to Y$ be given by $\phi_t(g) = g(\psi^{-1}(t))$. Define ν_t to be the image of ν under ϕ_t; ν_t is the required measure. Then

$$\mu^2(A) = \int_G \nu(\pi_1(A \cap Y_t))\, d\kappa(t)$$

for all Borel sets A of Y. Note that κ is unique, and μ^2 is invariant under permutation of the co-ordinates of Y, so κ is symmetric.

Now suppose G is Abelian. Then $T = G^2/D$ is a group and ψ is a homomorphism. T is an LCD space (Bourbaki (1966) III, §2.5) and the group S acts continuously on T, so ψ is a homeomorphism (Bourbaki (1963) Appendix 1, Lemma 2). Thus we identify T with S and G topologically. The choice of (T_n) shows κ is a Radon measure on T. This result can be improved. The following generalizes a result of Vere-Jones (1974). For $h \in G$ we have

$$\kappa(hK) = \mu^2(K(\{e\} \times hK)) \leqq \mu^2(K \times hKK) = \mathbf{E}(Z(K)Z(hKK))$$
$$\leqq \mathbf{E}(Z(K)^2)^{\frac{1}{2}}\mathbf{E}(Z(hKK)^2)^{\frac{1}{2}} \leqq \mathbf{E}(Z(KK)^2).$$

For $G = \mathbf{R}^n$ this shows κ is a tempered measure, which is easily seen to be positive definite, and so has a similar measure as its Fourier transform (in the sense of generalized functions, (Gel'fand and Vilenkin (1964))). Another interesting special case is $\mathbf{S}$. Here κ may be characterized by its cosine Fourier series.

5. Examples from stochastic geometry

The basic transformation of stochastic geometry (Kendall (1974), Davidson (1970), Ripley (1976c)) brings random collections of geometrical objects within our scope. The appropriate class $\mathcal{B}$ need not be the class of relatively compact sets (although it usually is), but it will always contain that class, so our decomposition of μ^m is still valid.

One of the first applications of the decomposition was to oriented line processes (Krickeberg (1970)). We will discuss unoriented lines to illustrate the differences and the greater simplicity of our approach. Our like processes differ from those of Bartlett (1967) in that we assume stationarity under the group G of rigid motions rather than in a preferred direction only. Bartlett's line processes are better considered as point processes on the real line marked with the angle.

The most natural parametrization of the set X of lines in the plane is by the direction and length of the normal to the origin. This is not uniquely defined for lines passing through the origin. Thus we limit the angle of the normal to $[0, \pi)$, and let the length be signed, so identifying X with $\mathbf{R} \times \mathbf{S}$. The group G acts continuously on X with the induced topology, so this is the topology given by the general theory (Ripley (1976c)), and $\mathcal{B}$ is the class of relatively compact sets (*loc. cit.*, Proposition 5). The space T of equivalence classes splits into two parts; T_0, the classes of pairs which intersect, indexed by the included angle, and T_1, the classes of pairs of parallel lines, indexed by the separation. Thus T can be identified with the sum of $[0, \pi)$ and $(0, \infty)$. We give both T_0 and T_1 the induced topology, and take $\mathcal{T}$ as the sum of the Borel σ-fields. It is easily verified that $p^{-1}(T_0)$ is open, and p is continuous both on this set and its complement, so (a) and (b) hold. Let B denote $[-1, 1] \times \mathbf{S}$, and h be the indicator of $((B \times B) \cap p^{-1}(T_0)) \cup ((B \times X) \cap p^{-1}(T_1))$. Then (c)–(e) hold. For the classes other than the diagonal in T_0 the pair of lines forms a symmetric rigid body, so ν_t may be found from the kinematic measure. For a class in T_1, ν_t may be found by identifying the pair of lines with an oriented line coincident with the first, the orientation being chosen so that the second line lies on the left. The invariant measure for oriented lines is known (Santaló (1953)).

Notice that for processes a.s. without parallel lines (equivalently, μ^2 is concentrated on $p^{-1}(T_0)$, so this is a measurable event), the off-diagonal part of the second moment measure depends only on the distribution of the directions of the lines.

We now consider half-infinite rays in the plane. Such a ray may be parametrized by its end-point and direction, so we identify the set X of rays with $\mathbf{R}^2 \times \mathbf{S}$. As a topological homogeneous space X is identified with G, so ν is proportional to the left Haar measure on G. Now $\mathscr{B}$ is the ideal of subsets of X generated by $\{\{x \mid x \text{ hits } K\} \mid K \text{ compact in } \mathbf{R}^2\}$, which strictly contains the relatively compact sets of X, and the left Haar measure is not finite on $\mathscr{A} \cap \mathscr{B}$. Thus $\nu \equiv 0$, and there are no second-order stationary processes of rays.

As our final example we consider the class of finite oriented line segments in the plane. Such a line segment is parametrized by its origin, direction and length. We take $X = \mathbf{R} \times \mathbf{S}$ to be the set of origins and directions, and $K = (0, \infty)$ to be the mark space of lengths. Under this parametrization the group G of rigid motions acts continuously on X, which is identified as a topological homogeneous space with G. (This is a case in which we may wish to replace $\mathscr{B} \times \{K\}$ in the definition of locally finite. One possible choice would be the ideal of subsets of $X \times K$ generated by $\{\{(x, k) \mid (x, k) \text{ hits } C\} \mid C \text{ compact in } \mathbf{R}^2\}$, so that in each realization of the point process a finite number of line segments meets each compact set, rather than just have finitely many segments with origins in any compact set.) The group G acting on $X \times K$ in the obvious manner acts continuously but not transitively. However, the theory of Section 2 still applies, and we can identify T with $G \times K^2$ as in Section 4, and find ν_t, where $t = (g, k, h)$ from the ν_g of Section 4. Thus

$$\mu^2(A_1 \times B_1 \times A_2 \times B_2) = \int_G \nu_g(A_1 \cap g^{-1}A_2)\kappa(dg \times B_1 \times B_2)$$

for a unique measure κ on T, $A_i \in \mathscr{A}$ and $B_i \in \kappa$.

6. Discussion

We have shown how the theoretical second-order properties of a stationary point process may be summarized by λ and κ. The interpretation of λ as the intensity is obvious, and its estimation is well-known. In the case of $\mathbf{R}^1$ κ is closely related to the more familiar covariance density γ (Cox and Lewis (1966), p. 74); if the latter exists κ has a density $(y + \lambda^2)$.

We have seen that in several cases κ may be characterized by a Fourier transform. This seems to be more difficult to interpret than κ itself, especially as no decomposition of the process into *point process* components seems possible.

For point processes in the plane, κ is a measure of the distribution of the inter-point distances. Experiments (Julesz (1975)) suggest that the eye perceives only the second-order properties of a spatial pattern, giving support to second-order analysis.

Suppose we observe that part of a single realization of a process within a bounded set E, which in our numerical examples is a square. For E a square or circle the distribution of the inter-point distances for a Poisson process is known (Bartlett (1964)), and, as Bartlett suggested, we can plot the observed inter-point distance distribution F' and compare this with the theoretical distribution F under the Poisson hypothesis. The observed distribution has a covariance which can be derived from the fourth-moment measure of the Poisson process. It may be shown that $\sqrt{n}\,(F'-F)$, where n is the number of points, converges weakly to a Gaussian process (Silverman (1976)).

This procedure can be used only to test the Poisson hypothesis. A more general procedure is based on an unbiased estimator of $\kappa(t)=\kappa((0,t))$. For $\mathbf{x}\in E$ and $r>0$ we define $k(\mathbf{x},r)=2\pi r/m(E\cap\partial b(\mathbf{x},r))$ if this exists, otherwise 0 (where m denotes Lebesgue measure on $\partial b(\mathbf{x},r)$). Then we define

$$F=\{(\mathbf{x},\mathbf{y})\,|\,\mathbf{x},\mathbf{y}\in E, 0<d(\mathbf{x},\mathbf{y})<t\},$$

$$G=F\cap(\{\mathbf{x}\,|\,Z(\{\mathbf{x}\})>0\})^2,\quad\text{and}\quad \hat{\kappa}(t)=\sum_G k(\mathbf{x},d(\mathbf{x},\mathbf{y}))/\nu(E).$$

Let $f(\mathbf{x},\mathbf{y})=I_F k(\mathbf{x},d(\mathbf{x},\mathbf{y}))$. Then $\mathbf{E}(\hat{\kappa}(t))=\mu^2(f)=\int \nu_s(f)\,d\kappa(s)$. Now $\nu_s(f)=0$ unless $s\in(0,t)$, when $\nu_s(f)=\nu(\{\mathbf{x}\,|\,\mathbf{x}\in E, \partial b(\mathbf{x},s)\cap E\neq\emptyset\})/\nu(E)$, so if E is the unit square then $\hat{\kappa}(t)$ is unbiased for $t\leqq 2^{-\frac{1}{2}}$.

For a Poisson process $\mu^2=\lambda^2(\nu\times\nu)$ off the diagonal, so $\kappa(t)=\pi\lambda^2t^2$. It will be most convenient to plot $\hat{\kappa}/\hat{\lambda}^2$ where $\hat{\lambda}$ is an estimate of λ (usually $Z(E)/\nu(E)$).

The distribution of $\hat{\kappa}$ is not known exactly, even for a Poisson process, although its covariance can be found from the fourth-moment measure.

Our first example shows the effect of inhibition between points; the points are the centres of cells of an insect epithelium. The plot of $\hat{\kappa}$ in Figure 1 clearly differs from the smooth curve, the theoretical value for a Poisson process with the observed intensity. This is confirmed by the dashed lines, the envelope of $\hat{\kappa}$ for twenty simulations of the Poisson process producing the same number of points (42). The shape of $\hat{\kappa}$ suggests fitting some form of 'hard-core' model. The model of Matérn ((1960), §3.6) in which *both* points of a primary Poisson process are deleted if they are within $2R$ of each other has a maximum possible intensity $1/4\pi eR^2$, which for the observed value of the interaction radius is a quarter of the observed intensity. Matérn's second, dynamic model does not seem appropriate. Other models have been suggested in liquid state theory, but these depend on boundary conditions and so do not yield *stationary* point processes.

In investigations of inhibition between points Bartlett (1974) plotted the distribution of the nearest-neighbour distances. Unless the data is collected in that form, our method has advantages in fitting models (κ can be computed) and easier simulation.

Our second example illustrates the opposite phenomenon, clustering. The data are taken from Strauss (1975), from the clustered groups of redwood seedlings of his Figure 1. Figure 2 shows the shape of $\hat{\kappa}$, again with the theoretical curve and envelope of twenty simulations for the Poisson process. The envelope provides the acceptance region for a non-parametric test of the hypothesis that the process is Poisson; this is rejected in both examples. The seedlings data show both local inhibition (at about one foot) and clustering.

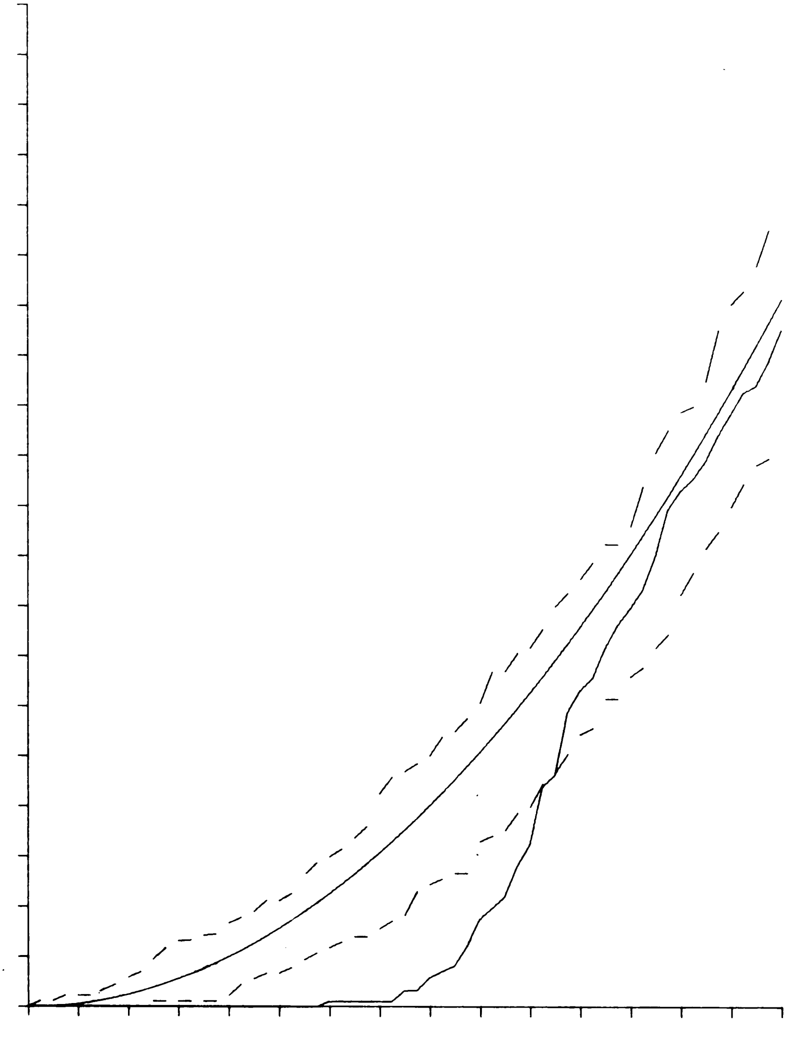

Figure 1
κ for 42 cell centres. The smooth curve is the theoretical value and the dashed lines the envelope of 20 simulations of a Poisson process. The other curve is the observed $\hat{\kappa}$.

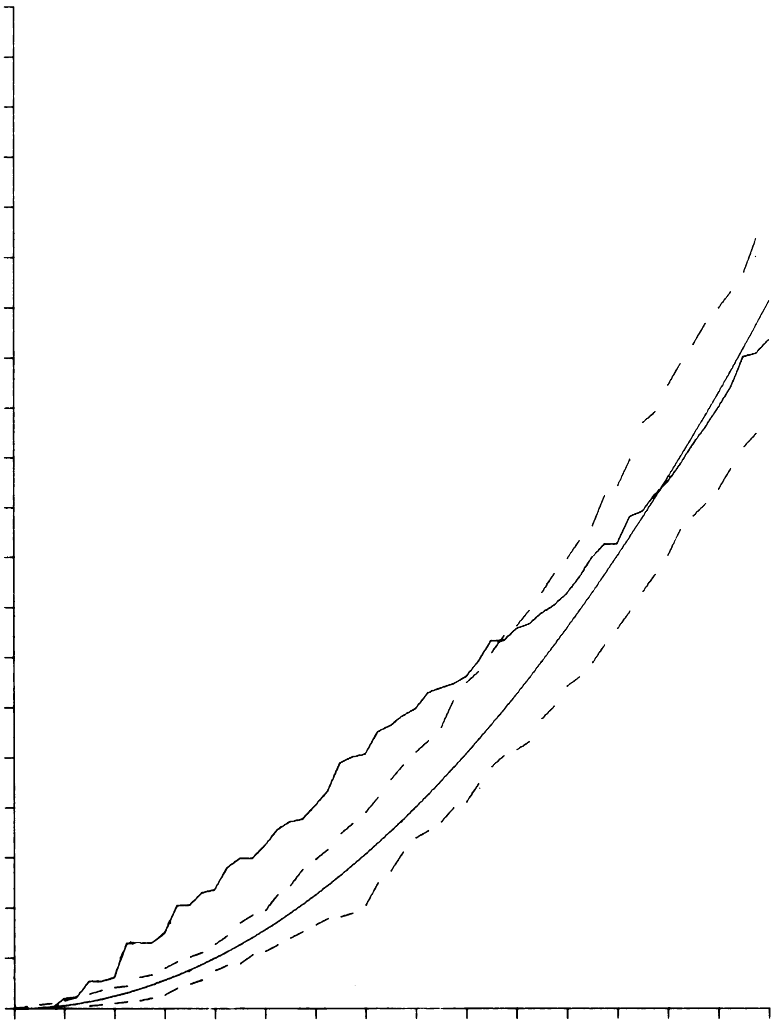

Figure 2
κ for 62 redwood seedlings.

Our approach is closely related to that of Glass and Tobler (1971). They worked with the radial distribution function of liquid state theory, essentially the density of κ with respect to the radial measure $r\,dr$, and plotted a histogram estimate of this function. They failed to consider the sampling errors; simulation shows the variance of the estimate to be considerable, and to increase as t decreases, explaining their conclusions. Our approach shows there is just evidence of inhibition up to $3\frac{1}{2}$ miles, as shown by Figure 3.

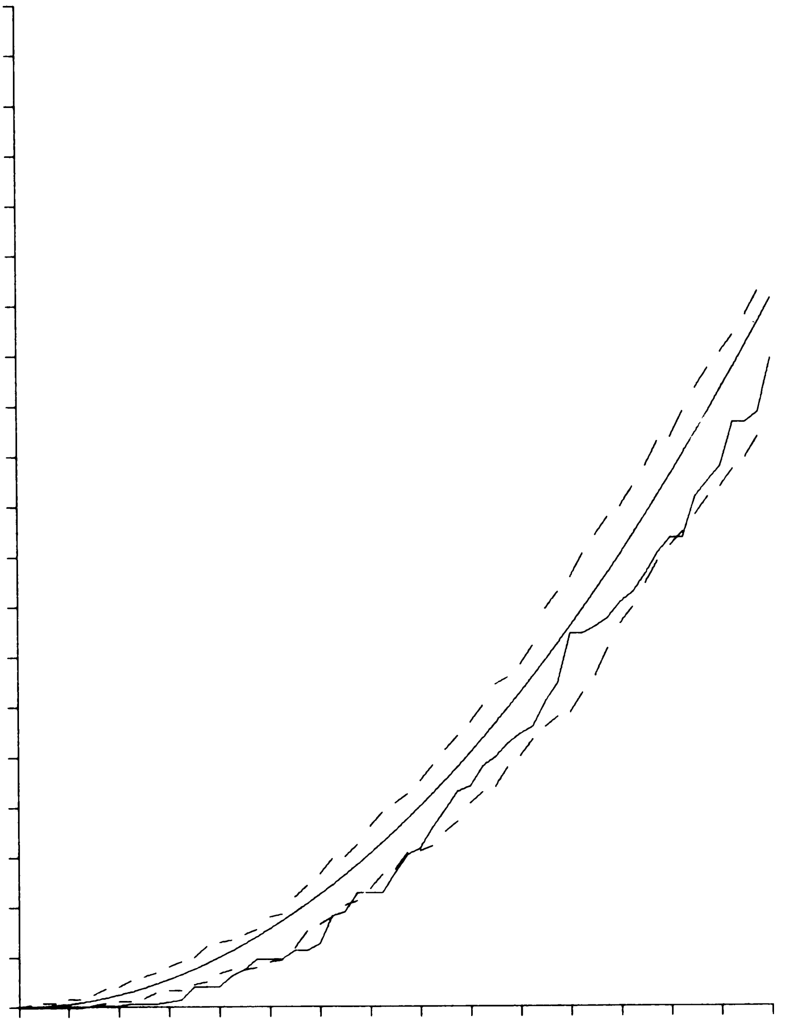

Figure 3
κ for the centres of 69 towns on the Spanish plateau.

Our technique is also related to the analysis of crystalline metals discussed by Serra (1972).

We can also use second-order methods to investigate interactions between two point processes Z_1 and Z_2 on the same space. We define a cross-moment by $\mu^{12}(A \times B) = \mathbf{E}(Z_1(A)Z_2(B))$ for $A, B \in \mathscr{C}$, and extend this to a Radon measure on the Borel sets of X^2. (Remember all processes are second-order.) Suppose the processes are jointly stationary. Then μ^{12} is an invariant Radon measure which can be represented by a measure κ_{12} as before.

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Suppose our point processes are on the real line, and we observe that part of a single realization within an interval E. Then we estimate $\kappa_{12}((r,s))$ by $\hat{\kappa}(r,s) = \Sigma_G(|E| - d(x,y))^{-1}$ for

$$G = (E \times E) \cap \{(x,y) \mid Z_1(\{x\})Z_2(\{y\}) > 0, r < y - x < s\}.$$

Then

$$\mathbf{E}(\hat{\kappa}(r,s)) = \kappa_{12}((r,s)) \quad \text{if} \quad |r|,\ |s| \leqq |E|.$$

A similar procedure (without the edge-correction) has been used in neurophysiology, the cross-correlation function of Perkel, Gerstein and Moore (1967) being the density (if this exists) of κ_{12} with respect to $\lambda_1\nu$. The reader is referred to this paper for an example of correlated point processes on the line; the difficulties mentioned there may be overcome to some extent by simulation.

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References

BARTLETT, M. S. (1964) The spectral analysis of two-dimensional point processes. *Biometrika* **51,** 299–311.

BARTLETT, M. S. (1967) The spectral analysis of line processes. *Proc. 5th Berkeley Symp. Math. Statist. Prob.* **3,** 135–153.

BARTLETT, M. S. (1974) The statistical analysis of spatial pattern. *Adv. Appl. Prob.* **6,** 336–358.

BHABHA, H. J. (1950) On the stochastic theory of continuous parameter systems and its application to electron cascades. *Proc. R. Soc. Lond.* A **202,** 301–322.

BOURBAKI, N. (1963) *Intégration* VII. Hermann, Paris.

BOURBAKI, N. (1966) *General Topology.* Hermann, Paris.

CHRISTENSEN, J. P. R. (1974) *Topology and Borel Structure.* North Holland, Amsterdam.

COX, D. R. AND LEWIS, P. A. W. (1966) *The Statistical Analysis of Series of Events.* Methuen, London.

DAVIDSON, R. (1970) Construction of line processes: second order properties. In *Stochastic Geometry*, ed. E. F. Harding and D. G. Kendall, Wiley, London, 55–75.

GEL'FAND, I. M. AND VILENKIN, N. YA. (1964) *Generalized Functions*, Vol. 4. Academic Press, New York.

GLASS, L. AND TOBLER, W. R. (1971) Uniform distribution of objects in a homogeneous field, cities on a plain. *Nature* **233,** No. 5314, 67–68.

JULESZ, B. (1975) Experiments in the visual perception of data. *Scientific American*, April, 34–43.

KENDALL, D. G. (1974) An introduction to stochastic geometry. In Harding and Kendall (1974), 3–9.

KRICKEBERG, K. (1970) Invariance properties of the correlation measure of line processes. In Harding and Kendall (1974), 76–88.

KRICKEBERG, K. (1973) Moments of point processes. In Harding and Kendall (1974), 89–113.

Matérn, B. (1960) Spatial variation. *Meddelanden från Statens Skogsforskningsinstitut* 49:5.

Mecke, J. (1967) Stationäre zufällige Maße auf lokalkompacten Abelschen Gruppen. *Z. Wahrscheinlichkeitsth.* **9,** 36–58.

Moyal, J. E. (1962) The general theory of stochastic population processes. *Acta. Math.* **108,** 1–31.

Nachbin, L. (1967) *The Haar Integral.* Van Nostrand, Princeton.

Perkel, D. H., Gerstein, G. L. and Moore, G. P. (1967) Neuronal spike trains and stochastic point processes II. Simultaneous spike trains. *Biophys. J.* **7,** 419–440.

Ramakrishnan, A. (1950) Stochastic processes relating to particles distributed in a continuous infinity of states. *Proc. Camb. Phil. Soc.* **46,** 595–602.

Ripley, B. D. (1976a) The disintegration of invariant measures. *Math. Proc. Camb. Phil. Soc.*, To appear.

Ripley, B. D. (1976b) Locally finite random sets. *Ann. Prob.* Submitted for publication.

Ripley, B. D. (1976c) The foundations of stochastic geometry. *Ann. Prob.* Submitted for publication.

Santaló, L. A. (1953) *Introduction to Integral Geometry.* Hermann, Paris.

Serra, J. (1972) Stereology and structuring elements. *J. Microscopy* **95,** 93–103.

Silverman, B. W. (1976) Limit theorems for dissociated random variables. *J. Appl. Prob.* Submitted for publication.

Strauss, D. J. (1975) A model for clustering. *Biometrika* **62,** 467–475.

Vere-Jones, D. (1968) Some applications of probability generating functionals to the study of input-output streams. *J. R. Statist. Soc.* B **30,** 321–333.

Vere-Jones, D. (1970) Stochastic models for earthquake occurrence. *J. R. Statist. Soc.* B. **32,** 1–62.

Vere-Jones, D. (1971) The covariance measure of a weakly stationary random measure. *J. R. Statist. Soc.* B **33,** 426–428.

Vere-Jones, D. (1974) An elementary approach to the spectral theory of stationary point processes. In Harding and Kendall (1974), 307–321.