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BERG'S TECHNIQUE FOR PSEUDO-ACTIONS WITH APPLICATIONS TO AF EMBEDDINGS

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ABSTRACT. Berg's interchange technique is generalized to the context of certain new objects called pseudo-actions. This is used to find a more geometric proof of the Pimsner-Voiculescu theorem on the AF embedding of the irrational rotation algebras. Connections with Berg's original results are briefly examined.

Embedding diagrams are introduced to provide a uniform way of describing embeddings of transformation group C^*-algebras $C(X) \rtimes \mathbb{Z}$ into AF algebras. Pimsner has classified the transformation group C^*-algebras which can be AF embedded. We present a new proof of this result using embedding diagrams and pseudo-actions. The need to calculate the join of an open cover with its iterates under the transformation has been eliminated.

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1. **Introduction and a history of Berg's technique.** Berg's technique has become a catch-all phrase describing certain approximation techniques in operator theory and operator algebras. The common thread in these techniques is that they involve perturbing some "shift-type" operator to become unitarily equivalent to a second "shift-type" operator. In addition, it is generally required that the unitary implementing this equivalence commute, or nearly commute, with a diagonal operator or algebra.

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In the paper [1] which spawned all this activity, Berg was interested in approximating weighted shifts T which have small self-commutators $[T, T*]$ by normal operators. His idea was to find sections of the shift with approximately equal, approximately constant weights. By gradually "rotating", or "interchanging", basic elements along these sections, he was able to obtain the direct sum of two shifts, each simpler than the original. In this way, he was able to ignore the local behavior of the weights, and concentrate on the behavior at infinity.

This technique of introducing a gradual flip between two shift segments works just as well when the segments are selected from different shifts. Pimsner and Voiculescu [8] took advantage of this when they embedded the irrational rotation algebras into the AF algebras associated to continued fractions.

Pimsner [7] generalized Berg's technique further when he classified the cross-product C^*-algebras $C(X) \rtimes \mathbb{Z}$ which can be embedded into AF algebras. Specifically, he found that the cross-product can be AF embedded exactly when every point in the underlying dynamical system is pseudo-non-wandering (chain recurrent). His construction begins with matrices that represent the pseudo-orbits of the system. In order to put these together into an AF algebra, he needed a way to "glue" together finite-dimensional shifts into longer shifts. He accomplished this by introducing a more complicated interchange technique involving several shift segments simultaneously. A similar technique was used by Veršik [7] (cf. [9], Section 6) to prove a related AF embedding theorem.

To illustrate the increasingly general interchanges described above, we offer the drawings in Figure 1.

The first goal of this paper is to find a precise, flexible formulation of Berg's technique which will cover the above situations.

(Berg himself, together with Davidson, has been investigating a generalization of the interchange technique. This is contained in [3], where it is used to find a constructive, quantitative version of the Brown-Douglas-Fillmore theorem. This version of Berg's technique involves tridiagonal matrices, not shift intervals. Therefore, we make no attempt to cover it here.)

The specific embeddings arising in the proof of Pimsner's theorem have been as interesting as the theorem itself. For example, the special cases studied in [5] and [6] have rather unusual K-theoretic and homotopy properties. For this reason, the second goal of this paper is to develop a uniform method of describing star-homomorphisms of $C(X) \rtimes \mathbb{Z}$ into AF algebras. This will, I hope, encourage people to investigate specific examples of cross-products which can be AF embedded to see what types of AF algebras and embeddings may arise.

The foundation for this work is a very simple object, called a pseudo-action. Pseudo-actions are a slight modification of, and I hope improvement on, pseudo-orbits. They give a compact means for describing simultaneously a permutation matrix and an algebra of diagonal matrices.

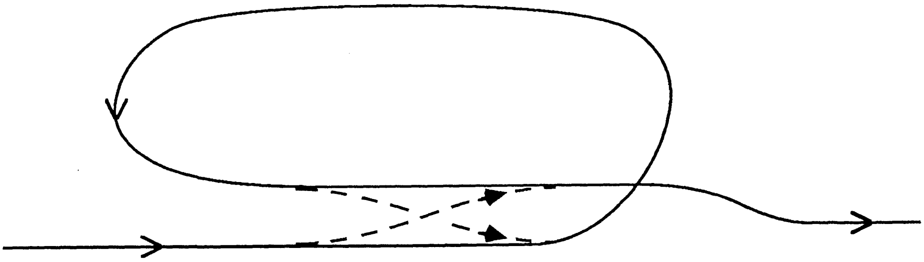

FIGURE 1a. Berg's interchange technique: original form.

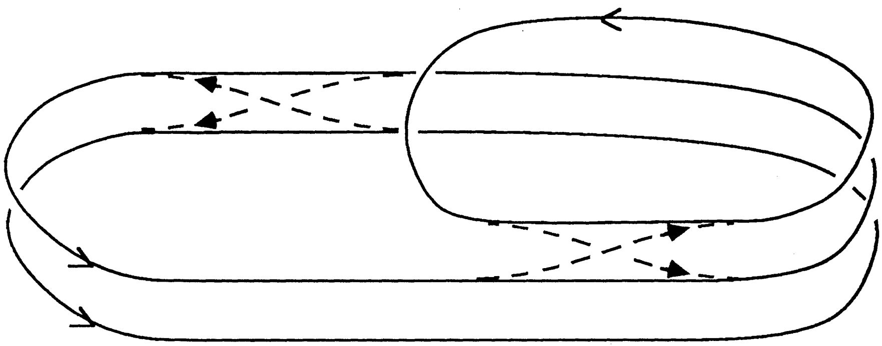

FIGURE 1b. Berg's interchange technique: as used by Pimsner and Voiculescu.

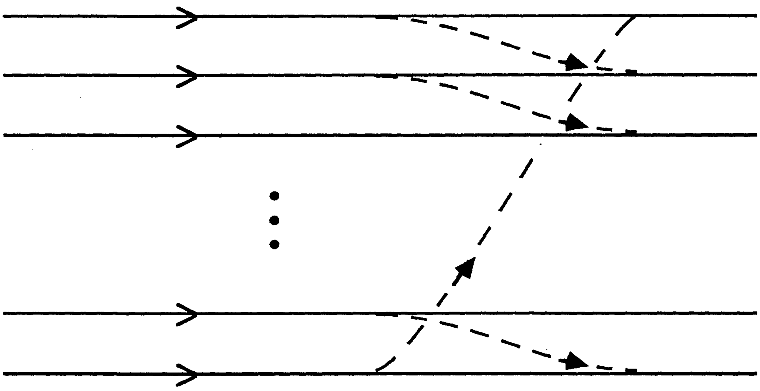

FIGURE 1c. Berg's interchange technique: as modified by Pimsner.

An embedding diagram is defined to be a Bratteli diagram whose vertices are labelled by pseudo-actions, subject to a summability condition. Once generalized to the context of pseudo-actions, Berg's technique can be used to show that embedding diagrams determine homomorphisms into the AF algebras of the underlying Bratteli diagrams.

Given this machinery, we obtain a slightly simplified proof of the Pimsner-Voiculescu result. With more work, in Sections 7–10, we are able to find a simplified version of Pimsner's construction of AF embeddings of $C(X) \rtimes \mathbb{Z}$. Our proof does not require taking the join of an open cover with its iterates, as does Pimsner's. This should be beneficial when looking at examples.

The AF algebras in Pimsner's original construction are far from generic. They always contain UHF subalgebras and they tend to be far from simple (cf. [5, Appendix]). A special class of embedding diagrams can be used to define a large collection of interesting AF embeddings into AF algebras, including simple ones. The author and George Elliott have investigated the K-theory of these embeddings. The results of this will appear elsewhere, although a very simple example is presented in Section 6.

The author would like to thank the operator theory group at Dalhousie for enduring a series of lectures that served as a rough draft of this article. He also wishes to thank George Elliott for his encouragement and motivating conversations during the planning stages of this work.

2. **A Form of Berg's technique.** The following lemma is implicitly contained in Pimsner's paper [7] and in Putnam's [9]. The idea of the proof, however, goes back to Berg.

LEMMA 2.1. *Suppose $\mathcal{H}$ is a Hilbert space which decomposes as $\mathcal{H} = \mathcal{H}_0 \oplus \cdots \oplus \mathcal{H}_n$, with p_j the projection onto $\mathcal{H}_j$. If U and V are unitary operators on $\mathcal{H}$ such that*

(i) $U\mathcal{H}_i = V\mathcal{H}_i = \mathcal{H}_{i+1}, \quad i = 1, \ldots, n-1$, *and*

(ii) $U|_{\mathcal{H}_i} = V|_{\mathcal{H}_i}, \quad i \neq n-1$,

then there exists a unitary W on $\mathcal{H}$ such that

(iii) $\|WVW^* - U\| < \pi / n$,

(iv) $W|_{\mathcal{W}_0} = \mathrm{id}_{\mathcal{H}_0}, W|_{\mathcal{H}_n} = \mathrm{id}_{\mathcal{H}_n}, \|(W - I)|_{\mathcal{H}_1}\| < \pi / n$, *and*

(v) $p_i W = W p_i, \quad i = 0, \ldots, n$.

PROOF. The statement of the lemma, and the proof, are clearest if we work with matrices in $p_0, p_1, \ldots, p_n$ coordinates. First, notice that we may use U to identify $\mathcal{H}_i$ with $\mathcal{H}_j$ for $i, j \geq 1$. Once we have done so, the hypotheses of the lemma state that U and V

are of the following form, when viewed in $\{p_j\}$ coordinates:

$$U = \begin{pmatrix} a & & & & & b \\ c & 0 & & & & d \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & 1 & 0 & \\ & & & & 1 & 0 \end{pmatrix}, \; V = \begin{pmatrix} a & & & & & b \\ c & 0 & & & & d \\ & 1 & 0 & & & \\ & & 1 & \ddots & & \\ & & & & & \\ & & & 1 & 0 & \\ & & & & v & 0 \end{pmatrix}.$$

Using the branch of $\sqrt[n]{}$ which sends the unit circle to $\exp(2\pi i[-1/2n, 1/2n))$, we obtain an n^{th} root $v^{1/n}$ of v such that $\|v^{1/n} - 1\| < \pi/n$. Let

$$W = \begin{pmatrix} 1 & & & & & \\ & v^{1/n} & & & & \\ & & v^{2/n} & & & \\ & & & \ddots & & \\ & & & & v^{1-1/n} & \\ & & & & & 1 \end{pmatrix}$$

By construction, W satisfies conditions (iv) and (v), and

$$WVW^* = \begin{pmatrix} 1 & & & & \\ & v^{1/n} & & & \\ & & v^{1/n} & & \\ & & & \ddots & \\ & & & & v^{1/n} \end{pmatrix} \cdot U,$$

so $\|WVW^* - U\| = \|v^{1/n} - 1\| < \pi/n$. ∎

REMARK. It is not necessary that U and V be unitary for the lemma to hold. We only require that U and V map $\mathcal{H}_i$ isometrically onto $\mathcal{H}_{i+1}, i = 1, \ldots, n-1$ and that $U(\mathcal{H}_n), V(\mathcal{H}_n)$ are contained in $\mathcal{H}_0 \vee \mathcal{H}_1$.

At first glance, this lemma may appear to be unrelated to Berg's original interchange technique. To see the connection, consider an operator S (for example a weighted shift) on a Hilbert space $\mathcal{H}$ which has two "shift sections" of weight 1 and length $n+1$. In other words, there are orthonormal vectors, not necessarily spanning,

$$\{\phi_0, \ldots, \phi_n, \psi_0, \ldots, \psi_n\}$$

such that

$$\begin{aligned} S\phi_i &= \phi_{i+1}, i < n, \\ S\psi_i &= \psi_{i+1}, i < n, \\ S\phi_n &\perp \{\phi_0, \ldots, \phi_n, \psi_0, \ldots, \psi_n\}, \\ S\psi_n &\perp \{\phi_0, \ldots, \phi_n, \psi_0, \ldots, \psi_n\}. \end{aligned}$$

Consider also the operator T defined by

$$\begin{aligned} T\phi_{n-1} &= \psi_n, \\ T\psi_{n-1} &= \phi_n, \\ T\xi &= S\xi \ \text{ if } \ \xi \perp \{\phi_{n-1}, \psi_{n-1}\}. \end{aligned}$$

A picture representing this situation is this:

$$S: \quad \begin{array}{l} \phi_0 \longrightarrow \phi_1 \longrightarrow \cdots \longrightarrow \phi_{n-1} \longrightarrow \phi_n \\ \\ \psi_0 \longrightarrow \psi_1 \longrightarrow \cdots \longrightarrow \psi_{n-1} \longrightarrow \psi_n \end{array}$$

$$T: \quad \begin{tikzcd} \phi_0 \arrow[r] & \phi_1 \arrow[r] & \cdots \arrow[r] & \phi_{n-1} \arrow[dr] & \phi_n \\ \psi_0 \arrow[r] & \psi_1 \arrow[r] & \cdots \arrow[r] & \psi_{n-1} \arrow[ur] & \psi_n \end{tikzcd}$$

Let $\mathcal{H}_i = \bigvee\{\phi_i, \psi_i\}, i = 1, \dots, n-1, \quad \mathcal{H}_0 = \left(\oplus_{i>0}\mathcal{H}_i\right)^{\perp}$. As remarked above, the lemma applies to S and T, so there exists a unitary W such that $\| WSW^* - T\| < \pi/(n-1)$.

A more concrete way to view this result is that there is a new orthonormal basis

$$\{\xi_0, \dots, \xi_n, \omega_0, \dots, \omega_n\}$$

of

$$\bigvee\{\phi_0, \dots, \phi_n, \psi_0, \dots, \psi_n\}$$

such that

$$\begin{aligned} \xi_0 &= \phi_0, \quad \xi_n = \psi_n, \\ \omega_0 &= \psi_0, \quad \xi_n = \phi_n, \end{aligned}$$

and S is close to T_0 where

$$\begin{aligned} T_0\xi &= \xi \ \textit{for} \ \xi \perp \{\phi_0, \dots, \phi_n, \psi_0, \dots, \psi_n\}, \\ T_0\xi_i &= \xi_{i+1}, \ i < n, \\ T_0\omega_i &= \omega_{i+1}, \ i < n. \end{aligned}$$

Of course, one such basis is

$$\begin{aligned} \xi_i &= W^*\phi_i, \ i = 0, \dots, n, \\ \omega_i &= W^*\psi_i, \ i = 0, \dots, n, \end{aligned}$$

but it is not the basis Berg exhibits in [1] or [2]. This basis does, however, arise in Davidson's proof of Berg's technique [4].

3. **Pseudo-actions.** Most applications of Berg's technique, and its generalizations, have involved diagonal operators and shift operators. For example, the principal object used in the Pimsner-Voiculescu AF embedding is a pair of unitary matrices, one diagonal and the other a cyclic shift. When constructing his more general AF embeddings, Pimsner works with algebras of diagonal matrices and cyclic shifts. A pseudo-action is simply a device for defining simultaneously permutation matrices and algebras of diagonal matrices.

DEFINITION 3.1. Let X be a topological space. A *pseudo-action* on X is a triple $\omega = (I, \omega, \alpha)$ such that

(i) I is a set,
(ii) $\omega: I \longrightarrow X$ is a function,
(iii) $\alpha: I \longrightarrow I$ is a bijection.

By the *order* of ω, denoted $|\omega|$, we mean the cardinality of I. We are mostly interested in finite pseudo-actions, $|\omega| < \infty$, so unless it is indicated otherwise, we shall assume that all pseudo-actions are finite.

Associated to ω is a C^*-algebra A_ω, a star-homomorphism $\pi_\omega: C_0(X) \longrightarrow A_\omega$ and a unitary $U_\omega \in A_\omega$. Consider the Hilbert space $\ell^2(I)$, with basis $\{ e_i \mid i \in I\}$. We let

$$\begin{aligned} A_\omega &= \mathcal{B}(\ell^2(I)), \\ \pi_\omega(f)e_i &= f(\omega(i))e_i \text{ for } i \in I, f \in C_0(X), \\ U_\omega e_i &= e_{\alpha(i)} \textit{ for } i \in I. \end{aligned}$$

There is an obvious notion of isomorphism of pseudo-actions. We say that ω is *isomorphic* to a second pseudo-action $\eta = (J, \eta, \beta)$ if there exists a bijection $f: I \longrightarrow J$ such that $\beta \circ f = f \circ \alpha$ and $\eta \circ f = \omega$. This implies that π_ω and U_ω are simultaneously unitarily equivalent to π_η and U_η. As in [7] and [8], we are not so much interested in this type of equivalence as in approximate simultaneous unitary equivalence.

Before proceeding further, we impose some restrictions on the space X. We shall assume that X is compact and metrizable. For convenience, we also assume that X is a compact subset of $\mathbb{R}^k$ or $\mathbb{C}^k$ with the induced metric d. This selects for us generators

$$g_1, \dots, g_k \in C(X),$$

namely the coordinate functions restricted to X. We use $g_1, \dots, g_k$ to establish a notion of distance between homomorphisms $\phi, \psi: C(X) \longrightarrow \mathcal{B}(\mathcal{H})$. We use the induced metric because then we have

$$|g_j(x) - g_j(y)| \leq d(x, y), \quad j = 1, \dots, k, \quad x, y \in X,$$

and this simplifies some formulas.

DEFINITION 3.2. Suppose $\omega = (I, \omega, \alpha)$ and $\eta = (J, \eta, \beta)$ are pseudo-actions on X. We define the *distance between ω and η*, denoted $d(\omega, \eta)$, as follows. If $|\omega| \neq |\eta|$ then $d(\omega, \eta) = \infty$. Otherwise

$$\begin{aligned} d(\omega, \eta) = \inf_W \max\{ \|U_\omega - WU_\eta W^*\|, \|\pi_\omega(g_1) - W\pi_\eta(g_1)W^*\|, \dots \\ \dots, \|\pi_\eta(g_k) - W\pi_\eta(g_k)W^*\| \}, \end{aligned}$$

where the infimum is taken over all unitaries of $\ell^2(J)$ onto $\ell^2(I)$.

A convenient way to describe a pseudo-action is by the sort of pictures shown in Figure 2. Here, we think of I as floating above X, with α represented by arrows between the points of I and $\omega\colon I \longrightarrow X$ downward projection. Notice that it does not suffice to draw the images in X as this fails, for example, to distinguish ω_1 from ω_2 (as defined in Figure 2), and $\omega_1 \not\simeq \omega_2$ since the spectrum of U_{ω_1} is not equal to the spectrum of U_{ω_2}. An upper bound on $d(\omega_1, \omega_2)$ can be obtained using Berg's technique, as we now show.

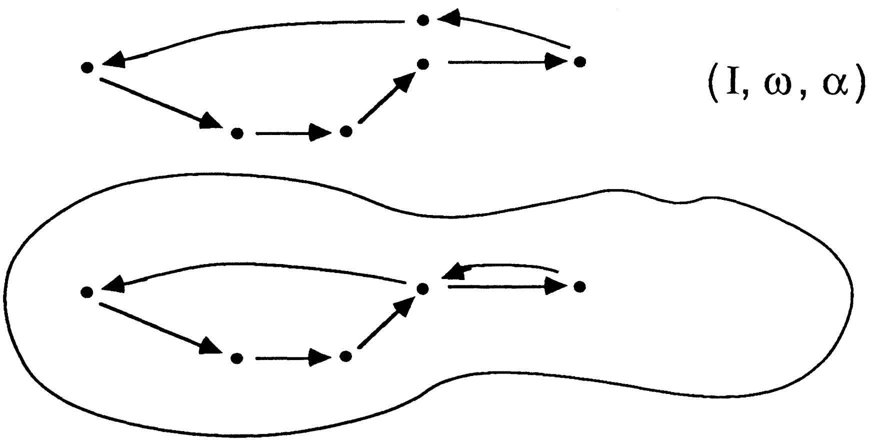

FIGURE 2a.

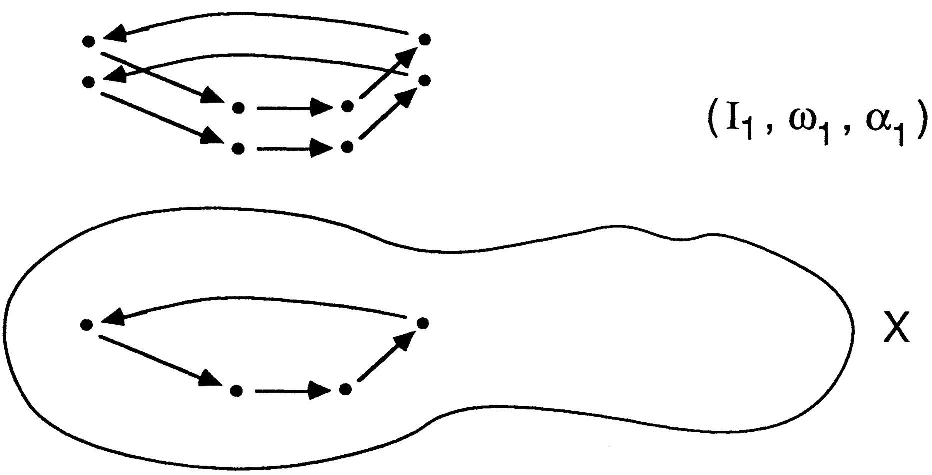

FIGURE 2b.

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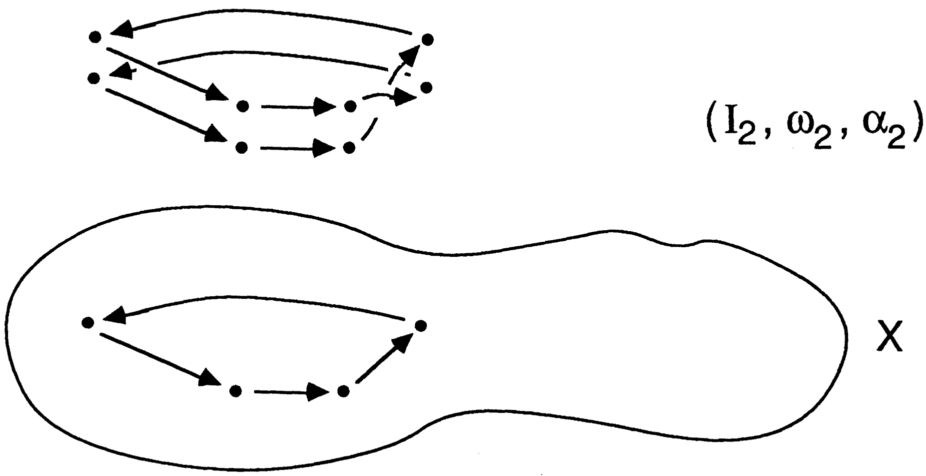

FIGURE 2c. Examples of pseudo-actions

Identifying A_{ω_1} and A_{ω_2} with $M_8(\mathbb{C})$ in an appropriate way, and letting

$$v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we find that

$$U_{\omega_1} = \begin{pmatrix} 0 & & & I \\ I & 0 & & \\ & I & 0 & \\ & & I & 0 \end{pmatrix}, U_{\omega_2} = \begin{pmatrix} 0 & & & I \\ I & 0 & & \\ & I & 0 & \\ & & v & 0 \end{pmatrix}.$$

For any $g \in C(X)$, there are scalars $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

$$\pi_{\omega_1}(g) = \pi_{\omega_2}(g) = \begin{pmatrix} \lambda_1 I & & & \\ & \lambda_2 I & & \\ & & \lambda_3 I & \\ & & & \lambda_4 I \end{pmatrix}.$$

By Lemma 2.1, there is a unitary W, which according to the proof of Lemma 2.1 is of the form

$$W = \begin{pmatrix} v^{1/4} & & & \\ & v^{2/4} & & \\ & & v^{3/4} & \\ & & & I \end{pmatrix},$$

such that

$$\| U_{\omega_1} - WU_{\omega_2}W^* \| < \pi/4.$$

Because of the block diagonal form of W, clearly W commutes with $\pi_{\omega_1}(g_j) = \pi_{\omega_2}(g_j)$, so

$$\|\pi_{\omega_1}(g_j) - W\pi_{\omega_2}(g_j)W^*\| = 0.$$

We have shown that $d(\omega_1, \omega_2) < \pi/4$.

We end this section with a few more definitions. Suppose (I, ω, α) and (J, η, β) are pseudo-actions on X. We define their *sum* $\omega + \eta$ by disjoint union. That is,

$$\omega + \eta = (I \sqcup J, \gamma, \delta),$$

$$\gamma(i) = \begin{cases} \omega(i) & \text{if } i \in I \\ \eta(i) & \text{if } i \in J \end{cases},$$

$$\delta(i) = \begin{cases} \alpha(i) & \text{if } i \in I \\ \beta(i) & \text{if } i \in J \end{cases}.$$

Notice that $A_{\omega+\eta} \neq A_\omega \oplus A_\eta$, but there is an obvious inclusion of $A_\omega \oplus A_\eta$ in $A_{\omega+\eta}$ which sends (U_ω, U_η) to $U_{\omega+\eta}$ and for which the following diagram commutes:

$$\begin{tikzcd} & A_{\omega+n} \\ C(X) \arrow[ur, "\pi_{\omega+\eta}"] \arrow[r, "\pi_\omega \oplus \pi_\eta"] & A_\omega \oplus A_\eta \arrow[u, hook] \end{tikzcd}$$

We say that (I, ω, α) is *principal* if ω is injective, and *cyclic* if α is a cyclic permutation. Every pseudo-action has a *cyclic decomposition*, unique up to order and isomorphism, as a sum of cyclic pseudo-actions. If $\omega, \omega_1, \omega_2$ are as depicted in Figure 2, then none of them is principal, ω and ω_2 are cyclic and ω_1 has a cyclic decomposition consisting of two isomorphic principal, cyclic pseudo-actions.

4. **Berg's technique for pseudo-actions.** Given a pseudo-action $\omega = (I, \omega, \alpha)$ on a space X, we would like to know which permutations δ on I can be considered small, meaning that $(I, \omega, \delta \circ \alpha)$ is close to ω. Essentially, the following theorem states that this is true so long as, for all $i, \delta(i)$ is close to i, i.e., $d(\omega(\delta(i)), \omega(i))$ is small, and also the "history" $\alpha^{-1}(\delta(i)), \alpha^{-2}(\delta(i)), \ldots$ of $\delta(i)$ stays close to the "history" of i for a long time. Recall that X is a space with certain assumptions on it, and $g_1, \ldots, g_k \in C(X)$ are specified.

THEOREM 4.1. *(The Pimsner-Berg technique). Suppose I is a finite set, ω and ω' are functions $\omega, \omega' \colon I \longrightarrow X$ and α and δ are permutations of I. Let I_0 be the set*

$$I_0 = \{ i \in I \mid \delta(i) \neq i \}.$$

If m is a positive integer such that

$$\alpha^{-j}(I_0) \cap I_0 = \emptyset \qquad \textit{for } j = 1, \ldots, m-1,$$

then

$$d((I,\omega,\alpha),(I,\omega',\delta\circ\alpha)) \leq \max\{\pi/m, E+F\}$$

where

$$E = 2\cdot\max\, d\Big(\omega\big(\alpha^{-j}(i)\big),\omega(\alpha^{-j}\delta^{r}(i))\Big),\quad j=0,\ldots,m-1,\quad r\in\mathbb{Z},\quad i\in I_0$$
$$F = \max\, d\big(\omega(i)\big),\big(\omega'(i)\big),(i\in I).$$

PROOF. First, we shall prove a weaker version of the theorem. For notation, we let

$$\omega = (I,\omega,\alpha),$$
$$\eta = (I,\omega',\delta\alpha).$$

Consider the following subspaces of $\mathcal{H} = \ell^2(I)$:

$$\mathcal{H}_j = \ell^2(\alpha^{j-m}(I_0)),\quad j=1,\ldots,m,$$
$$\mathcal{H}_0 = (\mathcal{H}_1\oplus\cdots\oplus\mathcal{H}_m)^{\perp}.$$

Again, let e_j denote the basis vector corresponding to $i\in I$. Notice that, for $i\in I_0, j = 1,\ldots,m-1$,

$$U_\omega e_{\alpha^{j-m}(i)} = e_{\alpha^{j-m+1}(i)}\in\mathcal{H}_{j+1},$$
$$U_\eta e_{\alpha^{j-m}(i)} = \left\{\begin{array}{ll} e_{\alpha^{j-m+1}(i)}, & j\neq m-1\\ e_{\delta(i)}, & j=m-1\end{array}\right\}\in\mathcal{H}_{j+1},$$

and

$$U_\omega e_i = U_\eta e_i \quad\text{for}\quad i\notin\alpha^{-1}(I_0).$$

Therefore, the hypotheses of Lemma 2.1 are satisfied, so there exists a unitary W on $\ell^2(I)$, which commutes with the projections on the $\mathcal{H}_i$, such that

$$\|WU_\eta W^* - U_\omega\| < \pi/m.$$

It should be clear that there is a diagonal operator on $\ell^2(I)$ which is constant on the blocks corresponding to $\mathcal{H}_1,\ldots,\mathcal{H}_m$ such that

$$\|\pi_\omega(g_k) - G\| \leq \max_{\substack{j=0,\ldots,m-1\\ i_1,i_2\in I_0}} d\Big(\omega\big(\alpha^{-j}(i_1)\big),\omega\big(\alpha^{-j}(i_2)\big)\Big),$$

for all k. Here we have used the fact that $|g_k(x)-g_k(y)| \leq d(x,y)$. Because W commutes with G, we have, for all k,

$$\|\pi_\omega(g_k) - w\pi_\omega(g_k)W^*\| \leq 2\|\pi_\omega(g_k) - G\|,$$

and

$$\begin{aligned}
&\|\pi_\omega(g_k) - W\pi_\eta(g_k)W^*\| \leq \\
&\leq \|\pi_\omega(g_k) - W\pi_\omega(g_k)W^*\| + \|W\pi_\omega(g_k)W^* - W\pi_\eta(g_k)W^*\| \\
&\leq \|\pi_\omega(g_k) - W\pi_\omega(g_k)W^*\| + \|\pi_\omega(g_k) - \pi_\eta(g_k)\| \\
&\leq 2 \cdot \max_{\substack{j=0,\ldots,m-1 \\ i_1,i_2\in I_0}} d\Big(\omega\big(\alpha^{-j}(i_1)\big), \omega\big(\alpha^{-j}(i_2)\big)\Big) + \max_{i\in I} d\big(\omega(i), \omega'(i)\big).
\end{aligned}$$

If δ is a cycle, then this is equal to the estimate stated in the theorem. Otherwise, it is much cruder.

One obtains the sharper estimate by working on each disjoint cycle of δ separately. The errors do not accumulate, essentially because the unitaries so obtained will be non-trivial on orthogonal subspaces, by condition (iv) of Lemma 2.1. We leave the details to the reader. ∎

REMARK. If I is an infinite set, this proof still works, exactly as stated if I_0 is finite, with minor modifications otherwise.

For an example which illustrates the way this theorem will be used when embedding cross-products into AF algebras, consider the two pseudo-actions illustrated in Figure 3. By Theorem 4.1, the distance between them is less that $\pi/4$. The permutation δ which converts one into the other is indicated by double lines.

For a more classical example, consider, for each integer $m > 0$, a two-sided shift T_m with weights as shown in Figure 4, graphed against $\mathbb{Z}$. It follows as a special case of [1; Theorem 2] that there is a normal operator at a distance at most $100\sqrt{m}$ from T_m. Let us see how to prove this in the language of pseudo-actions.

Let ω denote the pseudo-action depicted in Figure 5a. Up to unitary equivalence, T_m equals $U_\omega\pi_\omega(g)$, where $g\colon [0,1] \longrightarrow \mathbb{R}$ is ordinary inclusion. Let $\omega'\colon I \longrightarrow [0,1]$ and $\delta : I \longrightarrow I$ be as depicted in Figure 5b. Here δ, drawn in double lines, consists of $[\sqrt{m}]$ two-cycles, and ω' is constant on subsets of I of size approximately $2m/[\sqrt{m}]$. We may apply Theorem 4.1 with the m in the theorem equal to $[\sqrt{m}]$. Since

$$d\big(\omega(i), \omega'(i)\big) \leq \frac{m + [\sqrt{m}]}{2m[\sqrt{m}]}, \quad \text{for } i \in I$$

$$d\big(\alpha^{-r}(i), \alpha^{-r}\delta(i)\big) \leq 2r/m,$$

we find that, with $\eta = (I, \omega', \delta \circ \alpha)$, and $m > 4$,

$$d(\omega, \eta) < 8\sqrt{m}.$$

Consequently, there exists a unitary W such that

$$\|T_m - WU_\eta\pi_\eta(g)W^*\| \leq 16/\sqrt{m}.$$

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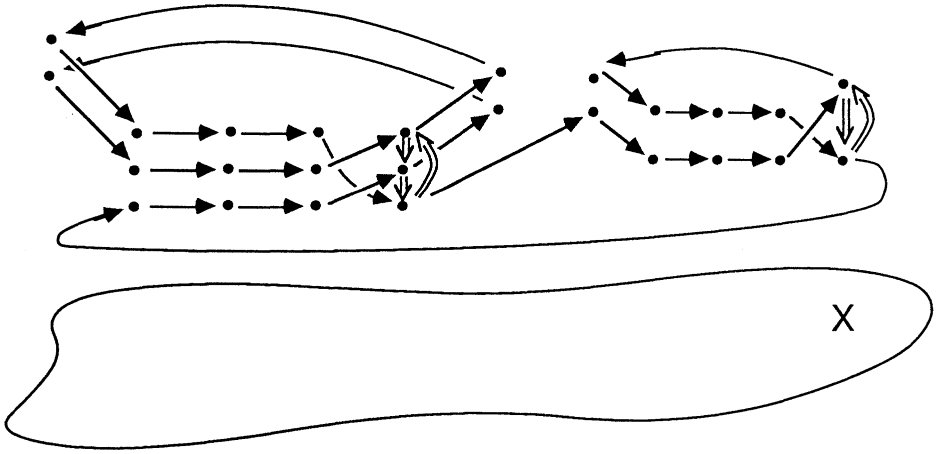

FIGURE 3a.

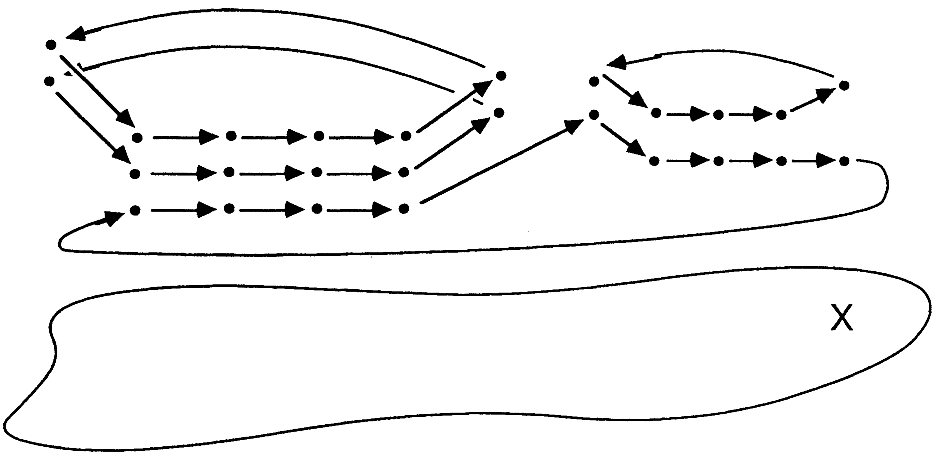

FIGURE 3b. An example of the Pimsner-berg interchange for pseudo-actions

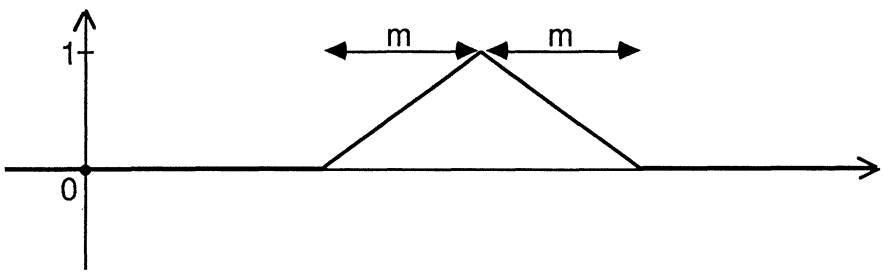

FIGURE 4. Weights for a nearly normal shift.

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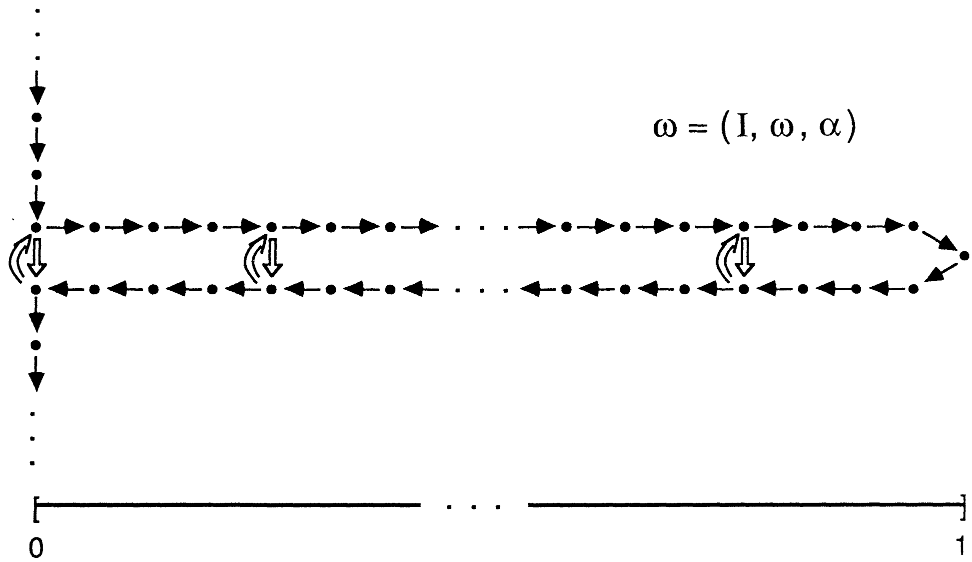

FIGURE 5a.

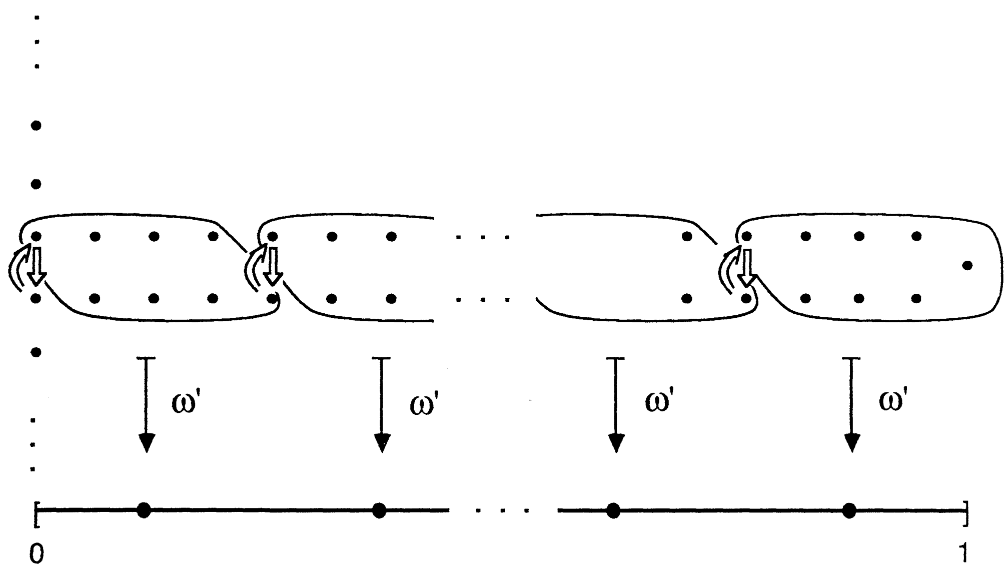

FIGURE 5b.

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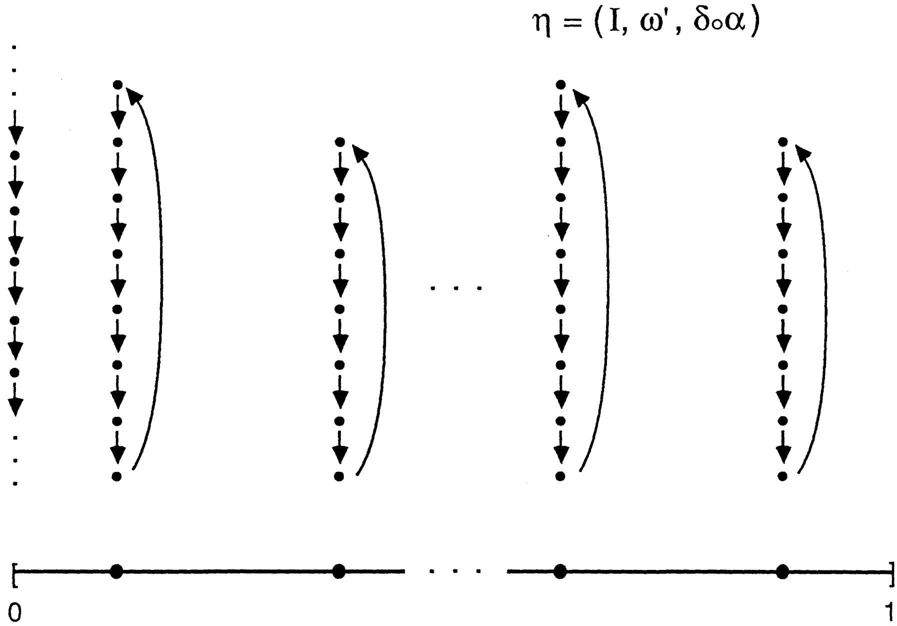

FIGURE 5c. A proof by pseudo-actions

Finally, notice that $\pi_\eta(g)$ commutes U_η, and therefore $U_\eta\pi_\eta(g)$ is normal.

We close this section with an example which illustrates the convenience of allowing δ in Theorem 4.1 to contain cycles other than two-cycles. Consider the weighted shifts T_m, m a positive integer, with weights as shown in Figure 6a. Using the classical interchange, we may "pinch off" the hump and show that, up to unitary equivalence, T_m is close to the shift R_m whose weights are shown in Figure 6b plus a finite-dimensional normal summand. Using Theorem 4.1, we can show that T_m is, up to unitary equivalence, close to the shift S_m whose weights are shown in Figure 6c, without needing to add on a direct summand.

Figure 7 shows a pseudo-action ω on $[0, 1]$ for which $T_m = U_\omega\pi_\omega(g)$. Define ω' equal to ω, except for points between x and y, where we define $\omega'(\alpha^j(x)) = j/3m$. Let δ be the permutation indicated by double lines. The reader can easily check that, for $\eta = (I, \omega', \delta \circ \alpha)$,

$$d(\omega, \eta) < 8/\sqrt{m},$$

which immediately shows that there exists a unitary W such that

$$\|T_m - WU_\eta\pi_\eta(g)W^*\| < 16/\sqrt{m},$$

and clearly $U_\eta\pi_\eta(g) = S_m$.

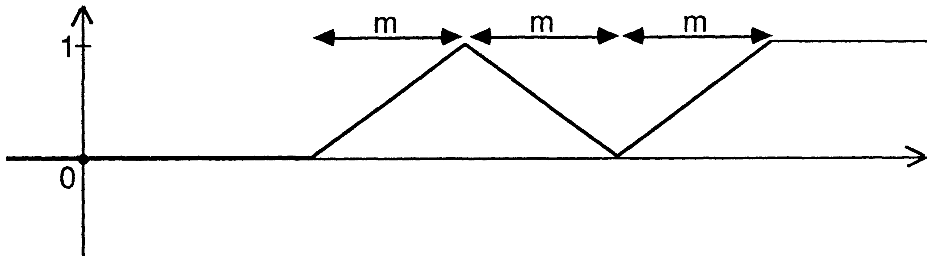

FIGURE 6a. Weight for the shift T_m

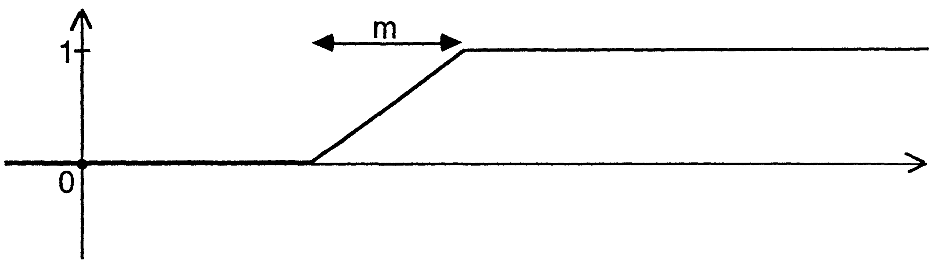

FIGURE 6b. Weight for the shift R_m

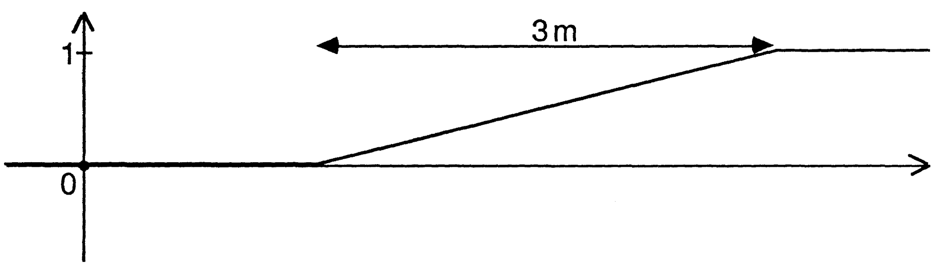

FIGURE 6c. Weight for the shift S_m

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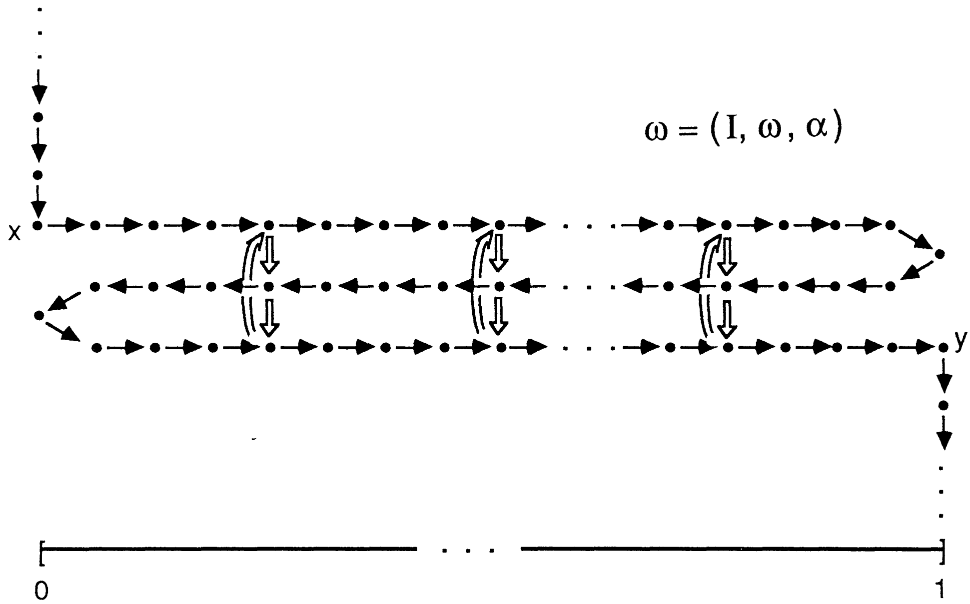

FIGURE 7. A pseudo-action proof that T_m is approximately unitarily equivalent to S_m.

5. **The Pimsner-Voiculescu embedding of A_θ.** Recall that, for θ irrational, the irrational rotation algebra A_θ is the universal C^*-algebra generated by two unitaries u, v subject to the relation $vu = e^{2\pi i\theta}uv$. Also recall that A_θ is simple. Two good references for these facts are [4] and [10].

Pimsner and Voiculscu [8] showed that A_θ could be embedded into an AF algebra in such a way that the induced map on K_0 is an isomorphism. In this section, we present a way to construct such an embedding which is a little simpler than the original.

Our construction is based on some pseudo-actions on the circle, associated to rational rotations. We regard S^1 as the unit circle in $\mathbb{C}$, and so our canonical list of generators of $C(S^1)$ is simply $g = e^{2\pi ix}$. Of course, any pseudo-action (I, ω, α) on $\mathbb{R}$ defines a pseudo-action on S^1, namely (I, ω', α) when $\omega'(x) = e^{2\pi i\omega(x)}$. Our pictures will all be of pseudo-actions on $\mathbb{R}$, regarded a pseudo-actions on S^1.

Suppose p and q are poisitive integers, not necessarily relatively prime. We define the p, q rotation $r_{p,q}$ to be the following pseudo-action on S^1:

$$\begin{aligned} r_{p,q} &= (I, \omega, \alpha), \\ I = [0, q) &= \{0, 1, \ldots, q-1\}, \\ \omega(j) &= \exp(2\pi ij/q), \\ \alpha(j) &= j + p - q[(j+p)/q]. \end{aligned}$$

Notice that $r_{p+q,q} = r_{p,q}$, but $r_{2p,2q} \neq r_{p,q}$. We have not defined $r_{0,q}$. These "zero-rotations" are useful when one is embedding $C(T^2)$ into an AF algebra, but the above definition is not the correct one. We shall discuss "zero-rotations" in a future paper with Elliott.

The notation

$$U_{p,q} = U_{r_{p,q}},$$
$$V_{p,q} = \pi_{r_{p,q}}(g)$$

is a useful shorthand. Clearly

$$V_{p,q}U_{p,q} = e^{2\pi ip/q}U_{p,q}V_{p,q}. \tag{5.1}$$

PROPOSITION 5.1. *Suppose p, q, s, t are positive integers such that $q \leq t$ and $0 \leq p/q, s/t \leq 1$. Suppose also that m, n are nonnegative integers, and let $p' = mp + ns$ and $q' = mq + nt$. We then have*

$$d(mr_{p,q} + nr_{s,t}, r_{p',q'}) \leq \max\{2\pi/q, E + F\},$$
$$E = 4\pi q\,|p/q - s/t|,$$
$$F = \max\{2\pi mq\,|p/q - p'/q'|,\ 2\pi nt\,|s/t - p'/q'|\}.$$

PROOF. Consider Figure 8. This illustrates a pseudo-action (I, ω, α) which is isomorphic to $mr_{p,q}+nr_{s,t}$. The double lines indicate a permutation δ such that $\delta \circ \alpha$ is cyclic. Let $\omega'\big((\delta \circ \alpha)^r(x)\big) = rp'/q'$. Regarded as a pseudo-action on S^1, $\eta = (I, \omega, \delta \circ \alpha)$ is isomorphic to $r_{p',q'}$. We may estimate $d(\omega, \eta)$ by Theorem 4.1.

Consider two points $j, k \in I_0$. If they are both from $mr_{p,q}$ or $nr_{s,t}$, they have identical histories going back $(q - 1)$ steps. If they lie one in $mr_{p,q}$, one in $nr_{s,t}$, then

$$d\big(\omega(\alpha^{-h}(j)), \omega(\alpha^{-h}(k))\big) \leq |\exp\big(2\pi i(-hp/q)\big) - \exp\big(2\pi i(-hs/t)\big)|$$
$$\leq 2\pi q\,|p/q - s/t|$$

for $h = 0, \ldots, q-1$. (We could use $2\pi(q-1)$ in place of $2\pi q$, but don't wish to make the statement of the proposition any murkier.) This explains E in the estimate. In a similar fashion, one sees that

$$d\big(\omega(j), \omega'(j)\big) \leq F,$$

when ω, ω' are considered as S^1 valued functions. ∎

Now consider the continued-fraction expansion $[a_1, a_2, \ldots]$ of θ, with p_n/q_n the partial fractions. Here are some basic facts about p_n and q_n.

$$p_n = a_n p_{n-1} + p_{n-2},\ q_n = a_n q_{n-1} + q_{n-2}, \tag{5.2}$$
$$p_n q_{n-1} + p_{n-1} q_n = \pm 1, \tag{5.3}$$

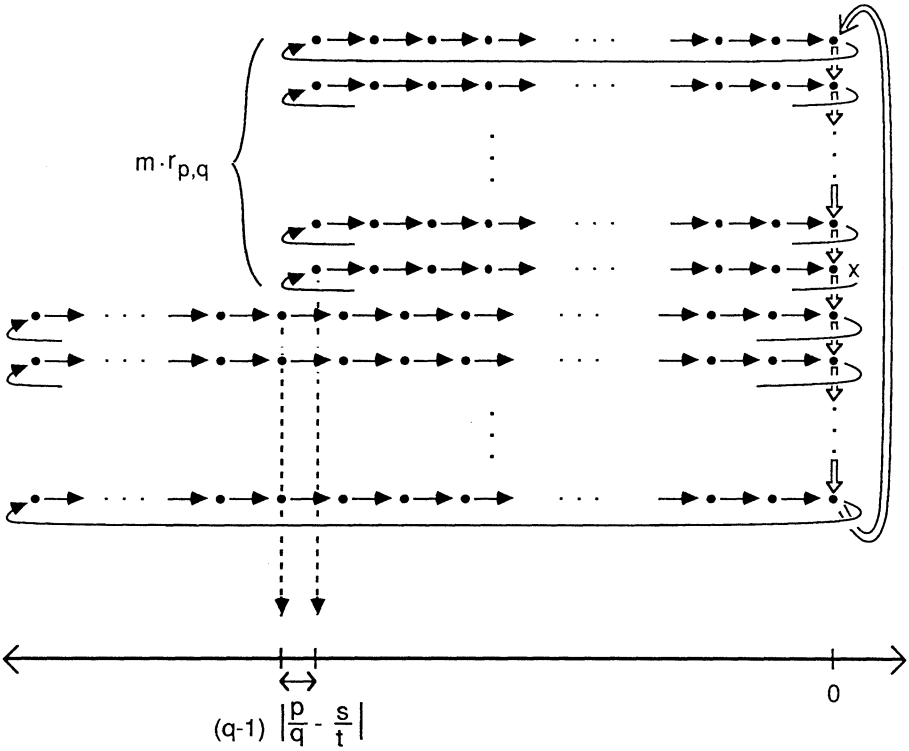

FIGURE 8. An illustration of why $mr_{p,q} + nr_{s,t}$ is close to $r_{mp+ns,mq+nt}$.

$$\left|\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}\right| < \left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}}\right| = \frac{1}{q_{n-1}q_{n-2}}, \tag{5.4}$$

$$\left|\theta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2}. \tag{5.5}$$

By Proposition 5.1 and (5.4), we find that

$$d(r_{p_n,q_n}, a_n r_{p_{n-1},q_{n-1}} + r_{p_{n-2},q_{n-2}}) \leq \max\{\pi / q_{n-2}, E + F\},$$

$$E = 4\pi q_{n-2}\left|\frac{p_{n-1}}{q_{n-1}} - \frac{p_{n-2}}{q_{n-2}}\right| = \frac{4\pi}{q_{n-1}},$$

$$F = 2\pi \max\left\{a_n q_{n-1}\left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right|, q_{n-2}\left|\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}}\right|\right\}$$

$$\leq 2\pi \max\left\{\frac{a_n}{q_n}, \frac{1}{q_{n-1}}\right\} = \frac{2\pi}{q_{n-1}}.$$

Therefore

$$d(r_{p_n,q_n}, a_n r_{p_{n-1},q_{n-2}}) \leq \max\left\{\frac{\pi}{q_{n-2}}, \frac{6\pi}{q_{n-1}}\right\}. \tag{5.6}$$

THEOREM 5.2. *There is an embedding of A_θ into B_θ, where B_θ is the AF algebra associated to the continued-fraction expansion of θ.*

PROOF. Recall that the Bratteli diagram shown in Figure 9a determines B_θ. Let us identify M_{q_n} with $A_{r_{p_n,q_n}}$ and let $U_n = U_{p_n,q_n}$, $V_n = V_{p_n,q_n}$. We may interpret (5.6) as saying that there exists a unitary $W = W_n$ such that, if we define γ_n as

$$\gamma_n\colon M_{q_n} \oplus M_{q_{n-1}} \longrightarrow M_{q_{n+1}} \oplus M_{q_n}$$

$$\gamma_n\big((S,T)\big) = \left(W \begin{pmatrix} S & & & & \\ & S & & & \\ & & \ddots & & \\ & & & S & \\ & & & & T \end{pmatrix} W^*, S\right),$$

we have the estimates

$$\|\gamma_n\big((U_n, U_{n-1})\big) - (U_{n+1}, U_n)\| \leq \max\left\{\frac{\pi}{q_{n-1}}, \frac{6\pi}{q_n}\right\},$$
$$\|\gamma_n\big((V_n, V_{n-1})\big) - (V_{n+1}, V_n)\| \leq \max\left\{\frac{\pi}{q_{n-1}}, \frac{6\pi}{q_n}\right\}.$$

The denominators q_n grow geometrically fast. Therefore, the sequences $\big((U_n, U_{n-1})\big)_1^\infty$ and $\big((V_n, V_{n-1})\big)_1^\infty$ are Cauchy sequences in

$$B = \varinjlim(M_{q_n} \oplus M_{q_{n-1}}, \gamma_n).$$

We let

$$U = \lim_{n\to\infty}(U_n, U_{n-1}) \in B$$
$$V = \lim_{n\to\infty}(V_n, V_{n-1}) \in B.$$

These are unitaries, and by (5.1) and (5.5), we also have $VU = e^{2\pi i\theta}UV$. There is an induced unital star-homomorphism

$$\phi\colon A_\theta \longrightarrow B,$$

sending u, v to U, V, and because A_θ is simple, ϕ is an embedding.

All that remains to do is to observe that $B \cong B_\theta$ because the γ_n have the correct K-theory. It should be pointed out, however, that if we define B_θ as the inductive limit of $M_{q_n} \oplus M_{q_{n-1}}$ with some canonical choice of connecting maps, the isomorphism between B and B_θ will be somewhat complicated, as it must involve the W_n. Another way to put this is that we do not have a very good idea of how $M_{q_n} \oplus M_{q_{n-1}}$ sits inside B. ∎

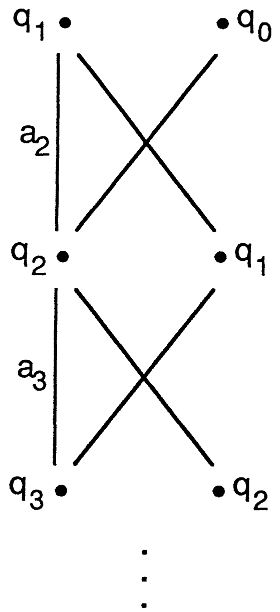

FIGURE 9a. The Bratteli diagram for the continued-fraction algebra.

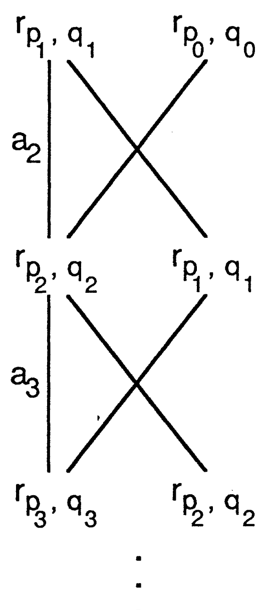

Figure 9b. A representation of the Pimsner-Voiculescu embedding.

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The preceding construction is summarized in Figure 9b. This will serve as a model for the definition of embedding diagram, given in the next section.

THEOREM 5.3. *If $\phi : A_\theta \longrightarrow B_\theta$ is a unital embedding then*

$$\phi_*: K_0(A_\theta) \longrightarrow K_0(B_\theta)$$

is an isomorphism.

PROOF. This is a triviality in the high-tech world of modern K-theory. Both A_θ and B_θ have unital traces, say σ and τ, which induce isomorphisms of $K_0(A_\theta)$ and $K_0(B_\theta)$ with $\mathbb{Z} + \theta\,\mathbb{Z}$. Since σ is the unique unital trace on A_θ, and ϕ is unital, ϕ must pull τ back to σ. Therefore, ϕ_* must be the identity map on $\mathbb{Z} + \theta\,\mathbb{Z}$. ∎

6. **Embedding Diagrams.** In this section we define embedding diagrams. These are embellished Bratteli diagrams which define simultaneously a homomorphism $\pi: C(X) \longrightarrow A$ and a unitary $U \in A$, where A is an AF algebra. The most interesting cases are those for which (π, U) is the covariant form of a homomorphism $C(X) \rtimes \mathbb{Z} \longrightarrow A$, but we do not yet limit ourselves to these cases.

Before we define embedding diagrams, we must clear up a few minor points about Bratteli diagrams. For technical reasons, it will be useful to allow diagrams which correspond to limit systems of finite-dimensional C^*-algebras

$$\cdots \longrightarrow A_n \xrightarrow{\phi_n} A_{n+1} \longrightarrow \cdots \tag{6.1}$$

which have non-injective connecting maps ϕ_n. For example, we consider the diagram in Figure 10 to be a Bratteli diagram.

Consider the system (6.1) and suppose $A_n = \oplus A_n^{(k)}$, with each $A_n^{(k)}$, isomorphic to $M_r(\mathbb{C})$ for some r. As usual, the connecting maps define homomorphisms

$$\phi_{n,m}: A_n \longrightarrow A_m,\ m \geq n,$$
$$\phi_{n,\infty}: A_n \longrightarrow A = \varinjlim A_n.$$

Consider the restrictions of these maps to $A_n^{(k)}$,

$$\phi_{n,m}^{(k)}: A_n^{(k)} \hookrightarrow A_n \longrightarrow A_m,\ m \geq n,$$
$$\phi_{n,\infty}^{(k)}: A_n^{(k)} \hookrightarrow A_n \longrightarrow A.$$

Since $A_n^{(k)}$ is simple, either $\phi_{n,\infty}^{(k)}$ is an embedding or zero. Also, $\phi_{n,\infty}^{(k)}$ will be nonzero if, and only if, $\phi_{n,m}^{(k)}$ is non-zero for all $m \geq n$.

We hope that the reader will forgive the following excessive formality which is necessary to allow easy statements of theorems.

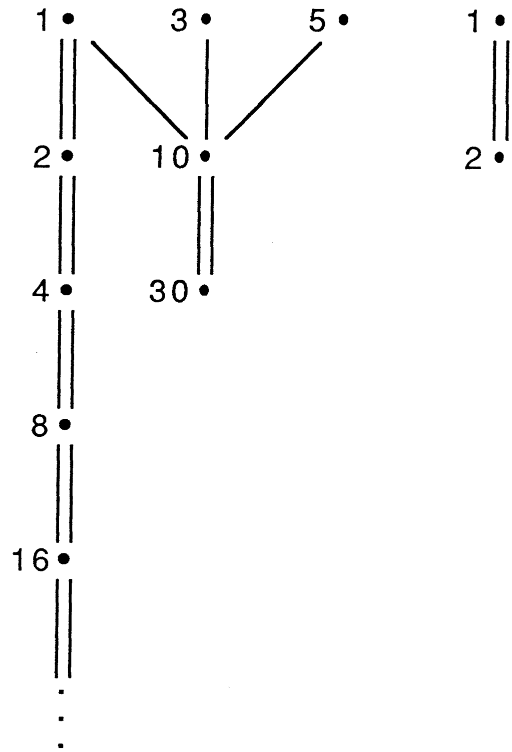

FIGURE 10. A Bratteli diagram with non-injective connecting maps.

DEFINITION 6.1. A *Bratteli diagram* $\mathcal{D}$ is a sequence $\mathcal{D}_n$ of finite sets (of vertices) together with positive integers (sizes and embedding multiplicities)

$$\begin{cases} s(v) & \text{for } v \in \mathcal{D}_n, \\ m(v,w) & \text{for } v \in \mathcal{D}_{n+1},\ w \in \mathcal{D}_n. \end{cases}$$

$\mathcal{D}$ is *unital* if

$$s(v) = \sum_{w \in \mathcal{D}_n} m(v,w)s(w), \ \text{for } v \in \mathcal{D}_{n+1}.$$

DEFINITION 6.2. A Bratteli diagram $\mathcal{D}$ is called an *embedding diagram* over a topological space X if, for each n,

(i) $\mathcal{D}_n$ is a finite set of pseudo-actions on X,

(ii) $s(\omega) = |\omega|$, for $\omega \in \mathcal{D}_n$

and

$$(iii) \qquad \sum_n \sup_{\omega \in \mathcal{D}_{n+1}} d\left(\omega, \sum_{\eta \in \mathcal{D}_n} m(\omega,\eta)\eta\right) < \infty.$$

Suppose A is an AF algebra, U is a unitary in A and $\pi\colon C(X) \longrightarrow A$ is a unital homomorphism. By universal nonsense, these define a unital homomorphism.

$$\phi\colon C(X) *_{\mathbb{C}} C(S^1) \longrightarrow A$$

sending f to $\pi(f)$ and $u = e^{2\pi i x}$ to U. Our real interest is in those cases where (U, π) is covariant for some action, i.e., cases where ϕ drops to a homomorphism

$$C(X) \rtimes \mathbb{Z} \longrightarrow A.$$

We will take up, in the next section, the question of when a homomorphism associated to an embedding diagram is covariant.

Suppose the $\mathcal{D}$ is an embedding diagram and that A is isomorphic to the limit of

$$\cdots \longrightarrow A_n \xrightarrow{\phi_n} A_{n+1} \longrightarrow$$

where

$$A_n = \bigoplus_{\omega \in \mathcal{D}_n} A_\omega$$

and A_ω is embedded into A_η with multiplicity $m(\eta, \omega)$. If, in addition, we have a homomorphism

$$\phi\colon C(X) *_{\mathbb{C}} (S^1) \longrightarrow A$$

such that

$$\phi(f) = \lim_{n \longrightarrow \infty} \phi_{n,\infty}(\rho_n(f)),$$
$$\phi(u) = \lim_{n \longrightarrow \infty} \phi_{n,\infty}(U_n)$$

where

$$U_n \in \mathcal{U}(A_n),$$
$$\rho_n\colon C(X) \longrightarrow A_n$$

are simultaneously unitarily equivalent to

$$\bigoplus_{\omega \in \mathcal{D}_n} U_\omega,$$
$$\bigoplus_{\omega \in \mathcal{D}_n} \pi_\omega,$$

then we say that ϕ is *associated to* $\mathcal{D}$.

PROPOSITION 6.3. *There always exists a homomorphism associated to an embedding diagram* $\mathcal{D}$.

PROOF. This is guaranteed by condition 6.2.iii. Specifically, we know that, given any connecting maps

$$\cdots \longrightarrow A_n \xrightarrow{\phi_n} A_{n+1} \longrightarrow \cdots,$$

which have the proper embedding multiplicities, we may correct them by defining $\phi'_n = Ad_{W_n} \circ \phi_n$, where W_n is a unitary in A_{n+1} chosen so that

$$\left\| \phi'_n \left(\bigoplus_{\omega \in \mathcal{D}_n} U_\omega \right) - \bigoplus_{\eta \in \mathcal{D}_{n+1}} U_\eta \right\|,$$
$$\left\| \phi'_n \left(\bigoplus_{\omega \in \mathcal{D}_n} \pi_\omega(g_j) \right) - \bigoplus_{\eta \in \mathcal{D}_{n+1}} \pi_\eta(g_j) \right\|, \; j = 1, \ldots, k,$$

are summable sequences. This means that

$$\lim_{n \to \infty} \phi'_{n,\infty}(\oplus U_\omega),$$
$$\lim_{n \to \infty} \phi'_{n,\infty}(\oplus \pi_\omega(g_j))$$

exist. We may define a homomorphism associated to $\mathcal{D}$ by sending $u, g_1, \ldots, g_k$ to these elements of $\varinjlim(A_n, \phi'_n)$. ∎

We do not know to what extent these associated homomorphisms are unique. Certainly, any two will induce the same map on $K_0(C(X) *_{\mathbb{C}} C(S^1))$, (or on $K_0(C(X) \rtimes \mathbb{Z})$ if the maps drop to the cross-product.) We also do not know what is the appropriate notion of equivalence for embedding diagrams. We will make do with the following weak notion of equivalence.

Suppose $\mathcal{D}$ and $\mathcal{E}$ are embedding diagrams with identical underlying Bratteli diagrams. Thus, we have bijections

$$\Gamma_n : \mathcal{D}_n \longrightarrow \mathcal{E}_n$$

which preserve embedding multiplicities and vertex sizes. If

$$\lim_{n \to \infty} \sup_{\omega \in \mathcal{D}_n} d(\omega, \Gamma_n(\omega)) = 0$$

then we shall say that $\mathcal{D}$ and $\mathcal{E}$ are *weakly equivalent*. In this case, it follows that a homomorphism is associated to $\mathcal{D}$ if, and only if, it is associated to $\mathcal{E}$.

We have already seen one embedding diagram, Figure 9b. Let us consider another. This will be one of the simplest examples of the type of embedding diagram considered in joint work with Elliott.

Consider the diagram in Figure 11a. To see that the summability condition is fulfilled, we must evaluate the maximum of the distances

$$d(r_{1,3^{n+1}}, 2r_{1,3^n} + r_{-1,3^n}),$$
$$d(r_{-1,3^{n+1}}, r_{1,3^n} + 2r_{-1,3^n}).$$

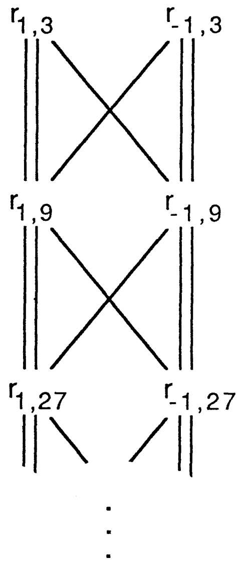

FIGURE 11a. An embedding diagram for mapping $C(T^2)$ into a simple AF algebra.

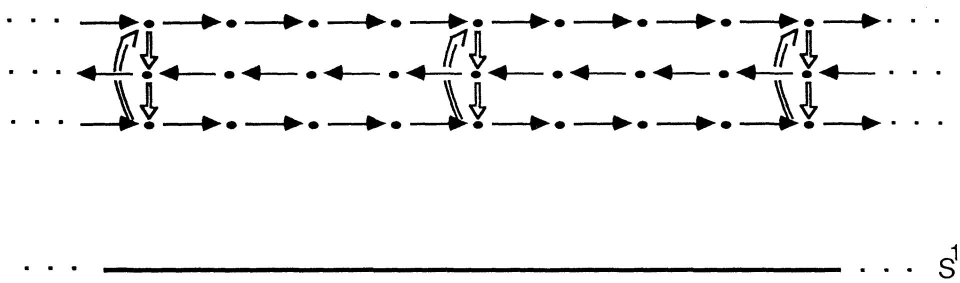

FIGURE 11b. Pseudo-actions and an interchange showing why the summability condition 6.2.iii holds.

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By symmetry, these are equal. Figure 11b illustrates how one may apply Theorem 4.1 to conclude that, for all n,

$$d(r_{1,3^{n+1}}, 2r_{1,3^n} + r_{-1,3^n}) \leq C/3^{n/2}$$

for some constant C (cf. Figure 7). As this is summable, we have an embedding diagram.

Any associated homomorphism will drop to the trivial cross-product $C(S^1) \rtimes \mathbb{Z}$ which is isomorphic to $C(T^2)$. Let $\phi : C(T^2) \longrightarrow A$ be such a map.

Consider the underlying Bratteli diagram and the associated system

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}} \cdots.$$

The homomorphisms

$$\mathbb{Z}^2 \xrightarrow{\begin{pmatrix} 1/3^n & 1/3^n \\ 1 & -1 \end{pmatrix}} \mathbb{R}^2$$

fit together to map the limit group into $\mathbb{R}^2$. It is now easy to see that $K_0(A)$ is a subgroup of $\mathbb{Z}[1/3] \oplus \mathbb{Z}$, namely

$$K_0(A) \cong \{(m/3^n, k) \mid m \equiv k \pmod 2\}$$

with order unit $(2,0)$ and the strict order from the first component. Notice that A is simple. Finally, arguing as in the proof of [6; Theorem 3.4], one may show that the induced homomorphism

$$\phi_*: K_0(C(T^2)) \cong \mathbb{Z} \oplus \mathbb{Z} \longrightarrow K_0(A)$$

is an injection, sending (a,b) to $(2a, 2b)$.

7. **Approximating actions by pseudo-actions.** For the rest of this paper, we shall assume that $T: X \longrightarrow X$ is a homeomorphism. We let $\alpha: C(X) \longrightarrow C(X)$ denote the associated aciton, and are interested in mappings of $C(X) \rtimes_\alpha \mathbb{Z}$ into AF algebras. Pimsner attacks this problem by considering sequences of open sets, taken from a cover of X, which approximate T. We are able to work in a similar fashion by considering a cover of X to be a discrete topological space and investigating those pseudo-actions over the cover which mimic the action of T.

For our purposes, a cover $\mathcal{V}$ is a finite cover of X by open sets. By the diameter of $\mathcal{V}$, we mean

$$\operatorname{diam} \mathcal{V} = \sup_{V \in \mathcal{V}} \sup_{x,y \in V} d(x,y).$$

The order $|\mathcal{V}|$ of $\mathcal{V}$ is the number of open sets it contains.

DEFINITION 7.1. Let $\mathcal{V}$ be a cover of X, and let $\omega = (I, \omega, \alpha)$ be a pseudo-action over $\mathcal{V}$. We say that $\mathcal{V}$ *respects* T if

$$T(\omega(j)) \cap \omega(\alpha(j)) \neq \emptyset$$

for all $j \in I$.

By themselves, pseudo-actions over a cover $\mathcal{V}$ are not useful. We are interested in $C(X)$, not $C(\mathcal{V})$. By choosing a point out of $\omega(j)$, for each j, we obtain a pseudo-action back on X.

DEFINITION 7.2. Suppose $\mathcal{V}$ is a cover of X and $\omega = (I, \omega, \alpha)$ is a pseudo-action over $\mathcal{V}$. If $\bar{\omega}$ is a pseudo-action over X, then $\bar{\omega}$ is *contained in* ω, denoted

$$\bar{\omega} \in \omega,$$

if $\bar{\omega}$ is (isomorphic to) $(I, \bar{\omega}, \alpha)$ where $\bar{\omega}: I \longrightarrow X$ is such that

$$\bar{\omega}(j) \in \omega(j)$$

for all $j \in I$.

Our strategy is to produce a diagram of pseudo-action on covers such that, by choosing pseudo-actions on X contained in these, we obtain an embedding diagram. The following lemmas will be used to show that the resulting homomorphism is covariant, if the pseudo-actions respect T, and does not depend on the choices made.

LEMMA 7.3. *For each $F \in C(X)$, and each $\varepsilon > 0$, there exists $\delta > 0$ such that, whenever ω is a pseudo-action over a cover $\mathcal{V}$ for which* diam $\mathcal{V} < \delta$ *and ω respects T, and given $\bar{\omega} \in \omega$, we have*

$$\|\pi_{\bar{\omega}}(\alpha(f)) - U_{\bar{\omega}}\pi_{\bar{\omega}}(f)U_{\bar{\omega}}^*\| < \varepsilon.$$

PROOF. The operator to be estimated is diagonal, so

$$\|\pi_{\bar{\omega}}\big(\alpha(f)\big) - U_{\bar{\omega}}\pi_{\bar{\omega}}(f)U_{\bar{\omega}}^*\| = \sup_i \left| f\left(T^{-1}\big(\bar{\omega}(i)\big)\right) - f\left(\bar{\omega}\big(\alpha^{-1}(i)\big)\right)\right|.$$

Let $i \in I_\omega$. Since ω respects T, there exists a

$$y \in T\Big(\omega\big(\alpha^{-1}(i)\big) \cap \omega(i)\Big).$$

Therefore,

$$\begin{aligned}
&\left| f\Big(T^{-1}\big(\bar{\omega}(i)\big)\Big) - f\Big(\bar{\omega}\big(\alpha^{-1}(i)\big)\Big)\right| \\
&\qquad \leq \left| f \circ T^{-1}\big(\bar{\omega}(i)\big) - f \circ T^{-1}(y)\right| + \left| f\big(T^{-1}(y)\big) - f\Big(\bar{\omega}\big(\alpha^{-1}(i)\big)\Big)\right| \\
&\qquad \leq \sup_{d(x,y) < \operatorname{diam} \mathcal{V}} |f \circ T^{-1}(x) - f \circ T^{-1}(y)| + \sup_{d(x,y) < \operatorname{diam} \mathcal{V}} |f(x) - f(y)|
\end{aligned}$$

and by the uniform continuity of f and $f \circ T^{-1}$, this converges to zero as diam $\mathcal{V}$ tends to zero. ∎

LEMMA 7.4. *Suppose that ω is a pseudo-action over a cover $\mathcal{V}$. If $\bar{\omega} \in \omega$ and $\bar{\bar{\omega}} \in \omega$, then*

$$d(\bar{\omega}, \bar{\bar{\omega}}) \leq \operatorname{diam} \mathcal{V}.$$

The proof is obvious, so we omit it.

8. **Decomposing pseudo-actions.** The more closely a pseudo-action approximates the homeomorphism T, the larger it tends to be. If we are to be able to construct a covariant embedding diagram out of pseudo-actions which approximate T, we must have a way of decomposing, at least approximately, a large pseudo-action into smaller ones. In this section, we introduce a simple notion, called the m-decomposition of pseudo-actions on covers, which translates, via the Berg-Pimsner technique, into an approximation theorem for pseudo-actions on X.

DEFINITION 8.1. A pseudo-action ω on a finite set S *1-decomposes* into the tuple $(\eta_1, \ldots, \eta_\ell)$ of pseudo-actions on S if there exists a permutation $\delta : I_\omega \longrightarrow I_\omega$ such that

$$\omega(\delta(j)) = \omega(j),\ j \in I_\omega,$$
$$(I_\omega, \omega, \delta \circ \alpha_\omega) \cong \sum \eta_k.$$

If, in addition, there exists such a δ for which

$$\omega\Big(\alpha^{-r}\big(\delta(j)\big)\Big) = \omega\big(\alpha^{-r}(j)\big),\ j \in I_\omega,\ r = 0, \ldots, m-1$$
$$\alpha^{-r}(I_0) \cap I_0 = \emptyset,\ r = 1, \ldots, m-1,$$

where $I_0 = \{\, i \mid \sigma(i) \neq i \,\}$, then we say that ω *m-decomposes* into $(\eta_1, \ldots, \eta_\ell)$.

Consider the pseudo-action illustrated in Figure 12a. Figure 12b illustrates a 1-decomposition of ω into three pseudo-actions; the double lines illustrate the implementing δ. This ω can only be 2-decomposed into two pseudo-actions, as illustrated in Figure 12c.

Our interest in m-decompositions is limited to the case $S = \mathcal{V}$, a cover of X, and where the pseudo-actions respect T.

LEMMA 8.2. *Suppose ω and $\eta_1, \ldots, \eta_\ell$ are pseudo-actions over $\mathcal{V}$, a cover of X, and T is a homeomorphism of X. If ω m-decomposes into $(\eta_1, \ldots, \eta_\ell)$, then ω respects T if, and only if, each η_j respects T.*

PROOF. It suffices to observe that if (I, ω, α) respects T and $\delta : I \longrightarrow I$ is a permutation such that

$$\omega\big(\delta(i)\big) = \omega(i),\ i \in I,$$

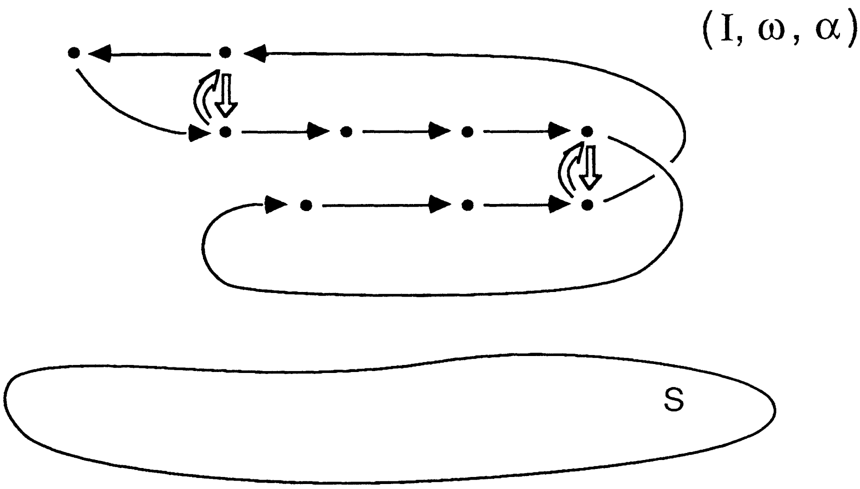

FIGURE 12a. A pseudo-action ω.

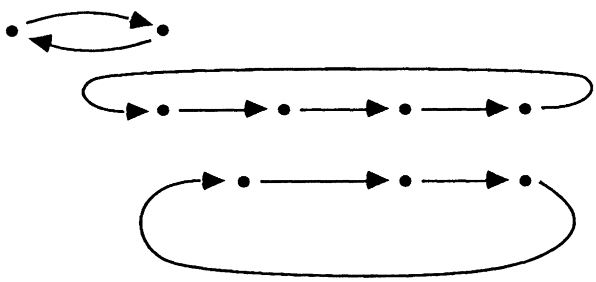

FIGURE 12b. A 1-decomposition of ω.

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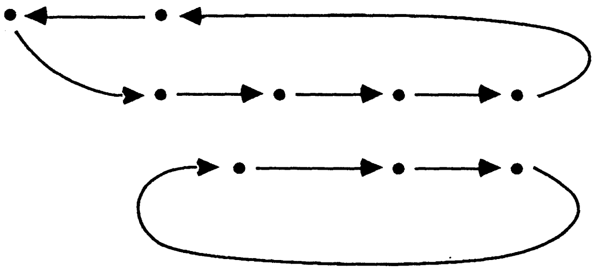

FIGURE 12c. A 2-decomposition of ω.

then $(I, \omega, \delta \circ \alpha)$ also respects T. This is true because

$$T\big(\omega(i)\big) \cap \omega\big(\delta \circ \alpha(i)\big) = T\big(\omega(i)\big) \cap \omega\big(\alpha(i)\big) \neq 0.$$

■

We now show that sufficiently large pseudo-actions may be m-decomposed into smaller ones.

LEMMA 8.3. *Let m be a non-negative integer, and let S be a finite set of cardinality k. Every pseudo-action over S can be m-decomposed into pseudo-actions of order at most* $2m \cdot k^m$.

PROOF. This proof is basically a simple induction based on the pigeon-hole principle. However, there is a minor detail arising when $m > 1$ which forces us to prove a stronger claim to facilitate the induction. Notice it suffices to prove the lemma for cyclic pseudo-actions of order at least m.

Given a pseudo-action ω over S, by a *marking* of ω we shall mean a subset $M(\omega)$ of I_ω, called the set of marked points, such that the "distances"

$$\min\{r > 0 \mid \alpha^r(i) \in M(\omega)\}, \; i \in M(\omega)$$

between marked points are in the interval [m,2m-1]. Markings always exist for cyclic pseudo-actions of order at least m.

What we shall prove is that given a marked pseudo-action ω over S, there is a permutation δ of I_ω, fixing the unmarked points $I_\omega \setminus M(\omega)$ and implementing an m-decomposition of ω into pseudo-actions of order at most $2m \cdot k^m$. This is trivially true for $|\omega| \leq 2m \cdot k^m$.

Assume that ω is a marked pseudo-action with $|\omega| > 2m \cdot k^m$ and that our claim is true for marked pseudo-actions of lesser order. Because of the size of ω, there must be two sequences of the form

$$\omega\big(\alpha^{-m+1}(i)\big), \omega\big(\alpha^{-m+2}(i)\big), \ldots, \omega(i), \quad i \in M(\omega),$$
$$\omega\big(\alpha^{-m+1}(j)\big), \omega\big(\alpha^{-m+2}(j)\big), \ldots, \omega(j), \quad j \in M(\omega), \; i \neq j,$$

which are identical.

If i and j lie in distinct orbits of α, then let δ_0 be the trivial permutation of I_ω. Otherwise, let δ_0 switch i and j. Either way, δ_0 implements an m-decomposition of ω into two pseudo-actions ω_1, ω_2 of lesser order,

$$\omega_i = \left(I_i, \omega|_{I_i}, (\delta_0 \circ \alpha)|_{I_i}\right).$$

Mark ω_i by $M(\omega) \cap I_i$. By induction, there exists permutations $\delta_i : I_i \longrightarrow I_i$, fixing the unmarked points and implementing m-decompositions into sufficiently small pseudo-actions.

Let $\delta = (\delta_1 \cup \delta_2) \circ \delta_0$. Clearly,

$$\omega\left(\alpha^{-r}\big(\delta(j)\big)\right) = \omega\big(\alpha^{-r}(j)\big), \quad r = 0, \ldots, m-1$$

since this equation holds for δ_0, δ_1 and δ_2. Finally, δ fixes the unmarked points, so $I_0 \subset M(\omega)$ and

$$\alpha^{-r}(I_0) \cap I_0 \subset \alpha^{-r}\big(M(\omega)\big) \cap M(\omega), \quad r = 1, \ldots, m-1$$

which is empty because the marked points are "distance" at least m apart. ∎

THEOREM 8.4. *Suppose ω is a pseudo-action on the cover $\mathcal{V}$, and ω m-decomposes into $(\eta_1, \ldots, \eta_\ell)$. Then*

$$d(\bar{\omega}, \textstyle\sum \bar{\eta}_j) \leq \max\{\pi/m, 3 \cdot \operatorname{diam} \mathcal{V}\}$$

whenever $\bar{\omega} \in \omega$ and $\bar{\eta}_j \in \eta_j, j = 1, \ldots, \ell$.

PROOF. This is an immediate Corollary to Theorem 4.1. ∎

9. **Pimsner's Construction.** We now present a generic method of constructing homomorphisms of $C(X) \rtimes \mathbb{Z}$ to AF algebras. We do not impose any conditions on T until the next section where we discuss when the homomorphisms constructed are embeddings. While based on Pimsner's work [7], this construction produces a wider class of homomorphisms. The AF algebras produced in [7] always contain UHF subalgebras. This is not true of our construction.

We will use the following notation, borrowed from Pimsner. If $\mathcal{V}$ and $\mathcal{W}$ are covers, then the expression

$$\mathcal{V} \preceq_f \mathcal{W}$$

shall mean that $f : \mathcal{W} \longrightarrow \mathcal{V}$ is a function such that

$$W \subseteq f(W) \text{ for all } W \in \mathcal{W}.$$

Of course, this means in particular that $\mathcal{W}$ is finer than $\mathcal{V}$.

Now suppose that $\mathcal{V} \preceq_f \mathcal{W}$ and $\omega = (I, \omega, \alpha)$ is a pseudo-action over $\mathcal{W}$. We shall let $f\omega$ denote the pseudo-action over $\mathcal{V}$

$$f\omega = (I, f \circ \omega, \alpha).$$

Clearly, $\bar{\omega} \in \omega$ implies $\bar{\omega} \in f\omega$, and if ω respects T then $f\omega$ respects T.

DEFINITION 9.1. Let T be a homeomorphism of X. A *Pimsner diagram* for T consists of a sequence of covers $\mathcal{V}_n$ of X such that

$$\sum \operatorname{diam} \mathcal{V}_n < \infty,$$

a sequence of functions $f_n \colon \mathcal{V}_{n+1} \longrightarrow \mathcal{V}_n$ such that

$$\mathcal{V}_n \preceq_{f_n} \mathcal{V}_{n+1},$$

a sequence of positive integers m_n such that

$$\sum 1/m_n < \infty,$$

and finally a Bratteli diagram $\mathcal{D}$ such that each $\omega \in \mathcal{D}_n$ is a pseudo-action over $\mathcal{V}_n$ which respects T and, for each $\omega \in \mathcal{D}_{n+1}$,

$$f_n\omega \; m_n\text{-decomposes into } \sum_{\eta \in \mathcal{D}_n} m(\omega, \eta)\eta.$$

PROPOSITION 9.2. *Suppose that $(\mathcal{V}_n, f_n, m_n, \mathcal{D}_n)_n$ is a Pimsner diagram for T. For each $\omega \in \mathcal{D}_n$ choose $\bar{\omega} \in \omega$. Let $\bar{\mathcal{D}}$ denote the Bratteli diagram obtained from $\mathcal{D}$ by replacing each ω by $\bar{\omega}$.*

1. *$\bar{\mathcal{D}}$ is an embedding diagram.*
2. *Any homomorphism associated to $\bar{\mathcal{D}}$ drops to $C(X) \rtimes \mathbb{Z}$.*
3. *Up to weak equivalence, $\bar{\mathcal{D}}$ does not depend on the choices of the $\bar{\omega}$.*

PROOF. The three parts of this theorem are immediate consequences of Theorem 8.4, Lemma 7.3 and Lemma 7.4, respectively. ∎

Given a specific $T \colon X \longrightarrow X$, one should select a reasonably small collection of pseudo-actions over each cover $\mathcal{V}_n$ and then add to these any pseudo-actions that arise in the decomposition of higher-level pseudo-actions. This should produce a managable Pimsner diagram to study.

For generic T, about the only natural way to restrict the collection of all pseudo-actions to make it finite is by size. For any $T \colon X \longrightarrow X$, any cover $\mathcal{V}$ and any $m > 0$, we let $\Omega(\mathcal{V}, T, m)$ denote the set of isomorphism classes of pseudo-actions over $\mathcal{V}$, of order at most $2m|\mathcal{V}|^m$, which respect T. (Actually, we regard the elements of $\Omega(\mathcal{V}, T, m)$ as pseudo-actions, one representative from each class.)

THEOREM 9.3. *Given a homeomorphism $T \colon X \longrightarrow X$ and given sequences of open covers $\mathcal{V}_n$, functions f_n and positive integers m_n such that*

$$\mathcal{V}_n \preceq_{f_n} \mathcal{V}_{n+1},$$
$$\sum 1/m_n < \infty,$$
$$\sum \operatorname{diam} \mathcal{V}_n < \infty,$$

there exists a Bratteli diagram $\mathcal{D}$ with

$$\mathcal{D}_n = \Omega(\mathcal{V}_n, T, m_n)$$

such that $(\mathcal{V}_n, f_n, m_n, \mathcal{D}_n)_n$ is a Pimsner diagram for T.

PROOF. All the data in the Pimsner diagram is determined except for the multiplicities $m(\omega,\eta)$ for $\omega \in \mathcal{D}_{n+1}$ and $\eta \in \mathcal{D}_n$. Given $\omega \in \mathcal{D}_{n+1}$, Lemma 8.3 implies that $f_n\omega$ can be m_n-decomposed into a list $(\eta_1,\ldots,\eta_\ell)$ of pseudo-actions over $\mathcal{V}_n$ of order at most $2m_n|\mathcal{V}_n|^{m_n}$. By Lemma 8.2., each η_j respects T and so is an element of $\mathcal{D}_n$. For $\eta \in \mathcal{D}_n$, we let $m(\omega,\eta)$, equal the number of times η occurs in this list. ∎

Notice that there is no unique, or canonical, way to choose the multiplicities $m(\omega,\eta)$.

10. **Injectivity.** In the last section (Theorem 9.3), we introduced a generic method for constructing Pimsner diagrams, and thus embedding diagrams. The resulting homomorphisms of $C(X) \rtimes \mathbb{Z}$ into AF algebras are injective, as we show in this section, except when there is a topological obstruction to the cross-product being a subalgebra of any AF algebra. Everything in this section is due to Pimsner, although we have streamlined some of his proofs.

DEFINITION 10.1. A point $x \in X$ is called *pseudo-non-wandering* for the homeomorphism $T: X \longrightarrow X$ if, for every cover $\mathcal{V}$ and every $V \in \mathcal{V}$ such that $x \in V$, there exists a finite sequence $V_0, V_1, \ldots, V_\ell \in \mathcal{V}$, $\ell \geq 1$, with

$$V_0 = V_\ell = V,$$
$$T(V_j) \cap V_{j+1} \neq \emptyset \text{ for } j = 0,\ldots,\ell-1.$$

An equivalent way of saying that $x \in X$ is pseudo-non-wandering is that, for $x \in V \in \mathcal{V}$, there exists a (finite) pseudo-action ω over $\mathcal{V}$ that respects T with $\omega(i) = V$ for some i.

It is relatively straightforward to show that having every $x \in X$ be pseudo-non-wandering is a necessary condition for the existence of an AF embedding of $C(X) \rtimes \mathbb{Z}$. We shall simply state the relevant results from [7].

LEMMA 10.2 (PIMSNER). *A point $x \in X$ is not pseudo-non-wandering if, and only if, there exists an open set $U \subset X$, with closure $\bar{U}$, such that $T(\bar{U}) \subset U$ and $x \in U \setminus T(\bar{U})$.*

Given U as in Lemma 10.2, one may easily construct a non-unitary isometry in the cross-product.

LEMMA 10.3 (PIMSNER). *If there exists $x \in X$ which is not pseudo-non-wandering, then $C(X) \rtimes \mathbb{Z}$ contains a non-unitary isometry.*

Since all isometries in an AF algebra are unitary, this lemma implies the following.

LEMMA 10.4 (PIMSNER). *If there exists $x \in X$ which is not pseudo-non-wandering, then $C(X) \rtimes \mathbb{Z}$ cannot be embedded into an AF algebra.*

The key ingredient in the proof of the converse is the next lemma which shows that there are sufficiently many pseudo-actions that approximate the orbit of a pseudo-non-wandering point.

LEMMA 10.5. *Suppose $x \in X$ is pseudo-non-wandering and m is a positive integer. For any cover $\mathcal{V}$, and any sequence*

$$V_0, V_1, \ldots, V_{m-1} \in \mathcal{V}$$

with

$$T^j x \in V_j, j = 0, \ldots, m-1,$$

there exists a pseudo-action ω over $\mathcal{V}$ which respects T such that, for some $i_0 \in I_\omega$,

$$\omega\big(\alpha^j(i_0)\big) = V_j, j = 0, \ldots, m-1.$$

PROOF. If x is periodic, this is trivial, so assume that $T^j x = T^k x$ only if $j = k$.

Choose disjoint open sets $Q_j, j = 0, \ldots, m-1$, such that

$$T^j x \in Q_j, j = 0, \ldots, m-1,$$

Next, choose open sets R_{m-1}, S_{m-1} such that

$$T^{m-1} x \in S_{m-1},$$
$$\bar{S}_{m-1} \subset R_{m-1} \subset Q_{m-1} \cap V_{m-1}$$

and also choose open sets $R_j, S_j, j = m-2, \ldots, 1, 0$, such that

$$T^j x \in S_j,$$
$$\bar{S}_j \subset R_j \subset Q_j \cap T^{-1}(S_{j+1}) \cap V_j.$$

Consider the cover

$$\mathcal{W} = \{R_j \mid j = 0, \ldots, m-1\} \cup \{V \backslash \cup \bar{S}_\ell \mid V \in \mathcal{V}\}.$$

Notice that $\mathcal{V} \preceq_f \mathcal{W}$ where

$$f(R_j) = V_j, j = 0, \ldots, m-1,$$
$$f(V \backslash \cup \bar{S}_\ell) = V, V \in \mathcal{V}.$$

Since x is pseudo-non-wandering and $x \in R_0$, there is a pseudo-action η over $\mathcal{W}$ which respects T and such that $\eta(i_0) = R_{i_0}$ for some $i_0 \in I_\eta$. Now, for $j < m-1$,

$$T(R_j) \subset R_{j+1}$$

but

$$T(R_j) \cap R_k \subset Q_{j+1} \cap Q_k = \emptyset, k \neq j+1,$$
$$T(R_j) \cap (V \backslash \cup \bar{S}_\ell) \subset S_{j+1} \cap (V \backslash S_{j+1}) = \emptyset, V \in \mathcal{V},$$

so the only way for η to respect T is by satisfying

$$\eta\big(\alpha^j(i_0)\big) = R_j.$$

The desired pseudo-action is $\omega = f\eta$. ∎

Lemma 10.5 says nothing about the size of the pseudo-action ω which follows the orbit of x. However, the next lemma shows that by m-decomposing ω, for appropriate m, we may produce a reasonably small pseudo-action tracking x.

LEMMA 10.6. *Suppose ω is a pseudo-action over a finite set S and $s_0, s_1, \ldots, s_{n-1}$ is a sequence in S such that*

$$\omega\big(\alpha^j(i_0)\big) = s_j, j = 0, 1, \ldots, n-1$$

for some $i_0 \in I_\omega$. If $m \geq n$ and ω m-decomposes into $(\eta_1, \ldots, \eta_\ell)$ then, for some $\eta_k = (J_k, \eta_k, \beta_k)$,

$$\eta_k\big(\beta_k^j(i_1)\big) = s_j,\ j = 0, 1, \ldots, n-1$$

for some $i_1 \in J_k$.

PROOF. Suppose δ implements the m-decomposition, so

$$(I, \omega, \delta \circ \alpha) \cong \sum \eta_k.$$

Since $m \geq n$, at most one of

$$i_0, \alpha(i_0), \ldots, \alpha^{n-1}(i_0) \tag{10.1}$$

lies in the set $I_0 = \{\, i \mid \delta(i) \neq i \,\}$. If none lie in I_0, then let $i_1 = i_0$ and observe that

$$(\delta \circ \alpha)^j(i_1) = \alpha^j(i_0), j = 0, 1, \ldots, n-1.$$

In this case then, we are done since some η_k must contain a copy of the sequence (10.1) and must map it to $s_0, \ldots, s_{n-1}$.

In the remaining case, let $\alpha^d(i_0)$ be the unique element of the sequence (10.1) contained in I_0, and let $i_1 = \alpha^{-d} \circ \delta^{-1} \circ \alpha^d(i_0)$. Since $\alpha^d(i_1) = \delta^{-1}(\alpha^d(i_0))$ is in I_0, no other element of the sequence

$$i_1, \alpha(i_1), \ldots, \alpha^{n-1}(i_1)$$

is contained in I_0. Therefore

$$(\delta \circ \alpha)^j(i_1) = \begin{cases} \alpha^j(i_1) & \text{for } 0 \leq j < d \\ \alpha^j(i_0) & \text{for } d \leq j < n \end{cases}.$$

Since δ implements an m-decomposition,

$$\begin{aligned} \omega\big(\alpha^j(i_1)\big) &= \omega\Big(\alpha^{j-d}\big(\delta^{-1}\big(\alpha^d(i_0)\big)\big)\Big) \\ &= \omega\Big(\alpha^{j-d}\big(\alpha^d(i_0)\big)\Big) = \omega\big(\alpha^j(i_0)\big) \end{aligned}$$

for $j < d$. Therefore

$$\omega\big((\delta \circ \alpha)^j(i_1)\big) = \omega\big(\alpha^j(i_0)\big) = s_j, j = 0, 1, \ldots, n-1$$

and we are done as in the first case. ∎

THEOREM 10.7. *Suppose that $\mathcal{D} = (\mathcal{V}_n, f_n, m_n, \mathcal{D}_n)_n$ is a Pimsner diagram for T and $\mathcal{D}_n = \Omega(\mathcal{V}_n, T, m_n)$. If every point in X is pseudo-non-wandering and*

$$\phi : C(X) \rtimes \mathbb{Z} \longrightarrow A$$

is a homomorphism associated to $\mathcal{D}$, then ϕ is an embedding.

PROOF. For each $\omega \in \mathcal{D}_n$, we let $\bar{\omega}$ denote some choice of pseudo-action $\bar{\omega} \in \omega$. We shall write A_ω for $A_{\bar{\omega}}$ and so forth. By $i_{\omega,N}, N \geq n$, and $i_{\omega,\infty}$ we shall mean the canonical homomorphisms

$$\text{and} \qquad A_\omega \hookrightarrow A_n \longrightarrow A_{n+1} \longrightarrow \cdots \longrightarrow A_N,$$

$$\text{where} \qquad A_\omega \hookrightarrow A_n \longrightarrow A$$

$$A_n \cong \bigoplus_{\eta \in \mathcal{D}_n} A_\eta .$$

Given $\eta \in \mathcal{D}_n$ and $\omega \in \mathcal{D}_{n+1}$, we shall write $\eta < \omega$ if η is in the m_n-decomposition of $f_n\omega$, i.e., if $m(\omega, \eta) \neq 0$. Given $\eta \in \mathcal{D}_n$ and $\omega \in \mathcal{D}_{n+z}$, we shall write $n << \omega$ if there exist $\eta_1, \ldots, \eta_{z-1}$ such that

$$\eta < \eta_1 < \cdots < \eta_{z-1} < \omega .$$

Let

$$\mathcal{D}'_n = \{\, \omega \in \mathcal{D}_n \mid \forall N > n, \exists \gamma \in \mathcal{D}_N \text{ such that } \omega << \gamma \,\}.$$

Recall that $i_{\omega,\infty}$ is an isometry unless $i_{\omega,N} = 0$ for some N. Therefore, $i_{\omega,\infty}$ is an isometry if, and only if, $\omega \in \mathcal{D}'_n$.

If $x \in X$ and ω is a pseudo-action over a cover $\mathcal{V}$, we shall say that ω *tracks x for ℓ steps* if, for some $i_0 \in I_\omega$,

$$T^j(x) \in \omega\big(\alpha^j(i_0)\big) \text{ for } j = 0, \ldots, \ell - 1.$$

Given integers $n < N$ and $x \in X$, let

$$\Gamma(x, n, N) = \{\, \omega \in \mathcal{D}_n \mid \omega \text{ tracks } x \text{ for } m_n \text{ steps and } \omega << \eta \text{ for some } \eta \in \mathcal{D}_N \,\}.$$

Applying Lemma 10.5 to the cover $\mathcal{V}_N$ and integer m_n, we find a pseudo-action γ on $\mathcal{V}_N$ which respects T such that γ tracks x for m_n steps. By Lemmas 10.6, 8.2 and 8.3, we may also assume that $\gamma \in \mathcal{D}_N$. Repeatedly applying Lemma 10.6, one easily sees that there exists $\omega \in \mathcal{D}_n$ such that ω tracks x for m_n steps and $\omega << \gamma$. Thus $\Gamma(x, n, N)$ is nonempty. Finally,

$$\Gamma(x, n, n+1) \supseteq \Gamma(x, n, n+2) \supseteq \cdots$$

is a decreasing sequence of nonempty finite sets, so

$$\begin{aligned} \Gamma(x, n) &= \bigcap_{N > n} \Gamma(x, n, N) \\ &= \{\, \omega \in \mathcal{D}'_n \mid \omega \text{ tracks } x \text{ for } m_n \text{ steps} \,\} \end{aligned}$$

is nonempty for every x.

The rest of this proof follows Pimsner almost verbation ([7, pp. 624–625]). It is included for completeness.

Recall that, for $a \in C(X) \rtimes \mathbb{Z}$,

$$\|a\| = \sup_{x \in X} \|\lambda_x(a)\|$$

where $\lambda_x\colon C(X) \rtimes \mathbb{Z} \longrightarrow \mathcal{B}(\ell^2(\mathbb{Z}))$ is defined by

$$\lambda_x(f)e_n = f(T^n x)e_n,$$
$$\lambda_x(u)e_n = e_{n+1}.$$

We need only prove injectivity on elements of the form

$$a = \sum_{i=0}^{N} f_i u^i.$$

For any $\varepsilon > 0$, we can find $x \in X$ and a sequence $(\xi_k)_{k=1}^{\infty}$ of scalars such that

$$\sum |\xi_k|^2 = 1$$

and

$$\|a\|^2 \leq \sum_{k=1}^{\infty} \left| \sum_{i=0}^{N} \xi_{k-i} f_i(T^k x) \right|^2 + \varepsilon / 2.$$

We may also assume, perhaps by replacing x, that ξ^k is supported on $[0, M]$ for some M.

Let

$$\delta_n = \max_{i=0,\ldots,N} \sup_{d(x,y) < \operatorname{diam} \mathcal{V}_n} |f_i(x) - f_i(y)|.$$

Notice that $\delta_n \longrightarrow 0$ as $n \longrightarrow \infty$. Let n_0 be an integer such that

$$m_{n_0} > M + N.$$

If $\omega \in \Gamma(x, n)$, where $n \geq n_0$, then there is a $j_0 \in I_\omega$ such that

$$\bar{\omega}\left(\alpha^k(j_0)\right), T^k x \in \omega(k) \in \mathcal{V}_n,\ k = 0, 1, \ldots, m_{n_0} - 1,$$

and so

$$\left| f_i\left(\bar{\omega}\left(\alpha^k(j_0)\right)\right) - f_i(T^k x) \right| \leq \delta_n$$

for all $i, k = 0, \ldots M + N$. Therefore

$$\begin{aligned}
\left\| i_{\omega,\infty}\left(\sum_{i=0}^{N} \pi_\omega(f_i) U_\omega^i \right) \right\|^2 &= \left\| \sum_{i=0}^{N} \pi_\omega(f_i) U_\omega^i \right\|_{A_\omega}^2 \\
&\geq \left\| \sum_{i=0}^{N} \pi_\omega(f_i) U_\omega^i \left(\sum_{k=0}^{M} \xi_k e_k \right) \right\|^2 \\
&= \sum_{k=0}^{M+N} \left| \sum_{i=0}^{N} \xi_{k-i} f_i\left(\bar{\omega}\left(\alpha^k(j_0)\right)\right) \right|^2 \\
&\geq \sum_{k=0}^{M+N} \left(\left| \sum_{i=0}^{N} \xi_{k-i} f_i(T^k x) \right| - (N+1)\delta_n \right)^2 \\
&> \|a\|^2 - \varepsilon \text{ for large } n.
\end{aligned}$$

Since ε was arbitrary,

$$\left\| i_{\omega,\infty}\left(\sum_{i=1}^{N} \pi_\omega(f_i)U_\omega^i\right)\right\| = \|a\|$$

and

$$\begin{aligned}\|\psi(a)\| &= \lim_n \sup_{\omega\in\mathcal{D}_n} \left\| i_{\omega,\infty}\left(\sum_{i=0}^{N} \pi_\omega(f_i)U_\omega^i\right)\right\| \\ &= \|a\|.\end{aligned}$$

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